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A Mathematical Study on a Steady MHD Flow in Double Stratification Medium

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ABSTRACT

The detailed analysis on the flow of MHD fluid in double stratification medium across a stretching sheet with exponential permeability is examined in this research. The approximate analytical solution for the governing equations in steady state is found by using a new approximate analytical method called Ananthaswamy-Sivasankari Method (ASM) and Modified Homotopy Analysis Method (MHAM). The approximate analytical expressions for the dimensionless velocity, dimensionless temperature and dimensionless concentration are derived using these methods. The analytical and numerical results (previous work) are compared and there is a good agreement between our analytical results and numerical works. The impacts of several parameters including porosity, magnetic, suction and heat source parameters are shown in graphical representation. The error table for the physical parameter namely Nusselt number for various values of Prandtl number has been provided. Both ASM and MHAM are very useful to solve some other non-linear boundary value problems especially in MHD fluid flow.

1. Introduction

Many researchers focused their attention on the various aspects of the problem of Magnetohydrodynamic (MHD) flow. Nanofluids play a vital role in biomedical sciences, industrial, and engineering processes. It is used in a variety of applications, including nuclear reactor cooling systems, blood flow monitoring, and accelerators. Sa'adAldin *et al.,* [1] researched and addressed the problem of unsteady MHD flow in porous media within the existence of a magnetic field between two parallel flat plates using the Finite Element Method. Acharya *et al.,* [2] investigated the spatial and temporal effects of medium porosity coupled with plate temperature fluctuations. He also indicated that the inclusion of porous media has no effect on the flow dynamics and that variable viscosity accounts for the heating and cooling of the plate induced by convective current.

Ali *et al.,* [3] solved numerically the heat transfer boundary layer flow across an angled stretched sheet in the existence of a magnetic field. Barik *et al.,* [6] investigated mass and heat transmission

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upon MHD flow within a porous media stretched over a heat source. Chamkha *et al.,* [7] presented a numerical solution (implicit FDM) to steady natural convection boundary-layer flow of a nanofluid consisting of a pure fluid with nanoparticles along a permeable vertical plate in the context of a magnetic field, heat generation or absorption, and suction or injection effects. Choudhary *et al.,* [9] focused on and described a two-dimensional unsteady flow of such an incompressible viscous fluid that exhibits electrical conductivity over a stretching permeable surface having a transverse magnetic field of uniform strength and numerically solved.

Dessie *et al.,* [10] analyzed the effects of the MHD on energy transfer through stretched sheets embedded in porous media involving variable viscosity, viscous dissipation as well as heat source/sink by using Lie's scaling group transformation and obtained the numerical solution using RK4 with the shooting method. Goud *et al.,* [11] examined the thermal and mass transport consequences just on the boundary layer flow of MHD with viscous, incompressible and radiating fluid across an exponentially stretching sheet. Ibrahim [12] studied the mass and heat transmission implications on persistent MHD flow via an exponentially stretched surface having heat generation, viscous dissipation and radiation numerically by RK4 with the shooting method. Ibrahim and Suneetha [13] have numerically explored the impact of Soret as well as the heat source for steady MHD mixed convective mass and energy transfer flow along an infinite vertical plate immersed in a porous media under the conditions of chemical reaction, viscous flow, and Joule's dissipation.

The velocity, temperature, and concentration of the fluid decrease with an increase in the suction parameter and an increase in the thermal stratification parameter, and the Schmidt number decreases the temperature and concentration, as reported by Faudzi *et al.,* [19]. In the work of Mukhopadhyay [20], the effects of suction and stratification parameter of a steady MHD flow together in substantially stratified surface was discussed and also, Mukhopadhyay *et al.,* [21] considered the first-order destructive/ constructive chemical reactions. Nazari *et al.,* [22] considered radiation and they concluded that the temperature increased with a higher radiation parameter using HAM. Reddy [24] examined the influences of thermal radiation, chemical reactions and Caisson fluid parameters in magneto hydrodynamic flow.

Saidulu *et al.,* [25] investigated the slip effects of MHD flow on Caisson fluid over an exponentially stretching sheet in the presence of thermal radiation, a heat source/sink and chemical reactions and the double slip effects were evaluated by Zaman *et al.,* [26]. Sekhar [27] described the boundary layer phenomena of the MHD flow problem. Singh and Kumar [28] examined the free convection flow on a vertical plate in porous media with variable wall temperature and concentration in a doubly stratified and viscous dissipating micro polar fluid with chemical reaction, heat generation and ohmic heating. Swain *et al.,* [30] looked at the impacts of different flow parameters of MHD flow in an exponentially stretching sheet through a porous medium with a heat source/sink. The radiation effect of MHD boundary layer flow due to an exponentially stretching surface embedded in a porous medium has been studied by Yusuf *et al.,* [31]. Numerous authors have found solutions to other MHD flow problems using HAM, the keller box method, numerically with the bvp4c solver in MATLAB, and the shooting method [32-36].

The primary goal of this study is to use ASM and MHAM to produce an approximate analytical solution to the flow of MHD fluid in a thermal and chemical stratification medium across a stretching sheet with exponential permeability. The resultant analytical and numerical findings are then compared and represented graphically. To interlined the impacts of several parameters such as porosity, magnetic, suction and heat source parameters.

2. Methodology

Consider a steady two-dimensional MHD motion in a viscous, electrically conducting and incompressible fluid over an exponentially extending surface, as reported by Nur Suhaida Aznidar Ismail *et al.,* [23]. The surface is considered to be stretched with velocity U along the $x-axis$, and the $y-axis$ normal to $x-axis$. A variable magnetic field $B = B_0\, e^{\frac{x}{2L}}$ was transmitted properly for this surface where B_0 is constant. The surface consists of both heat $T_w(x)$ and concentration $C_w(x)$ which are completely immersed in a temperature segregated media with a changing ambient temperature $T_\infty(x)$ and fluctuating ambient concentration $C_\infty(x)$ where $T_w(x) > T_\infty(x)$ and $C_w(x) > C_\infty(x)$ respectively. It is assumed that $T_w(x) = T_0 + b\, e^{\frac{x}{2L}}$, $T_\infty(x) = T_0 + c\, e^{\frac{x}{2L}}$, $C_w(x) = C_0 + m\, e^{\frac{x}{2L}}$ and $C_\infty(x) = C_0 + n\, e^{\frac{x}{2L}}$ where T_0 is the reference temperature and C_0 is the reference concentration where b, c, m and n are positive constants.

The governing partial differential equations for this flow are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho}u - \frac{\nu}{K'}u \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{Q}{\rho c_p}\left(T - T_\infty\right) \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - k_1\left(C - C_\infty\right) \tag{4}$$

where u and v represent the elements for velocity on the x and y directions correspondingly, μ is the coefficient of fluid viscosity, $\nu = \frac{\mu}{\rho}$ indicates the kinematic viscosity, ρ is the fluid density, σ is the fluid's electrical conductivity, and B is a variable of the magnetic field. The permeability is in the form of $K' = k^* e^{\frac{-x}{L}}$ where k^* is a constant. α is the thermal diffusivity, T is the temperature of fluid, $Q = Q_0\, e^{\frac{x}{L}}$ is the dimensional heat generation where Q_0 is a constant. c_P is the temperature under static pressure, C is the fluid concentration, T_w is the temperature of the surface and D is the solute diffusion coefficient. k_1 is the variable rate of chemical conversion of the first-order irreversible reaction where $k_1 = \frac{1}{2}k_0\, e^{\frac{x}{L}}$, k_0 is a constant.

The boundary conditions are given as follows:

$$u = U = U_0\, e^{\frac{x}{L}}, \quad v = -V(x) = -V_0\, e^{\frac{x}{L}}, \; T = T_w(x) = T_0 + b\, e^{\frac{x}{2L}}, \; C = C_w(x) = C_0 + m\, e^{\frac{x}{2L}} \quad at \quad y = 0 \tag{5}$$

$$u \to 0, \quad T = T_\infty(x) = T_0 + c\, e^{\frac{x}{2L}}, C = C_\infty(x) = C_0 + n\, e^{\frac{x}{2L}} \;\; as \;\; y \to \infty \tag{6}$$

where U_0 is the reference velocity. $V(x) > 0$ is the suction velocity while $V(x) < 0$ is the blowing velocity. The initial strength of suction is denoted by $V_0 > 0$ and $V_0 < 0$ is the initial strength of blowing.

The dimensionless similarity variables used in the mathematical analysis are:

$$\left.\begin{array}{l} \eta = \sqrt{\dfrac{\nu U_0}{2L}} e^{\frac{x}{2L}} y, \quad u = U_0\, e^{\frac{x}{2}}\, f'(\eta), \quad v = -\sqrt{\dfrac{\nu U_0}{2L}} e^{\frac{x}{2L}} \{ f(\eta) + \eta\, f'(\eta) \}, \\ \theta(\eta) = \dfrac{T - T_\infty}{T_w - T_0} \quad and \quad \phi(\eta) = \dfrac{C - C_\infty}{C_w - C_0} \end{array}\right\}. \tag{7}$$

The ordinary differential equations found by Ismail *et al.,* [23] are provided below by substituting Eq. (7) into Eq. (2), Eq. (3) and Eq. (4):

$$f''' + f\, f'' - 2\, f'^2 - K\, f' - M\, f' = 0 \tag{8}$$

$$\theta'' + \Pr\left(Q_H\, \theta - f'\theta - (St)\, f' + f\, \theta' \right) = 0 \tag{9}$$

$$\phi'' + Sc\left(f\, \phi' - \phi\, f' - (St_1)\, f' - \beta\, \phi \right) = 0 \tag{10}$$

The transformed boundary conditions are given by:

$$f(0) = S, \;\; f'(0) = 1, \;\; \theta(0) = 1 - St, \;\; \phi(0) = 1 - St_1, \quad at \quad \eta = 0 \tag{11}$$

$$f'(\infty) \to 0, \; \theta(\infty) \to 0, \;\; \phi(\infty) \to 0, \;\; as \quad \eta \to \infty \tag{12}$$

where the prime indicates a differential with respect to η, $K = \dfrac{2L\nu}{k^* U_0}$ signifies porosity coefficient, $M = \dfrac{2\sigma B_0^2\, L}{\rho U_0}$ represents the magnetic variable, $\Pr = \dfrac{\nu}{\alpha}$ is the Prandtl number, $Q_H = \dfrac{2L\, Q_0}{U_0 \rho\, c_P}$ is the heat source parameter, $St = \dfrac{c}{b}$ is the thermal stratified variable , $Sc = \dfrac{\nu}{D}$ is the Schmidt number , $St_1 = \dfrac{n}{m}$ is the chemically stratified parameter and $\beta = \dfrac{k_0\, L}{U_0}$ is the rate of reaction parameter. $S = \dfrac{V_0}{\sqrt{\dfrac{\nu U_0}{2L}}}$ is the suction or blowing parameter. $S > 0$ denotes the suction variable while $S < 0$ is the blowing variable. The physical aspects for basic flow are the skin friction coefficient C_f, the local Nusselt number Nu and the local Sherwood number Sh outlined below:

$$C_f = \frac{\tau_w}{\rho U^2/2}, \quad Nu = \frac{x\,q_w}{k(T_w - T_\infty)}, \; Sh = \frac{x\,J_w}{D(C_w - C_\infty)} \tag{13}$$

where the wall shear stress τ_w, the surface heat flux q_w and the surface mass flux are given by:

$$\tau_w = \mu\left((\partial u/\partial r)_{y=0}\right), \; q_w = -k\left((\partial T/\partial r)_{y=0}\right), \; J_w = -D\left((\partial C/\partial r)_{y=0}\right) \tag{14}$$

Through the use of the dimensionless variables in Eq. (7), we discovered

$$f''(0) = \frac{C_f}{\sqrt{\frac{2}{\mathrm{Re}}}\sqrt{\frac{x}{L}}}, \; -\theta'(0) = \frac{Nu\,(1-St)}{\sqrt{\frac{x}{L}}\sqrt{\frac{\mathrm{Re}}{2}}} \;\; and \;\; -\phi'(0) = \frac{Sh\,(1-St_1)}{\sqrt{\frac{\mathrm{Re}}{2}}\sqrt{\frac{x}{L}}} \tag{15}$$

where $\mathrm{Re} = \frac{U\,x}{\nu}$ is the local Reynolds number.

3. Approximate Analytical Solution using ASM and MHAM

A new approach called the Ananthaswamy-Sivasankari method (ASM) is presented for the evaluation of the third order non-linear ordinary differential equations [8,29]. It can be used to solve differential equations, both linear and non-linear. This technique can also be easily extended to address several additional non-linear problems in the physical, chemical, and biological sciences, especially MHD boundary layer problems. However, the new technique presented here is appropriate for dealing with boundary value issues. Additional boundary conditions for the differential equation and its derivatives can be constructed. Below explains the basic concept of ASM [8,29].

3.1 Basic Concept of Ananthaswamy-Sivasankari Method (ASM)

Let us consider the non-linear boundary value problem

$$q : f(y, y', y'', y''') = 0 \tag{16}$$

where q represents the third order non-linear differential equation such that $y = y(x, c, d, \ldots)$ in which c, d are given parameters and $x \in [L, U]$ can be finite or infinite with the following boundary conditions:

$$\left.\begin{array}{l} At \quad x = L, \; y(x) = y_{L_0} \,(or)\; y'(x) = y_{L_1} \,(or)\; y''(x) = y_{L_2} \\ At \quad x = U, \; y(x) = y_{U_0} \,(or)\; y'(x) = y_{U_1} \,(or)\; y''(x) = y_{U_2} \end{array}\right\} \tag{17}$$

Assume that the approximate analytical solution of the non-linear equations is an exponential function of the form

$$y(x) = l + m e^{ax} + n e^{-ax} \tag{18}$$

The unknown coefficients l, m and n are obtained by solving the non-linear differential equations as follows:

$$\left.\begin{aligned} y(L) &= l + m e^{aL} + n e^{-aL} = y_{L_0} \\ y'(L) &= a m e^{aL} - a n e^{-aL} = y_{L_1} \\ y''(L) &= a^2 m e^{aL} + a^2 n e^{-aL} = y_{L_2} \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} y(U) &= l + m e^{aU} + n e^{-aU} = y_{U_0} \\ y'(U) &= a m e^{aU} - a n e^{-aU} = y_{U_1} \\ y''(U) &= a^2 m e^{aU} + a^2 n e^{-aU} = y_{U_2} \end{aligned}\right\} \tag{20}$$

Eq. (19) and Eq. (20) may be used to get the unknown parameters l, m and n .

The following non-linear differential equations are obtained by substituting Eq. (18) into Eq. (16).

$$q : f(y(x,l,m,n,a,c,d), y'(x,l,m,n,a,c,d), y''(x,l,m,n,a,c,d), y'''(x,l,m,n,a,c,d)) = 0 \tag{21}$$

This equation is valid at x where $x \in [L,U]$. Solving Eq. (21), the unknown parameter a can be obtained in terms of the given parameters c and d .

The Homotopy analysis approach has been successfully used to solve a wide range of issues in science and engineering. It is an analytical method for obtaining series solutions to non-linear equations that is non-perturbative. In comparison to other perturbative and non-perturbative analytical approaches, HAM allows us to adjust and govern the convergence of a solution using the so-called convergence-control parameter. As a result, HAM has emerged as the most efficient technique for finding analytical solutions for the unknown function and its derivatives. As seen in the work of Liao [14-18], previous applications of HAM have mostly focused on non-linear differential equations whose non-linearity is a polynomial. Liao [14-18] developed the Homotopy analysis methodology, which is a powerful analytical technique for non-linear problems. Below explains the basic concept of MHAM.

3.2 Basic Concept of Modified Homotopy Analysis Method

Consider the following differential equation:

$$N[u(t)] = 0 \tag{22}$$

where N is a nonlinear operator, t denotes an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. By means of generalizing the conventional Homotopy method, Liao constructed the so-called zero-order deformation equation as follows:

$$(1-p)\,L[\varphi(t;p)-u_0(t)] = p\,h\,H(t)\,N[\varphi(t;p)] \tag{23}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter and $h \in [-1, 1]$, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$, $\varphi(t;p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p = 0$ and $p = 1$, it holds:

$$\varphi(t;0) = u_0(t) \text{ and } \varphi(t;1) = u(t) \tag{24}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have:

$$\varphi(t;p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)\,p^m \tag{25}$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \varphi(t;p)}{\partial p^m}\Big|_{p=0} \tag{26}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series, Eq. (25) converges at $p = 1$ then we have:

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) \tag{27}$$

By differentiating Eq. (23) for m times with respect to the embedding parameter p, and then setting $p = 0$ and finally dividing them by $m!$, we will have the so-called m^{th}-order deformation equation as:

$$L[u_m - \chi_m\,u_{m-1}] = h\,H(t)\,\Re_m(\vec{u}_{m-1}) \tag{28}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varphi(t;p)]}{\partial p^{m-1}} \tag{29}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{30}$$

Applying L^{-1} on both sides of Eq. (28), we get

$$u_m(t) = \chi_m\, u_{m-1}(t) + h\, L^{-1}[H(t)\, \Re_m(\vec{u}_{m-1})] \tag{31}$$

In this way, it is easily to obtain u_m for $m \ge 1,$ at m^{th} order, we have

$$u(t) = \sum_{m=0}^{M} u_m(t) \tag{32}$$

When $M \to +\infty$, we get an accurate approximation of the original Eq. (22). For the convergence of the above method, we refer the reader to Liao [15-18] and Faudzi *et al.,* [19]. If Eq. (22) admits a unique solution, then this method will produce the unique solution.

The approximate analytical solution for steady-state velocity is given below.

3.3 Approximate Analytical Solution of Velocity using ASM

The approximate analytical solution of the velocity distribution in Eq. (8) that satisfies the boundary condition is as follows:

$$f(\eta) = l + q e^{a\eta} + r e^{-a\eta} \tag{33}$$

$$f(\eta) = a q e^{a\eta} - a r e^{-a\eta} \tag{34}$$

Utilizing the boundary conditions in Eq. (11) and Eq. (12), we obtain the values of the parameters $l, q\ \&\, r$ as follows:

$$l = S + \frac{1}{a},\ q = 0 \ \ and \ \ r = \frac{-1}{a} \tag{35}$$

Thus, Eq. (33), becomes

$$f(\eta) = S + \frac{1}{a} - \frac{1}{a} e^{-a\eta} \tag{36}$$

Now, by putting Eq. (36) into Eq. (9), and then simplifying, we obtain

$$a^2 e^{-a\eta} + \left[\left(S + \frac{1}{a} - \frac{1}{a} e^{-a\eta}\right)\left(-a\, e^{-a\eta}\right)\right] - 2\, e^{-2a\eta} - K\, e^{-a\eta} - M\, e^{-a\eta} = 0 \tag{37}$$

Now, taking $\eta = 0$, Eq. (37) becomes

$$a^2 - S\,a - 2 - K - M = 0 \tag{38}$$

On solving the Eq. (38), we get the value of the parameter a which is given by

$$a = \frac{S \pm \sqrt{S^2 + 8 + 4K + 4M}}{2} \tag{39}$$

Hence, an approximate analytical solution of the velocity is obtained by substituting Eq. (39) into Eq. (36) as follows:

$$f(\eta) = S + \frac{1}{a} - \frac{1}{a} e^{-a\eta} \tag{40}$$

where a is obtained in Eq. (39). The approximate analytical solution for temperature and concentration is given below [4,5].

3.4 Approximate Analytical Solution of Temperature and Concentration using MHAM

The transformed temperature and concentration equations are obtained by substituting Eq. (40) into Eq. (9) and Eq. (10), we get:

$$\theta'' + \Pr\begin{bmatrix} Q_H\,\theta - e^{-a\eta}\,\theta - (St)\,e^{-a\eta} \\ + \left(S + \frac{1}{a} - \frac{1}{a} e^{-a\eta}\right)\theta' \end{bmatrix} = 0 \tag{41}$$

$$\phi'' + Sc\begin{bmatrix} \left(S + \frac{1}{a} - \frac{1}{a} e^{-a\eta}\right)\phi' - \phi\,e^{-a\eta} \\ -(St_1)e^{-a\eta} - \beta\,\phi \end{bmatrix} = 0 \tag{42}$$

The Homotopy for Eqs. (41) and (42) is as follows:

$$\theta'' = h\left\{\theta'' + \Pr\left[Q_H\,\theta - e^{-a\eta}\,\theta - (St)\,e^{-a\eta} + \left(S + \frac{1}{a} - \frac{1}{a} e^{-a\eta}\right)\theta'\right]\right\} \tag{43}$$

$$\phi'' = h\left\{\phi'' + Sc\left[\left(S + \frac{1}{a} - \frac{1}{a} e^{-a\eta}\right)\phi' - \phi\,e^{-a\eta} - (St_1)e^{-a\eta} - \beta\,\phi\right]\right\} \tag{44}$$

The initial approximation for Eqs. (43) and (44) is given by

$$\theta_0(0) = 1 - St,\ \theta_0(\infty) \to 0 \ \ and\ \ \theta_i(0) = 0,\ \theta_i(\infty) \to 0,\ \ i = 1,2,3,... \tag{45}$$

$$\phi_0(0) = 1 - St_1,\ \phi_0(\infty) \to 0 \ \ and\ \ \phi_i(0) = 0,\ \phi_i(\infty) \to 0,\ \ i = 1,2,3,... \tag{46}$$

The approximate analytical solutions to Eq. (43) and Eq. (44) are as follows:

$$\theta=\theta_0+p\theta_1+p^2\theta_2+\dots \tag{47}$$

$$\phi=\phi_0+p\phi_1+p^2\phi_2+\dots \tag{48}$$

Substituting Eq. (47) and Eq. (48) into Eq. (43) and Eq. (44) and comparing the coefficients of like powers of p, we get the following equations

$$p^0:\theta_0''=0 \tag{49}$$

$$p^1:\theta_1''-\theta_0''=h\left\{\theta_0''+\Pr\left[Q_H\,\theta_0-e^{-a\eta}\theta_0-(St)e^{-a\eta}+\left(S+\frac{1}{a}-\frac{1}{a}e^{-a\eta}\right)\theta_0'\right]\right\} \tag{50}$$

$$p^0:\phi_0''=0 \tag{51}$$

$$p^1:\phi_1''-\phi_0''=h\left\{\phi_0''+Sc\left[\left(S+\frac{1}{a}-\frac{1}{a}e^{-a\eta}\right)\phi_0'-\phi_0e^{-a\eta}-(St_1)e^{-a\eta}-\beta\phi_0\right]\right\} \tag{52}$$

By using MHAM, the initial guessing solutions for Eq. (26) and Eq. (27) and utilizing the boundary conditions in Eq. (45) and Eq. (46) are given by:

$$\theta_0(\eta)=(1-St)\,e^{-\eta} \tag{53}$$

$$\phi_0(\eta)=(1-St_1)\,e^{-\eta} \tag{54}$$

On solving Eq. (50) and Eq. (52) with the use of Eq. (53) and Eq. (54) and utilizing the boundary conditions in Eq. (45) and Eq. (46), we get the following solution

$$\begin{aligned}\theta_1(\eta)=&-\left[\begin{pmatrix}h+1+h\Pr Q_H\\-h\Pr S-\dfrac{h\Pr}{a}\end{pmatrix}(1-St)-\frac{h\Pr St}{a^2}+\frac{h\Pr\left(\dfrac{1}{a}-1+St\right)}{(1+a)^2}\right]e^{-\eta}\\&+\left(h+1+h\Pr Q_H-h\Pr S-\frac{h\Pr}{a}\right)(1-St)e^{-\eta}-\frac{h\Pr St}{a^2}e^{-a\eta}+\frac{h\Pr\left(\dfrac{1}{a}-1+St\right)}{(1+a)^2}e^{-(1+a)\eta}\end{aligned} \tag{55}$$

$$\phi_1(\eta) = -\left[\left(h+1-h\,Sc\left(S+\frac{1}{a}\right)-h\,Sc\,\beta\right)(1-St_1)-\frac{h\,Sc\,St_1}{a^2}+\frac{h\,Sc(1-St_1)\left(\frac{1}{a}-1\right)}{(1+a)^2}\right]e^{-\eta}$$
$$+\left(h+1-h\,Sc\left(S+\frac{1}{a}\right)-h\,Sc\,\beta\right)(1-St_1)e^{-\eta}-\frac{h\,Sc\,St_1}{a^2}e^{-a\eta}+\frac{h\,Sc(1-St_1)\left(\frac{1}{a}-1\right)}{(1+a)^2}e^{-(1+a)\eta} \quad (56)$$

According to HAM technique, we have

$$\theta = \lim_{p\to 1}\theta(\eta) = \theta_0 + \theta_1 \quad (57)$$

$$\phi = \lim_{p\to 1}\phi(\eta) = \phi_0 + \phi_1 \quad (58)$$

Therefore, the approximate analytical solutions of the temperature and concentration equations are derived by substituting Eq. (53) to Eq. (56) into Eq. (57) and Eq. (58) and we get the results as follows:

$$\theta_1(\eta) = (1-St)e^{-\eta} - \left[\left(h+1+h\Pr Q_H - h\Pr S - \frac{h\Pr}{a}\right)(1-St)-\frac{h\Pr St}{a^2}+\frac{h\Pr\left(\frac{1}{a}-1+St\right)}{(1+a)^2}\right]e^{-\eta}$$
$$+\left(h+1+h\Pr Q_H - h\Pr S - \frac{h\Pr}{a}\right)(1-St)e^{-\eta}-\frac{h\Pr St}{a^2}e^{-a\eta}+\frac{h\Pr\left(\frac{1}{a}-1+St\right)}{(1+a)^2}e^{-(1+a)\eta} \quad (59)$$

$$\phi_1(\eta) = (1-St_1)e^{-\eta} - \left[\left(h+1-h\,Sc\left(S+\frac{1}{a}\right)-h\,Sc\,\beta\right)(1-St_1)-\frac{h\,Sc\,St_1}{a^2}+\frac{h\,Sc(1-St_1)\left(\frac{1}{a}-1\right)}{(1+a)^2}\right]e^{-\eta}$$
$$+\left(h+1-h\,Sc\left(S+\frac{1}{a}\right)-h\,Sc\,\beta\right)(1-St_1)e^{-\eta}-\frac{h\,Sc\,St_1}{a^2}e+\frac{h\,Sc(1-St_1)\left(\frac{1}{a}-1\right)}{(1+a)^2}e^{-(1+a)\eta} \quad (60)$$

4. Results and Discussion

We looked at the effects of porosity, magnetic, suction and heat source parameters in this portion. The comparison of the analytical results of non-dimensional velocity, temperature, and concentration in Eq. (40), Eq. (59) and Eq. (60) with the numerical solution stated by Ismail *et al.,* [23] is interlined graphically. Figure 1 to Figure 12 illustrate the comparison of numerical and analytical solutions for non-dimensional velocity, temperature and concentration with different amounts of the physical parameters.

Figure 1 to Figure 3 represent the comparison of the non-dimensional velocity using Eq. (23), with numerical result shown in the study of Ismail *et al.,* [23] for varying values of K, M and S . As in Figure 1, the velocity was reduced by increasing the value of the porosity parameter K . From Figure 2, it is noted that by increasing the value of the magnetic parameter M , the velocity decreases. As shown in Figure 3, the velocity decreased by raising the amount of the suction parameter S . Also, Table 1, clearly evaluates the analytical and numerical results of the velocity $f''(0)$ for different values of the physical quantities λ, Me and Fr .

Figure 4 to Figure 9 depicts the comparison of the non-dimensional temperature using Eq. (59), with the numerical result described by Ismail *et al.,* [23] for varying values of $K, M, S, \Pr, St$ and Q_H . It can be seen from Figure 4 that the temperature increases by raising the value of the porosity parameter K . According to Figure 5, it is observed that by increasing the amount of the magnetic parameter M , the temperature rises. As shown in Figure 6, the temperature decreased by increasing the value of the suction parameter S . Figure 7 and Figure 8 show that by increasing the values of the Prandtl number Pr and thermal stratification parameter St respectively, the temperature decreases. Figure 9 clearly describes that the temperature increases by raising the amount of the heat source parameter Q_H .

Figure 10 to Figure 15 show the comparison of the non-dimensional concentration using Eq. (60), with the numerical result reported in the study of Ismail *et al.,* [23] for several amounts of K, M, S, Sc, β and St_1 . Figure 10 and Figure 11 illustrate that the concentration rises by raising the value of the porosity parameter K and the magnetic parameter M . From Figure 12, it is known that by increasing the amount of the suction parameter S , the concentration decreases. Also, from Figure 13, Figure 14 and Figure 15, it is clear that the concentration decreased by raising the values of the Schmidt number Sc , rate of reaction parameter β and chemical stratification parameter St_1 respectively.

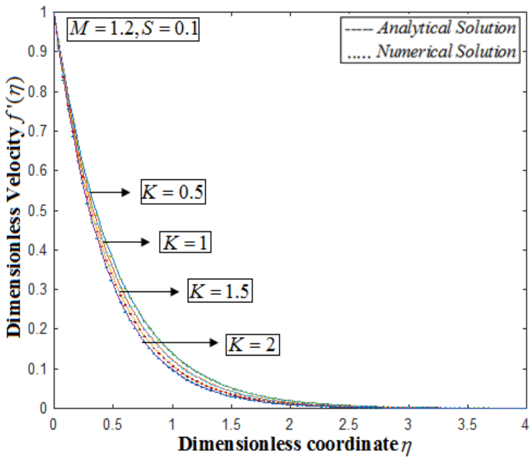

Fig. 1. Dimensionless velocity versus the dimensionless coordinate η for some fixed values of M, S and different values of K

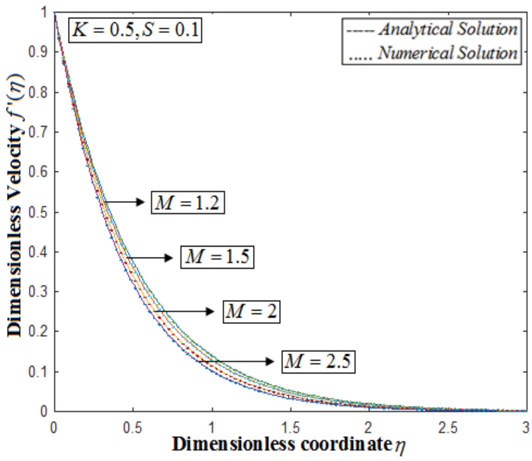

Fig. 2. Dimensionless velocity versus the dimensionless coordinate η for certain fixed values of K, S and varying values of M

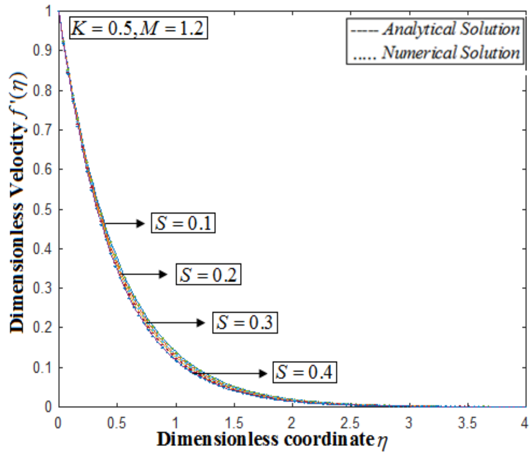

Fig. 3. Dimensionless velocity versus the dimensionless coordinate η for some fixed values of K, M and varying values of S

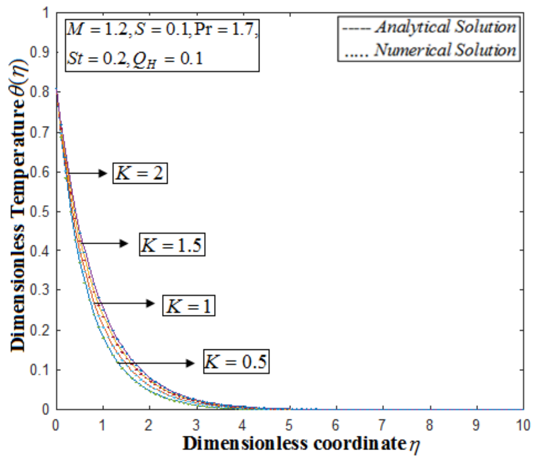

Fig. 4. Dimensionless temperature versus the dimensionless coordinate η for varying values of K and in some fixed amounts of $M, S, \mathrm{Pr}, St\ and\ Q_H$

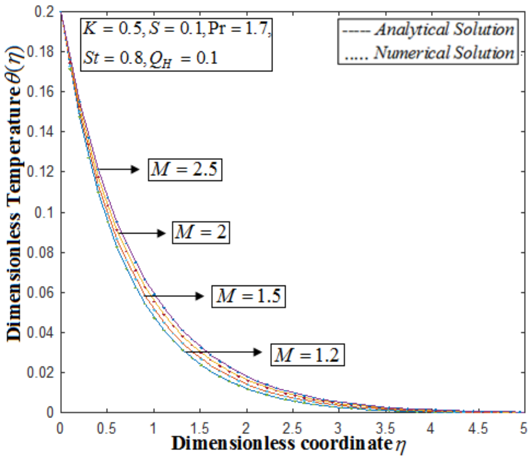

Fig. 5. Dimensionless temperature versus the dimensionless coordinate η for distinct amounts of M and in some fixed amounts of $K, S, \mathrm{Pr}, St\ and\ Q_H$

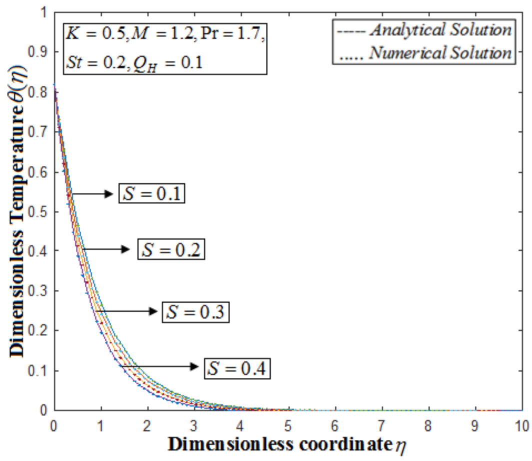

Fig. 6. Dimensionless temperature versus the dimensionless coordinate η for different values of S and in some fixed amounts of $K, M, \mathrm{Pr}, St\ and\ Q_H$

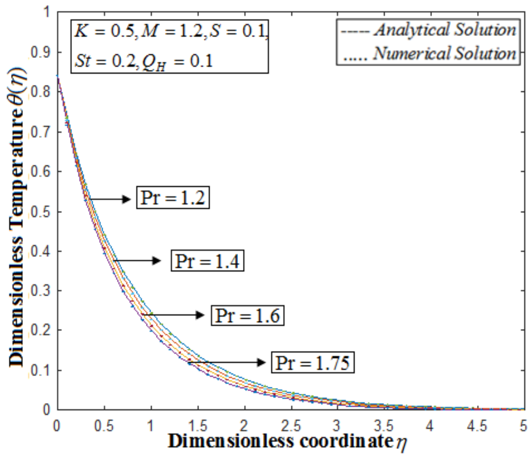

Fig. 7. Dimensionless temperature versus the dimensionless coordinate η for varying values of $\Pr$ and in some constant values of K, M, S, St *and* Q_H

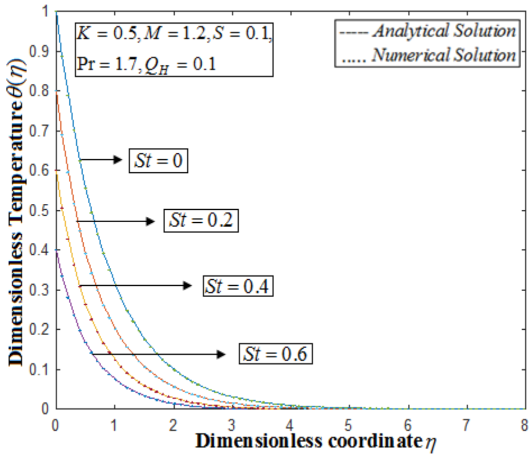

Fig. 8. Dimensionless temperature versus the dimensionless coordinate η for varying amounts of St and in certain constant amounts of $K, M, S, \Pr,$ *and* Q_H

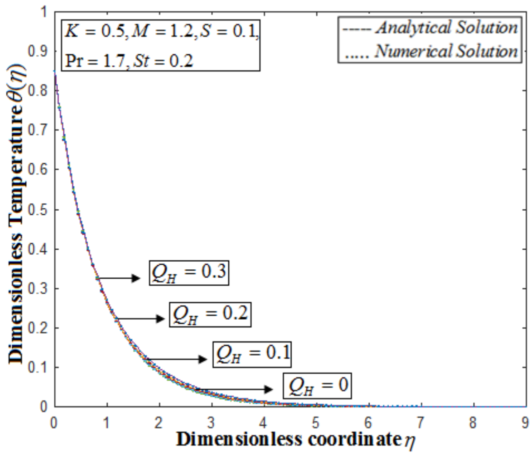

Fig. 9. Dimensionless temperature versus the dimensionless coordinate η for varying values of Q_H and in some fixed values of $K, M, S, \Pr,$ *and* St

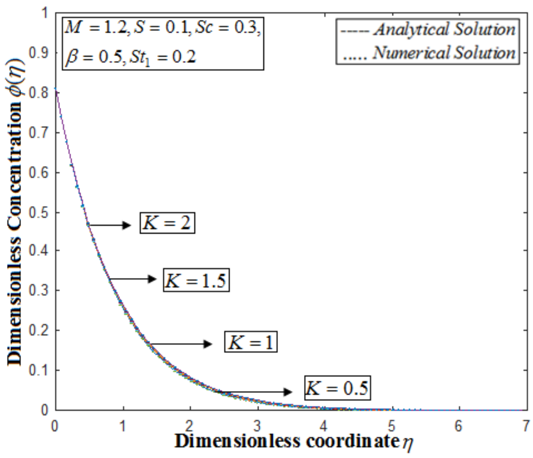

Fig. 10. Dimensionless concentration with the dimensionless coordinate η for varying values of K and some constant amounts of M, S, Sc, β *and* St_1

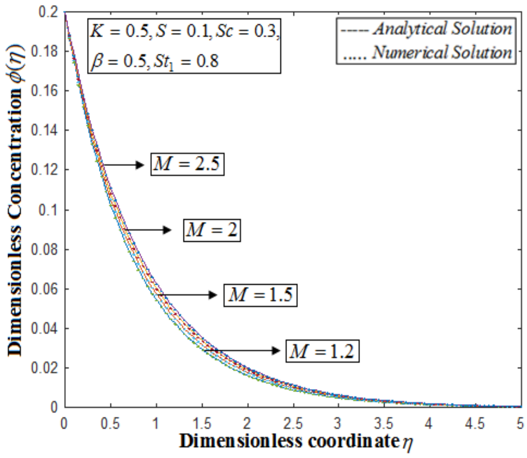

Fig. 11. Dimensionless concentration with the dimensionless coordinate η for several values of M and some particular values of K, S, Sc, β and St_1

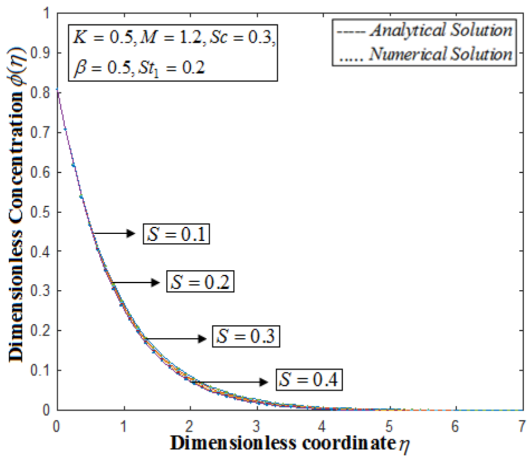

Fig. 12. Dimensionless concentration with the dimensionless coordinate η for varying values of S and some fixed amounts of K, M, Sc, β and St_1

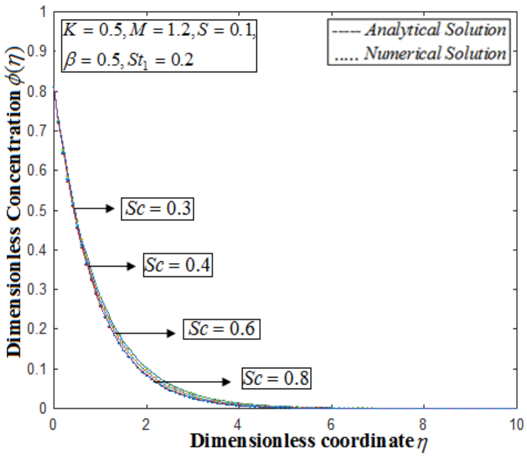

Fig. 13. Dimensionless concentration with the dimensionless coordinate η for several values of Sc and some fixed values of K, M, S, β and St_1

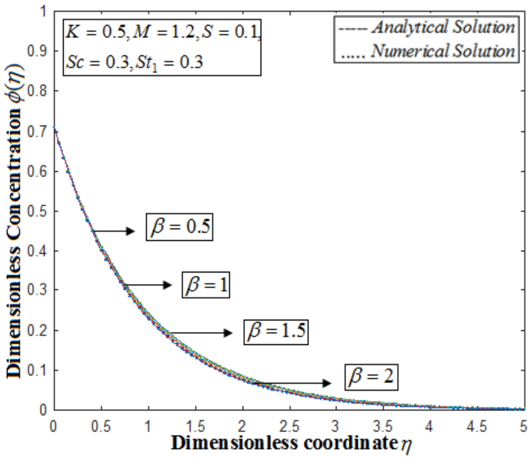

Fig. 14. Dimensionless concentration with the dimensionless coordinate η for varying amounts of β and certain constant amounts of K, M, S, Sc and St_1

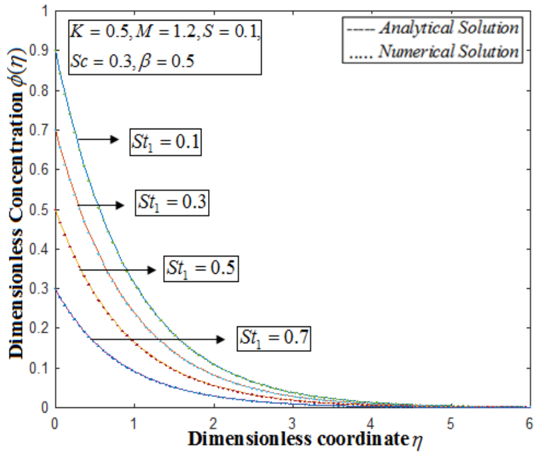

Fig. 15. Dimensionless concentration with the dimensionless coordinate η for distinct values of St_1 and some constant values of $K, M, S, Sc\ and\ \beta$

Table 1
Comparison of analytical solution with the numerical solution reported in Ismail *et al.,* [23] for different values of the parameters $\Pr$ and some fixed amounts of $K, M, S, St\ and\ Q_H$

Pr	Numerical -θ'(0)	Analytical -θ'(0)	Error %
1	0.9548	0.9538	0.1048
2	1.4715	1.4678	0.2521
3	1.8691	1.8745	0.2881
Average error percentage			0.2150

5. Conclusions

This study covered the flow of Magnetohydrodynamic (MHD) fluid in a double stratification medium across a stretching sheet with exponential permeability. In steady state, the approximate analytical solutions for the governing equations having thermal and chemical stratifications were solved with the help of a new approximate analytical method called the Ananthaswamy-Sivasankari Method (ASM), and the Modified Homotopy Analysis Method (MHAM). The analytical results were compared with the numerical solutions. The graphs were interlined to show the impacts of several parameters, including porosity, magnetic, suction, and heat source parameters.

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