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Numerical Solution of Burgers Equation using Finite Difference Methods: Analysis of Shock Waves in Aircraft Dynamics

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ABSTRACT

In this research, the Lax, the Upwind, and the MacCormack finite difference methods are applied to the experimental solving of the one-dimensional (1D) unsteady Burger's Equation, a Hyperbolic Partial Differential Equation. These three numerical analysis-solving methods are implemented for accurate modeling of shock wave behavior high-speed flows that are necessary for aerospace engineering design. This research analysis proves that the MacCormack technique is the one that treats the differential equations with second-order accuracy. This method is quite preferred when it comes to numerical simulations because of its advanced level of accuracy. Although the Upwind and Lax methods are slightly less accurate, they show the development of shock waves that give visualizations to better understand the flow dynamics. Also, in this study, the impact of varying viscosity coefficients on fluid flow characteristics by using the lax (a numerical method for solving the viscous Burgers equation) is investigated. This identification of the phenomenon sheds light on the behavior of boundary layers, which, in turn, can be used to improve the design of high-speed vehicles and lead to a greater understanding of the area of fluid dynamics.

1. Introduction

In terms of aerospace engineering, the analysis of the complicated dynamics of the unstable flow phenomena and shock waves is the critical business of the day. Consequently, utilizing advanced mathematical models and computational techniques becomes essential for the investigation and operation of high-speed aircraft by Gülsu and Wolff *et al.,* respectively [1]. One of the common partial differential equations, which represents the interaction between diffusion and convection in a complex manner, is known as the unsteady (one-dimensional) Burgers equation, a hyperbolic partial differential equation, by several authors [2, 3].

The usage of this equation is sensed deeply in the exploration of supersonic and hypersonic flows, in which the shock waves manifest as an important factor as stated by some researchers [4-6]. The unsteady Burgers equation that can be formulated mathematically, but with great physical knowledge, is the foundation of our research, by Ou and Jameson [7]. The objective of this study is

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to employ and compare three distinct finite difference methods: Lax, Upwind, and McCormack. These methods, known for their computational efficiency and flexibility, are on the one hand, a good solution for numerically resolving the Burgers equation, which implies on the other hand, the possibility of analyzing shock waves in the aerospace application as considered by Fertahi *et al.,* [8].

The one-dimensional (1D) Burger equation which is unsteady, has been analyzed in terms of the different attributes of the Lax, Upwind, and MacCormack methods:

The Lax method possesses first-order accuracy in space and time. Manshoor *et al.,* [8] studied that although it brings a degree of stability, its utility is better suited to scenarios where compliance with the shock wave dynamics is not the main factor. The Lax method could be used in aerospace where shock waves exert great influence on the supersonic and hypersonic vehicles by preliminary analyses or numerical applications wherein less refined resolutions are acceptable [8].

Conversely, the Upwind method, which is also of first-order precision, has become the preferred method to be used by aerospace engineers where shock wave imaging is very critical, as Berdyshev and Wijiatmoko *et al.,* reported [9, 10]. The method stands out in terms of its ability to depict the particular dynamics of shock waves, which is a key feature to consider in designing high-speed aircraft, spacecraft, or missile systems, as Abada considered [11]. The point is that its stability capability increases the credibility of the application of the shock wave in aerospace operations, where the accuracy of the shock wave is of great importance, by Le *et al.,* [11].

In 2023, Jiang *et al.,* [11] presented that the MacCormack method, with second-order accuracy in spatial and temporal terms, tends to provide more pronounced solutions. Such a method of improved resolution can be beneficial in aerospace cases requiring more accurate numerical results, although the distortions may be seen at the endpoints of shock waves. The MacCormack approach can be consequential for aerospace engineers in cases where it is necessary to keep the degree of accuracy balanced against computational cost, as explained by several authors [12-14].

The Lax, Upwind, and MacCormack methods provide various balance solutions as well as unique trade-offs in stability, accuracy, and computational complexity, making these methods powerful tools in the aircraft engineering field. The conversion of these types of medication into CFD models depends upon the specific demands of the application [15], where the accurate rendering of shock waves is a determining factor in the design and performance of aircraft systems [9, 13, 16].

The following sections will deepen into the theoretical aspects of the Burgers equation and the computational methods applied, which will then be combined together into a complete analysis of shock waves in aerospace applications.

2. Methodology

In this section the analysis of the Burgers Equation and the three distinct finite difference methods: Lax, Upwind, and McCormack will be discussed theoretically.

2.1 Burgers Equation

The unsteady 1-D Burgers equation describes the viscous fluid flow behavior in one space dimension as it develops over time. The non-conservation form is expressed in Eq. (1), as depicted by some researchers [17-19]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Where:

u (x,t) represents the velocity of the fluid at position x and time t.
t represents time.
x represents spatial position.
μ represents viscosity.

The unsteady 1-D Burgers equation plays a vital role in fluid dynamics studies, where wave-propagation and the formation of shock waves in one-dimensional flows are examined, as reported by some authors [20, 21]. Here's why it's significant [17, 22, 23]:

i. **Wave Propagation:** The equation can be regarded as the convective flow of fluid, in this case, the changes in velocity are the waves that are propagating through the fluid. These include the generation of shock waves and rarefaction waves that are essential in numerous physical aspects.
ii. **Shock Wave Physics:** Shock waves are sudden, substantial variations in fluid properties that only arise when flow speeds exceed the speed of sound. The 1D Burgers equation without dimension provides a somewhat idealized model to understand the formation and behavior of shock waves in shock wave physics and aerodynamics.
iii. **Numerical Analysis:** The 1D Unsteady Burgers equation is a paradigm problem for the comparison and development of numerical methods in solving partial differential equations. Due to non-linearity and the generation of shock waves, it serves as a complex yet reliable check for the accuracy of algorithms by Pawar and San [24].
iv. **Mathematical Properties:** The equation models a range of physical phenomena with diverse mathematical properties, for instance, the shock wave forming from the originally smooth conditions and the singularity developing in finite time by Zhang and Yang [25]. Analyzing these characteristics helps to understand how nonlinear partial differential equations behave and their solutions.

The initial conditions and boundary conditions are used to determine the numerical and mathematical analysis:

Initial Conditions: $u(x,0) = \sin(2\pi x)$ **Boundary Conditions:** $u(0,t) = 0$, $u(1,t) = 0$

Thus, Eq. (2) represents the **conservation form of** the Burgers equation (i.e. governing equation):

$$\frac{\partial u}{\partial x} + \frac{\partial F}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \tag{2}$$

Where: The term ∂F/∂x represents the change in the flux over a small change in space (x).

However, we will be using Eq. (2) in place of the original Eq. (1) due to the nature of the partial derivative equations derived in their conservative form. Here, the coefficients of the derivative parts of the partial differential equations can become different or constant resulting in the derives not affecting the equation.

The conservation form is usually preferable in numerical simulations due to various reasons [2, 14]:

i. **Conservation of Physical Quantities**: Through the use of conservation principles, the numerical solutions are capable of conserving thereby physical principles.
ii. **Stability and Accuracy**: This is especially important in numerical simulations where stability and accuracy are very important if we are to achieve meaningful results.
iii. **Discretization and Numerical Methods**: Conservation laws equations present a choice for strictly numerical approaches, like finite volume methods, that are mostly adopted to grid numerical solving of conservation laws. In general, such methods are more reliable and accurate than other numerical techniques.

Generally, by converting Burger's equation to its conservation form, the numerical solution keeps important physical properties and provides an opportunity to explore the use of numerical methods better suited for conservation laws.

2.2 Finite Difference Methods

The Burgers equation is solved numerically by utilizing several finite difference methods which are disclosed by several researchers [14, 26, 27]:

2.2.1 Naïve finite difference method [28]**:**

i. **Description**: The naïve finite difference method involves discretizing the spatial and temporal derivatives in Burgers equation directly using finite difference approximations.
ii. **Implementation**: Replace the derivatives in the original form of Burgers equation with finite difference approximations.
iii. **Discretization**: Commonly used schemes include central differencing for the spatial derivatives and forward or backward differencing for the temporal derivative.
iv. **Accuracy and Stability**: Although straightforward to apply, the naive finite difference method may be unstable in the case of a solution with a sharp feature or high-speed traveling waves.

2.2.2 Upwind method [9]

i. **Description**: The upwind method is one type of numerical scheme that replaces the spatial derivatives with their finite difference counterparts considering the direction of flow. It proves to be exceedingly efficient in resolving second-order hyperbolic conservation laws.
ii. **Implementation**: In the context of Burgers equation, the upwind method involves using backward differencing for negative velocities and forward differencing for positive velocities.
iii. **Discretization**: The flux term in the conservation form of Burgers equation is approximated using upwind differencing.
iv. **Advantages**: The upwind technique, in contrast, is more reliable regarding problems with discontinuity or shock waves as it deals naturally with the flow direction and guarantees the necessary stability and precision near discontinuities.
v. **Disadvantages**: The upwind method is likely to induce numerical diffusion in areas of gradual flow, where the level of accuracy may decrease.

2.2.3 Lax method [16, 29]**:**

i. **Description**: The Lax method is one of the numerical procedures for resolving the hyperbolic partial differential equations, including the Burgers equation. It puts both explicit and implicit techniques into play.
ii. **Implementation**: The Lax algorithm works by updating the solution at each time step by taking the average value of the solution at neighboring points with some weights.
iii. **Discretization**: The Lax method employs central difference schemes for spatial derivatives and a semi-implicit time-stepping scheme.
iv. **Advantages**: The stability of the Lax scheme is basically ensured for a wider range of time step sizes compared to explicit methods.
v. **Disadvantages**: The Lax method can imply the appearance of numerical diffusion and dispersion even in the exact solution, particularly near discontinuities.

2.2.4 MacCormack method [13, 14] **:**

i. **Description**: The MacCormack technique is an implicit numerical procedure for solving hyperbolic PDE (partial differential equations) with better accuracy compared to explicit methods.
ii. **Implementation**: The MacCormack technique contains a predictor-corrector approach, which is expressed as an explicit predictor followed by an implicit corrector.
iii. **Discretization**: The MacCormack method is based on central differencing for spatial differentiation and the two-step method for time advances.
iv. **Advantages**: The MacCormack process will likely show greater rates of mathematic accuracy and stability than explicit methods alone, therefore, making it the preferred method for problems containing steep gradients or shock waves.
v. **Disadvantages**: The MacCormack technique could expend extra numerical procedure because of its predictor-corrector structure.

In a typical case, each finite difference method behaves differently when solving Burger’s equation taken numerically. The technique is determined by unique attributes of this issue, for example, the issue of shockwaves, the smooth solution, and the computational efficiency. These studies might unveil the strengths and limitations of different approaches when they run under different conditions.

Presently, the Burgers equation has implemented the Finite Difference Method (FDM) (i.e. Space (CD) and Time (FD)) as represented in Eq. (3). Also, the upwind method is applied to the conservation form as described in Eq. (4):

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{1}{h}\left(F_{j+1}^n-F_{j-1}^n\right)=\frac{\mu}{h^2}\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right) \tag{3}$$

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\left(\frac{F_j^+-F_{j-1}^+}{\Delta x}\right)^n+\left(\frac{F_{j+1}^--F_j^-}{\Delta x}\right)^n=0 \tag{4}$$

The primary objective behind transitioning to a numerical flux function is: To make the already existing techniques more resilient to numerical uncertainties and improve the accuracy level. The distinction between the naive finite-difference (FD) scheme and the upwind scheme rests on the same principle of accuracy. The naive method possesses a first-order accuracy in time and second-order accuracy in space, while the upwind method retains the first-order accuracy in space and time by Gülsu and Wei *et al.,* [1, 30].

3. Results

In this section, the three finite-difference methods numerical results will be explained and disclosed below:

3.1 Upwind Method

The upwind scheme of the non-viscous Burgers equation updated finite-difference formula, as illustrated in Eq. (5):

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{4h}\left(u_{j+1}^2 - u_{j-1}^2\right) + \frac{\Delta t}{4h}\left[(|u|u)_{j+1} - 2(|u|u)_j + (|u|u)_{j-1}\right] \tag{5}$$

The upwind method is specifically employed for the solution of hyperbolic partial differential equations, making it the method of choice for addressing the Burgers equation in our study. This approach leverages finite difference stencils to solve the propagation wave equation within a fluid flow field. Notably, the upwind method is characterized by a first-order level of accuracy in both spatial and temporal dimensions, with the additional requirement of stability as expressed by the condition $\frac{\Delta t}{\Delta x}|u|_{max} \leq 1$, where '$u_{max}$' represents the maximum local advection speed for all j. This inequality represents a non-linear variant of the Courant-Friedrichs-Lewy (CFL) condition, a key determinant of method stability [29]. The results presented in Figure 1 by using MATLAB further illustrate the practical implications of this approach.

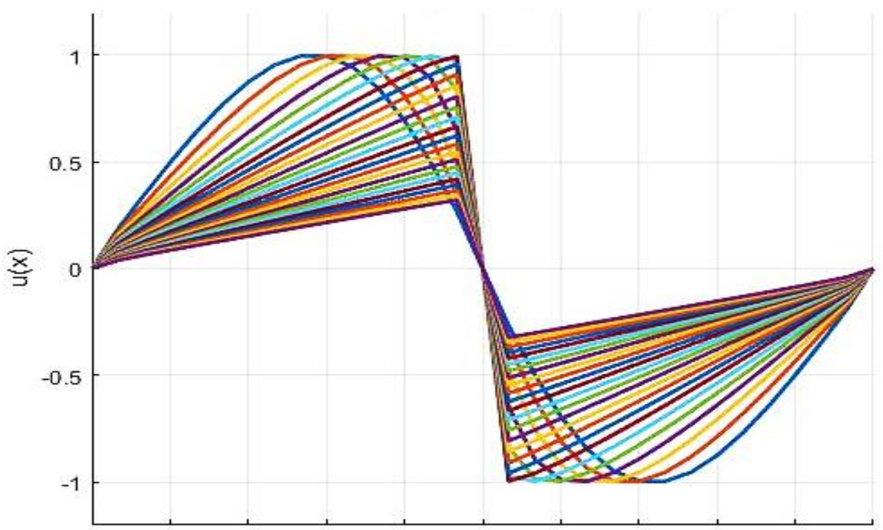

Fig. 1. Shock formation in inviscid Burger's equation after 0.5 seconds with a hundred number of grid points

A grid consisting of one hundred points with a time step of 0.01 seconds in our numerical solution has been employed. The observation from Figure 1 allows us to draw a significant conclusion: shock formation takes place approximately after twenty time-steps. Consequently, an increase in the number of grid points that adhere to the stability condition results in a smoother shock formation, whereas the opposite is true when the grid points are reduced.

3.2 Lax Method

Eq. (6) is the updated FD formula of the non-viscous Burgers equation using the Lax scheme:

$$u_j^{n+1} = \tfrac{1}{2}\left(u_{j+1} + u_{j-1}\right)^n - \frac{\Delta t}{4h}\left(u_{j+1}^2 - u_{j-1}^2\right)^n \qquad (6)$$

The Lax method, tailored for solving hyperbolic partial differential equations, is our chosen technique for addressing the Burgers equation. This method is characterized by first-order accuracy in both spatial and temporal dimensions, coupled with a stability requirement represented by $\frac{\Delta t}{\Delta x}|u|_{max} \leq 1$where '$u_{max}$' denotes the maximum value of the local advection speed for all j. This condition, depicted as a nonlinear variation of the Courant-Friedrichs-Lewy (CFL) condition, plays a pivotal role in method stability [16, 29]. A visual representation of our results can be found in Figure 2. In our numerical solution, we utilized fifty space steps and a time step of 0.01 seconds.

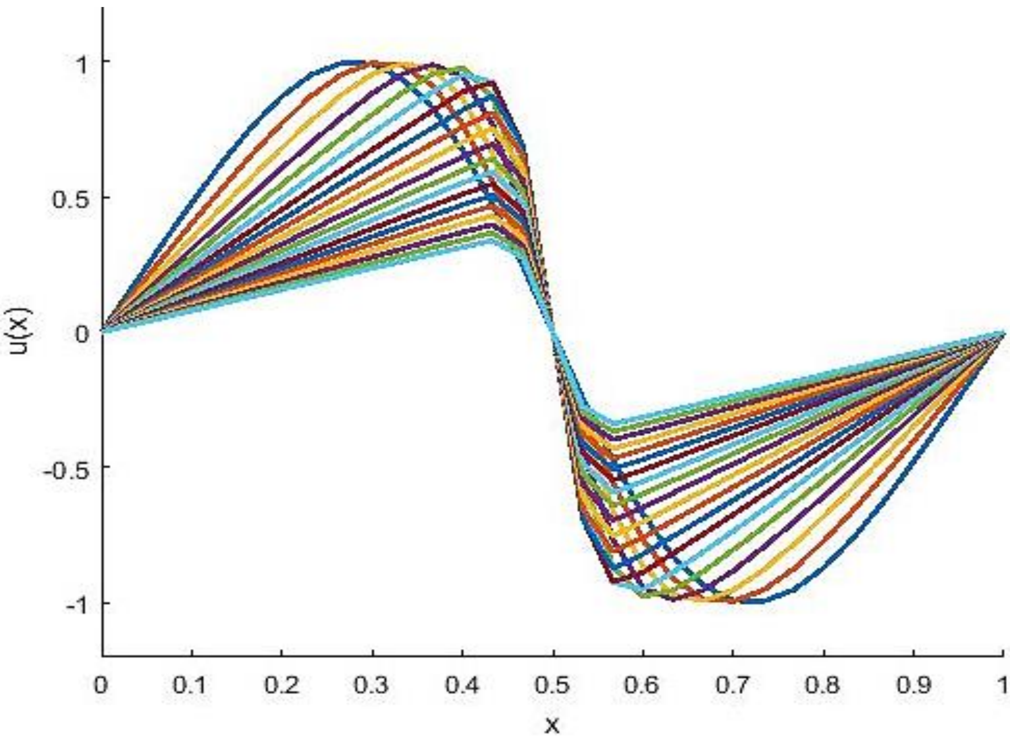

Fig. 2. Numerical solution of the inviscid Burger's equation by Lax method after 0.01 seconds with fifty number of grid points

With the lax method, there are limitations in our ability to comprehensively investigate and attain clarity when varying the number of grid points. Consequently, we settled on employing one hundred grid points to achieve accurate shock formation. However, upon attempting to increase or decrease the number of grid points, the numerical solution exhibited distortion. In response to this challenge, we captured multiple snapshots for varying grid point numbers, as depicted in the Figure 3 to Figure 6 below.

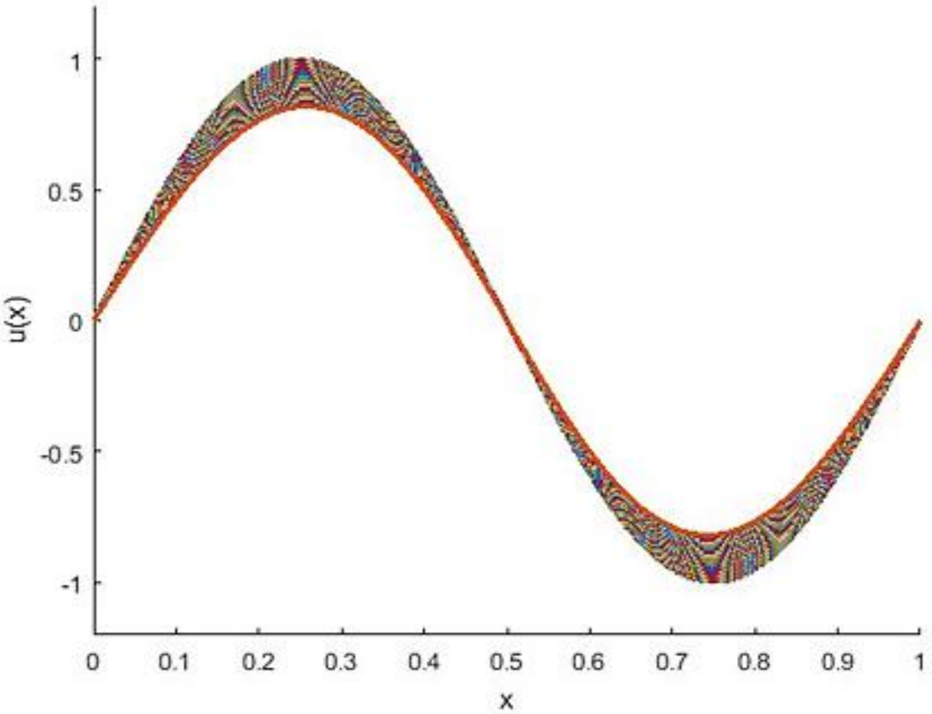

Fig. 3. Shock formation with one hundred grid points

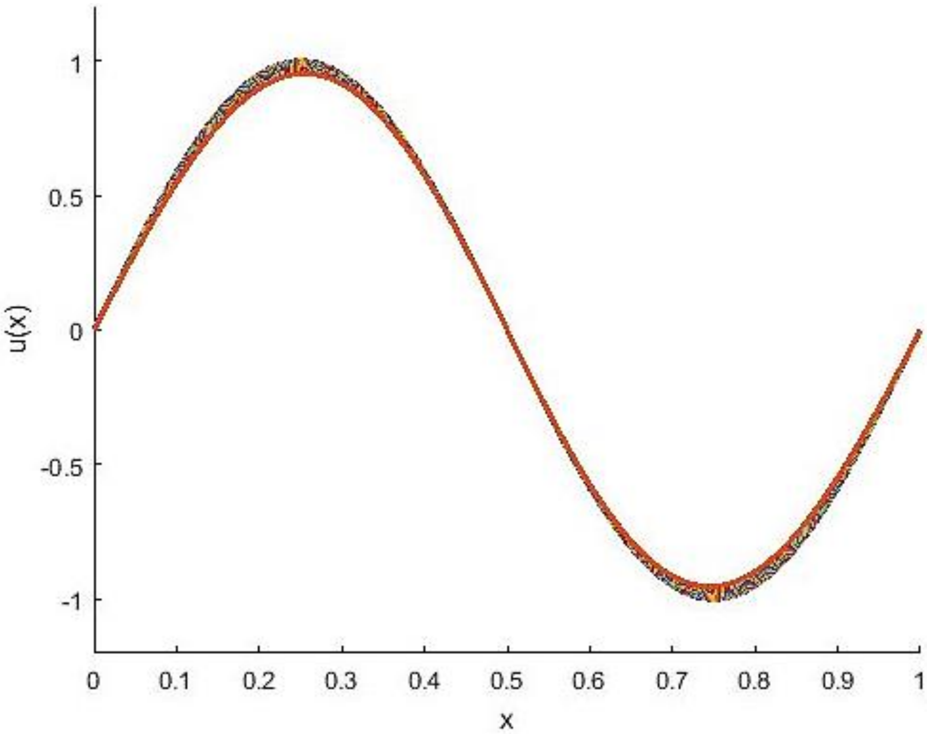

Fig. 4. Shock formation with two hundred grid points

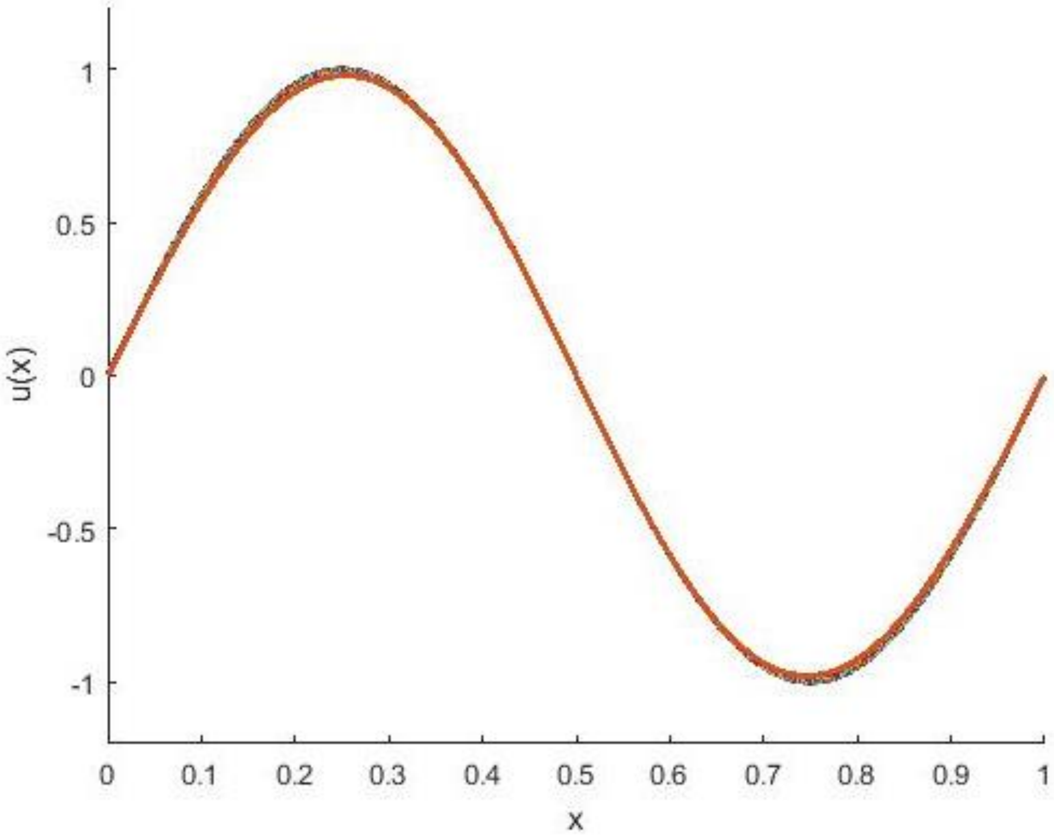

Fig. 5. Shock formation with three hundred grid points

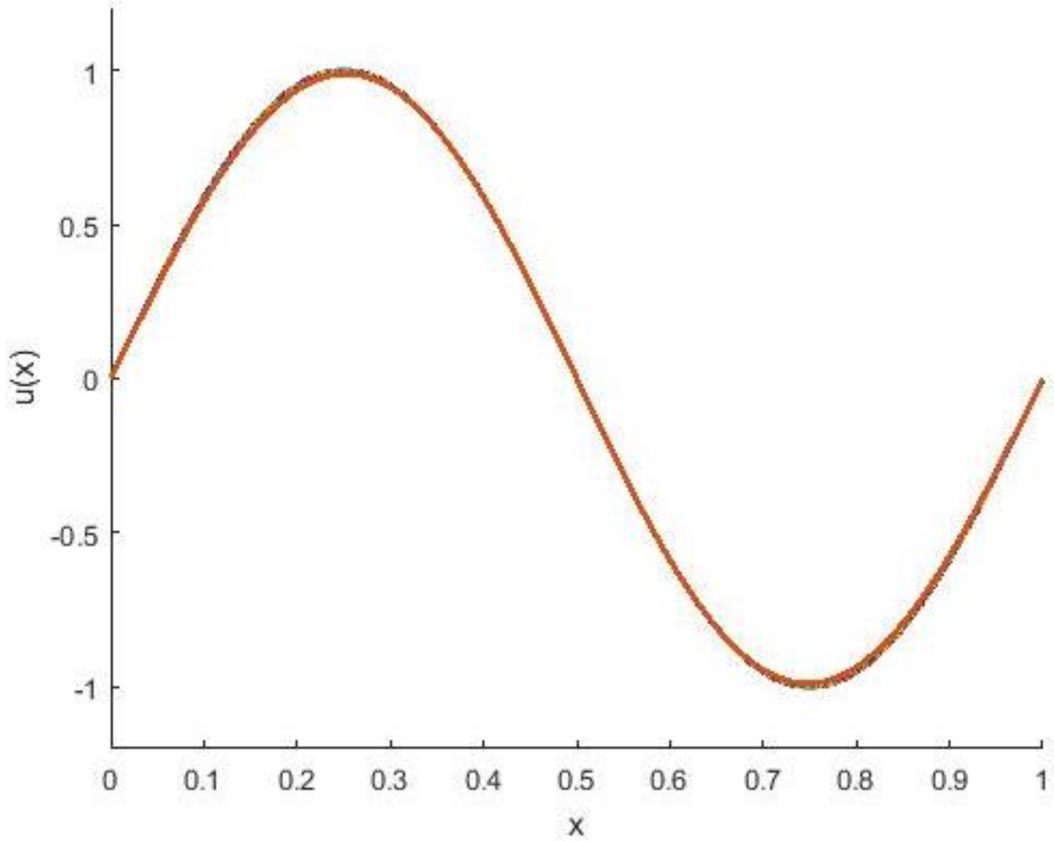

Fig. 6. Shock formation with four hundred grid points

3.3 MacCormack Method

The MacCormack scheme of the non-viscous Burgers equation updated the finite-difference formula for predictor step forward space and corrector step backward space, as shown in Eq. (7) and Eq. (8), respectively.

$$\frac{u_j^* - u_j^n}{\Delta t} + \frac{1}{h}\left(F_{j+1}^n - F_j^n\right) = 0 \tag{7}$$

$$\frac{u_j^{n+1} - \left(u_j^* + u_j^n\right)/2}{\Delta t/2} + \frac{1}{h}\left(F_j^* - F_{j-1}^*\right) = 0 \tag{8}$$

The MacCormack method, chosen for solving hyperbolic partial differential equations, stands as our method of preference for addressing the Burgers equation. This method distinguishes itself with second-order accuracy in both spatial and temporal dimensions, accompanied by a stability criterion articulated as $\frac{\Delta t}{\Delta x}|u|_{max} \leq 1$, where '$u_{max}$' represents the maximum value of the local advection speed for all j. This condition, presented as a nonlinear variant of the Courant-Friedrichs-Lewy (CFL) condition, assumes a pivotal role in preserving the method's stability [13, 14, 31]. A visual representation of our findings is provided in Figure 7. In our numerical solution, we employed fifty space steps and a time step of 0.01 seconds.

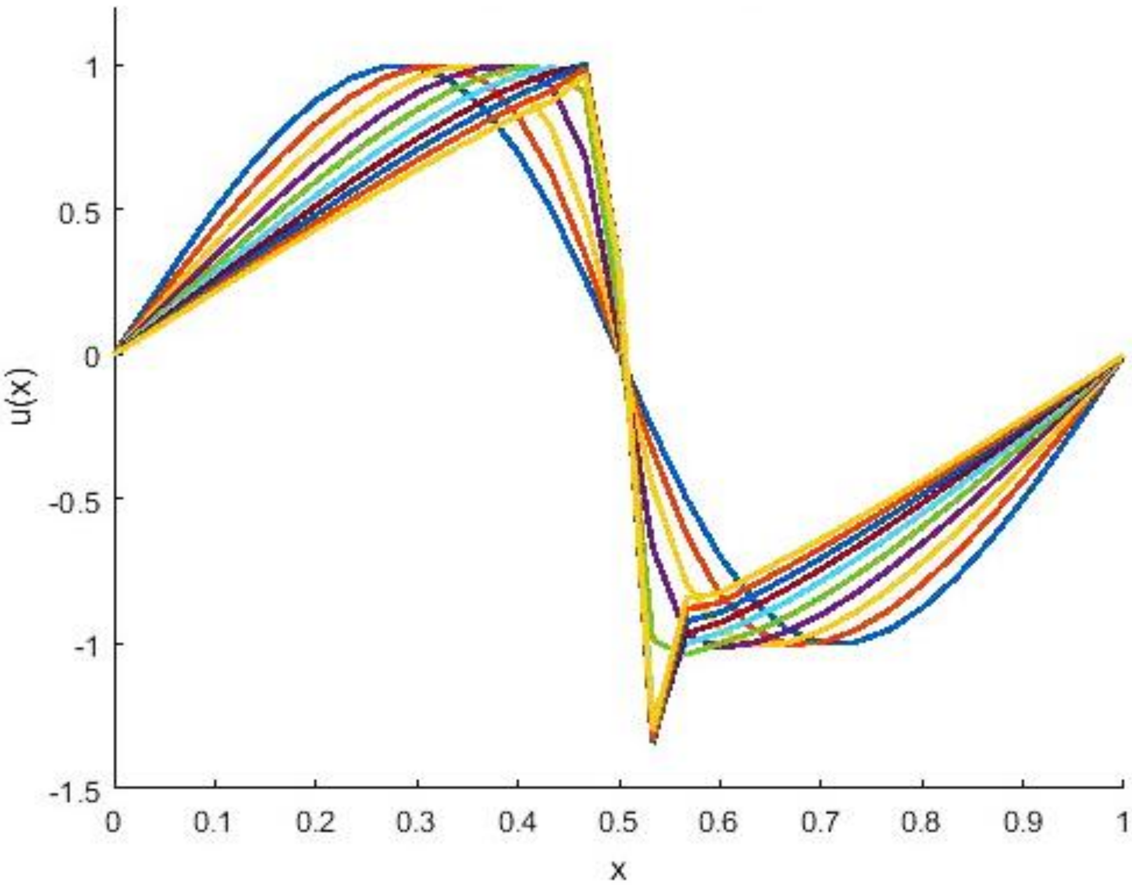

Fig. 7. Numerical solution of the inviscid Burger’s equation by MacCormack method

4. Discussion

The analysis presented in this study highlights the strengths and limitations of different numerical methods for solving fluid dynamics problems relevant to aerospace systems, particularly aircraft design and analysis.

Based on Figure 7 outcomes, the investigation reveals that the MacCormack method, with its second-order accuracy, excels in capturing sharper flow features compared to first-order methods like Upwind and Lax. This enhanced accuracy proves crucial in accurately predicting critical aerodynamic phenomena, such as shock waves, which significantly influence aircraft performance and stability. While the MacCormack method successfully captures shock formation, slight distortions observed at the shock's endpoints necessitate further investigation and potential refinement of the method. Notably, the MacCormack method demonstrates robustness in maintaining stability even under varying spatial and temporal discretization, a valuable characteristic for complex aerospace applications.

However, when considering viscous flows, as encountered in boundary layer analysis and other critical areas of aircraft design, the Lax method emerges as a suitable candidate. Its ability to handle varying viscosity coefficients allows for a detailed investigation of viscous effects on aerodynamic characteristics. Throughout this research planning to explore this further by employing the Lax method with different viscosity values (0.03, 0.02, 0.015, 0.01, and 0.001) to gain deeper insights into the interplay between viscous forces and flow behavior, which is essential in aircraft engineering design, particularly in contexts involving boundary layers.

It is important to acknowledge that the choice of an appropriate numerical method is highly dependent on the specific problem under consideration. While the MacCormack method offers superior accuracy for capturing shocks, its computational cost or complexity might be higher compared to the Lax method. In scenarios where computational efficiency is paramount, particularly for preliminary design and optimization studies, the Lax method could be a viable alternative, especially for analyzing viscous flow regimes.

4.1 Comparative and Validation Assessment

Aksan [19] proposed that the investigation of constructed the FEM on the discretization in time method be the objective of his approaches to provide an accurate numerical solution to the Burgers equation. However, this study investigates the effectiveness of utilizing three finite-difference methods, which are the Lax, Upwind, and MacCormack methods, in solving the unsteady Burger's equation. Also, these three numerical methods are performed for an accurate modeling of high-speed flow shock wave behavior. Additionally, the research focuses on the various finite difference methods, outlining the computational efficiency benefits for each and the ease of implementation.

Aksan [19] demonstrates in his study the potential of his approaches to handle complex geometries and achieve higher accuracy. While, the impact of varying viscosity coefficients on fluid flow characteristics by using the lax (a numerical method for solving the viscous Burgers equation) is investigated. Moreover, the other two approaches visualize the shock wave perfectly, and they can provide an advancement in accuracy in regions of smooth flow.

5. Observation

Figure 8 through Figure 12 provide valuable insights into the behavior of the Lax method across a range of viscosity coefficients due to the specification of the initial and boundary conditions. Where the purpose of specifying this initial condition is to provide a non-zero starting point and smooth solution to the Burgers equation. Also, it allows for interesting dynamics development, such as shock wave formation. The reason behind choosing these boundary conditions is to ensure a closed system, leading to a well-behaved and physically realistic solution, keeping the solution confined within the specified domain by preventing external influences. Furthermore, these initial and boundary conditions are standard practices for analyzing the Burgers equation, allowing for a clear comprehension of its essential behaviors. A key observation is the remarkable stability of the results, exhibiting minimal sensitivity to variations in viscosity. This stability is of paramount importance in aerospace engineering applications, where reliable and predictable numerical simulations are crucial for vehicle design and performance evaluation.

However, a notable phenomenon emerges when the viscosity coefficient is decreased: a considerable increase in the wave of propagation speed. Such a conclusion in turn emphasizes the complex interaction of viscosity and fluid dynamics, one with immediate consequences for the air vehicle's mobility and performance. This is because fluid behavior at different viscosities is critical hence learning and predicting fluid motion becomes necessary.

Through the comprehensive examination of the Lax scheme, the significance of viscosity alterations has been established and the gateway has been cleared for the extension of this study to other complex aspects of fluid flow dynamics within aerospace engineering. We expect that the understanding accrued in this study will be necessary for making appropriate decisions related to building and operating aerospace systems characterized by boundary layer behavior. Overall, shock wave behavior predictions under different viscous flow conditions can add to the understanding of and the efforts to avoid the undesirable effects of shock boundary layer interactions. These involvements may result in the development of flow separation, higher drag coefficients, and lower aerodynamic efficiency. The success of the mathematical model lies in the ability to give an account of the combined actions of viscosity and shock wave dynamics.

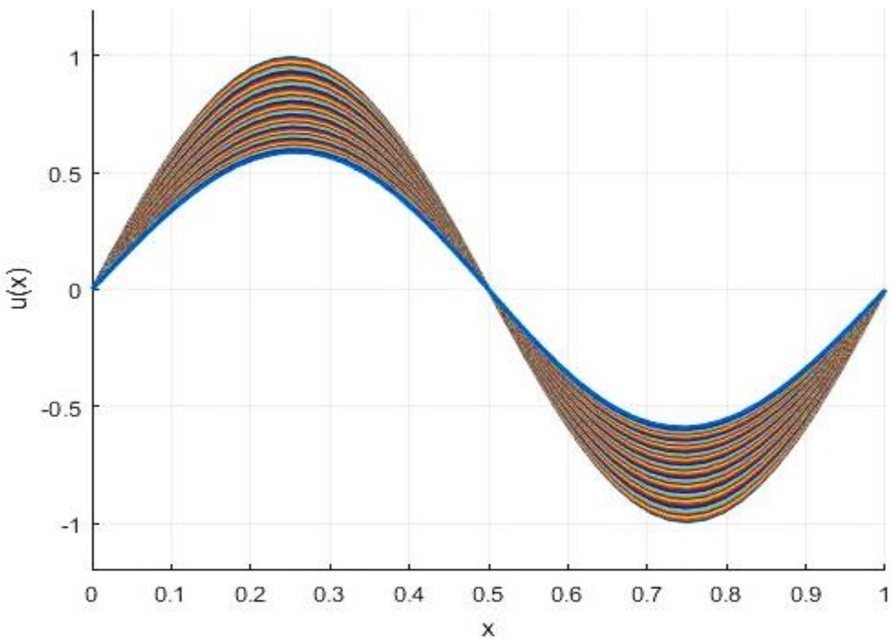

Fig. 8. Explicit solution of 1D viscous Burgers equation by Lax method at μ = 0.03

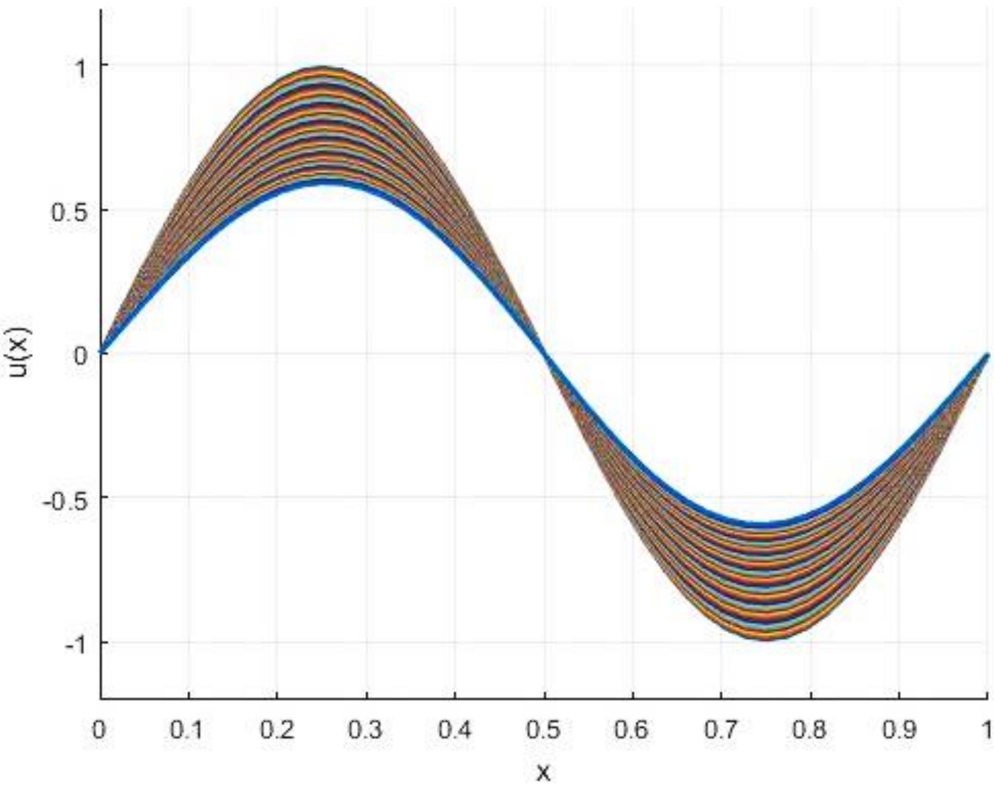

Fig. 9. Explicit solution of 1D viscous Burgers equation by Lax method at μ = 0.02

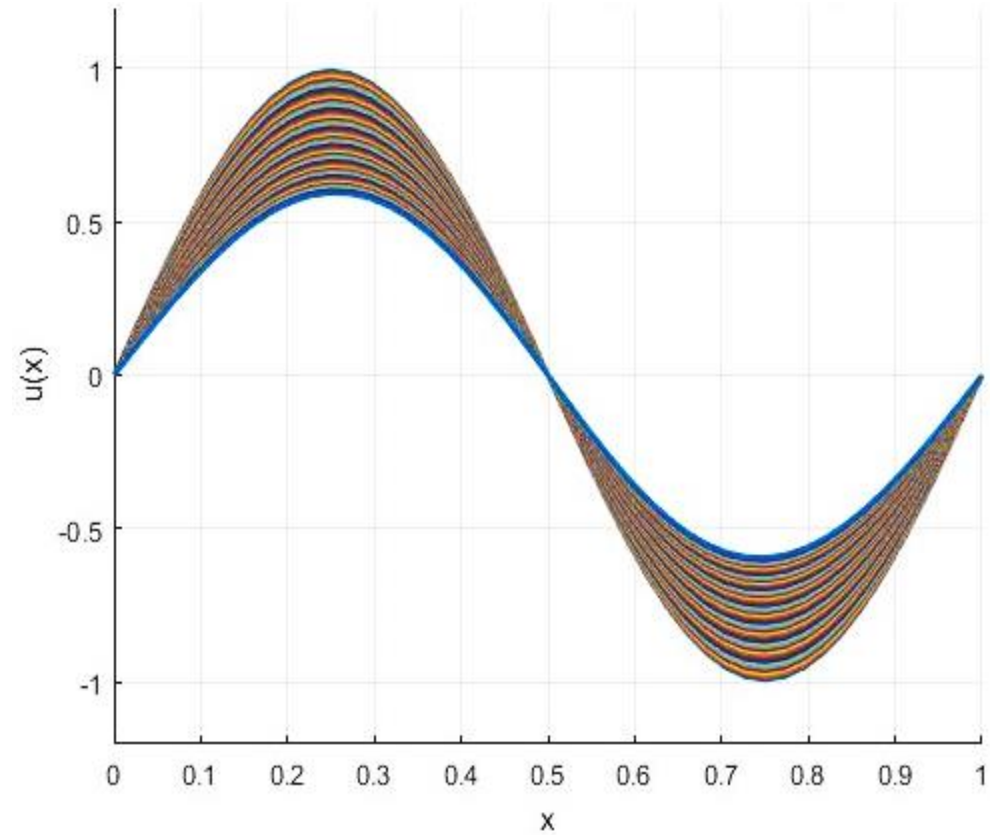

Fig. 10. Explicit solution of 1D viscous Burgers equation by Lax method at μ = 0.015

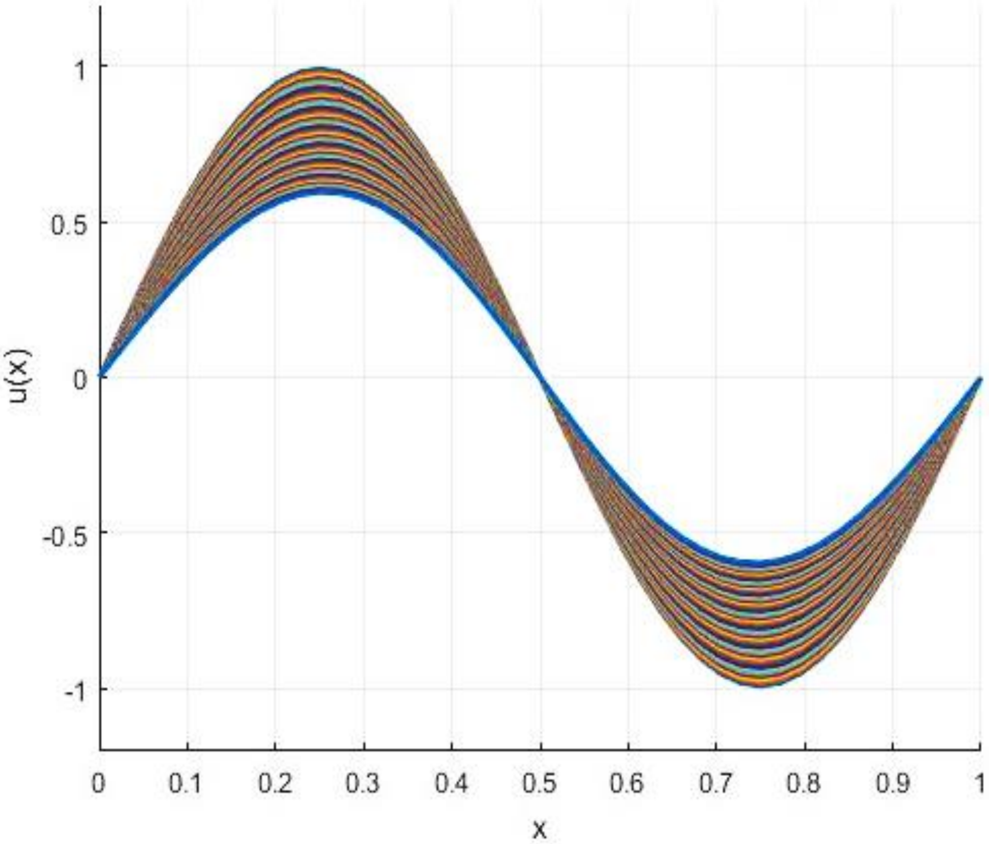

Fig. 11. Explicit solution of 1D viscous Burgers equation by Lax method at μ = 0.01

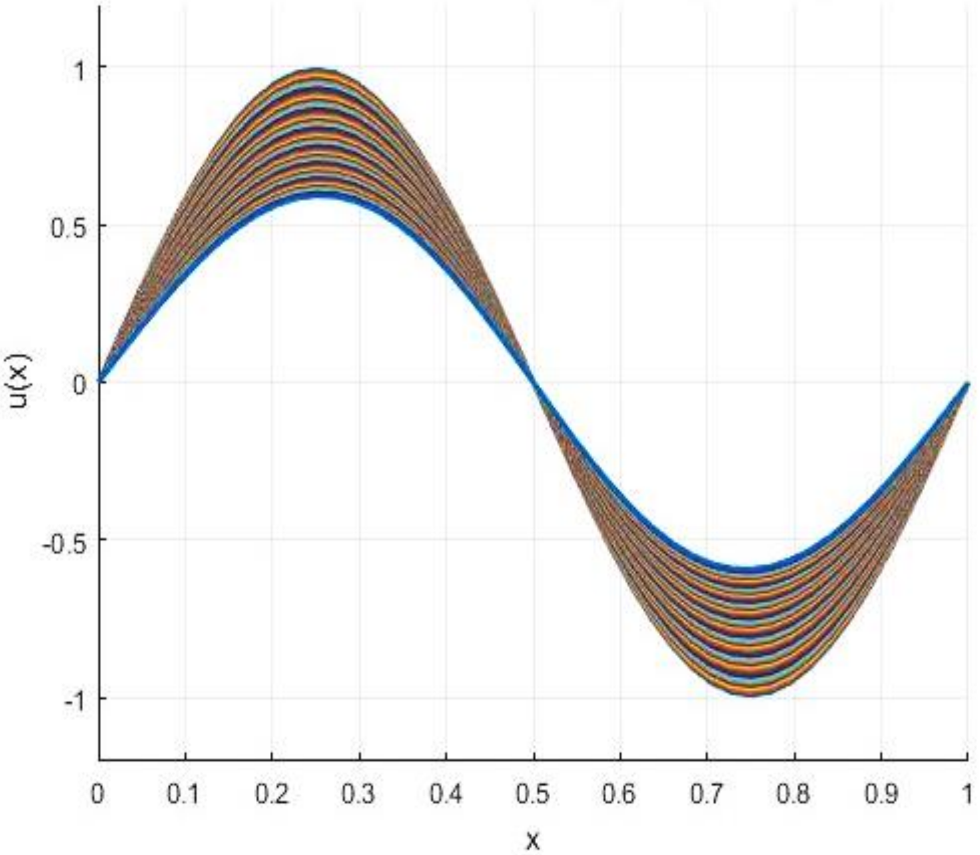

Fig. 12. Explicit solution of 1D viscous Burgers equation by Lax method at μ = 0.001

6. Conclusions and Future Work

This study has presented a detailed development of the 3 numerical methods – upwind, lax, and MacCormack – for solving the viscous and the inviscid unsteady Burger's equations. These methods are useful techniques for the numerical solving of hyperbolic partial differential equations with applications in a variety of different fields including aerospace engineering and fluid dynamics.

Based on our examination, especially on shock wave generation, the Upwind method arises as the best option in contrast to the other two of them. The feature of a CFD simulation that is most relevant to aerospace engineers is that it generates shock waves with great accuracy and clarity. The first-order accuracy and stability properties of the Upwind method make it a suitable technique for the simulations in which the shock wave accurate representation is important.

However, while Upwind performs very well in visualizing the shock, it is legitimate to note that there are higher-order methods, for instance, the MacCormack method, among others, that can provide an improvement in accuracy in regions of smooth flow. On the other hand, these methods can involve the generation of spurious oscillations in the proximity of discontinuities such as shock waves which can subsequently curtail the accuracy of the solution. Hence, the final choice of the best numerical technique depends on the particular application which in turn directly affects the accuracy and the stability of the numerical simulation.

The results and main conclusions of this study hold great importance for the aerospace industry that deals with the modeling of shock waves and unsteady flows because of the use of these models in the designing, analyzing, and optimizing of high-speed aircraft. The accuracy and reliability of these simulations can be considerably improved by selecting a suitable numerical method like the Upwind method concerning aerospace technology as well as fluid dynamics which is a developing field.

Future studies can take the method as a basis in order to realize the work with more complicated and realistic models, including the transonic flows and shock-boundary interaction, for instance. Moreover, exploring the feasibility of adaptive mesh refinement can be advantageous to increase the precision and resolution of the numerical simulations in the regions that are these types of regions. Through the utilization of the power of different numerical methods together with the continuous improvement of simulation techniques, aerospace engineers can possess a deeper understanding of complex flow-induced phenomena, which ultimately boosts airplane design and operations.

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