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Approximate Analytical Expression for the Influence of Flow of MHD Nanofluids on Heat and Mass Transfer

S. Punitha[1] , V. Ananthaswamy[2,*], V.K.Santhi[3]

[1] Research Scholar, Research Centre and PG Department of Mathematics, The Madura College (Affiliated to Madurai Kamaraj University), Madurai, Tamil Nadu, India
[2] Research Centre and PG Department of Mathematics, The Madura College (Affiliated to Madurai Kamaraj University), Madurai, Tamil Nadu, India
[3] PG Department of Mathematics, The Meenakshi Arts College for women (Affiliated to Madurai Kamaraj University), Madurai, Tamil Nadu, India

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ABSTRACT

In depth analysis on the boundary layer flow of magnetohydrodynamic nanofluids is conducted in this study. The analytical results are estimated for temperature profile, concentration profile, reduced Nusselt number and reduced sherwood number using Modified q-Homotopy analysis method**.** Also, the impacts of numerous physical parameters such as the magnetic field, the Eckert number, the thermophoresis parameter, Brownian parameter and Lewis number are discussed in detail. Comparing our obtained results with numerical solution results in a very good fit. Additionally, the findings are displayed graphically. Reduced skin friction, reduced Nusselt number and reduced Sherwood number are shown in table representation. This method can be extended to physical, chemical, and engineering sciences.

1. Introduction

In many engineering process with applications in industries such as extrusion, melt-spinning, heat rolling, and rubber sheet manufacturing, cooling of a large metal plate in a path that may be an electrolyte, this flow can be stretching in the surface these are the important issues. Kalaivanan *et al.,* [1] investigated that the effect of elastic deformation on the boundary layer flows of nanofluid over a stretching surface in the presence of slip boundary condition. In the study of Vleggaar *et al.,* [2] the polymers and filaments sheet are manufactured by continuous extrusion of the polymer from a die to a windup roller. There are many applications in engineering industries for boundary layer over the stretching surface (Fisher [3]). Having a low heat transfer in a fluid would cause limited heat transfer and can lead to limited heat transfer efficiency. Due to the high thermal conductivity of metal particles, adding them to a fluid would increase the thermal conductivity and also heat transfer of the resultant mixture fluid. Choi’s [4] initial analysis of the term "nanofluid" a liquid suspension containing ultra-fine particles. Nanoparticles (e.g., Copper (*Cu*), Silver (*Ag*), Alumina (Al_2O_3),

*Corresponding author.
E-mail address: ananthu9777@gmail.com (V. Ananthaswamy)

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Titanium ($TiO2$)) range from 1 to 100 nm in diameter defined by Oztop *et al.,* [5].The base liquid's thermal conductivity is improved by (10% − 50%) if it is suspended by a low volumetric fraction (less than 5%) of nanoparticles are discussed by Eastman *et al.,* [6–8]. In Khan *et al.,* [9], the authors examined improvements in the thermal conductivity of fluids (such as oil, water, and ethylene glycol mixture) which are poor heat transfer by suspending nano /micro or large particle materials in these fluids. Kuznetsov *et al.,* [10] explained that the effects of nanoparticle on the natural convection boundary layer flow through a vertical plate, and also described about the Brownian motion and thermophoresis.

In the work of Noghrehabad *et al.,* [11-14] it can be shown that the force on heat and mass transfer of nanofluid over linear stretching sheet. Hamad *et al.,* [15] explored an analytical solution of the natural convection flow of nanofluid past a semi-infinite vertical stretching sheet in the presence of a magnetic field. Buongiorno's model is employed by Niazi *et al.,* [16-18] for describing the behaviour of nanofluids, and introduced the flux conservation unique, and it is in coincidence with practical observations. During the recent days, convection heat transfer of nanofluids is a hot topic of academic and industrial research due to its various applications in industries processes such as thermal heating, power generation and chemical processing [19-22].

Pavlov *et al.,* [23] applied the polymer and metallurgy industries, hydro-magnetic techniques are used in magnetic fields. Takhar *et al.,* [24] influences the high application in physics, chemistry, and engineering. [25-28] analytic tool for nonlinear problem, namely heat transfer, MHD flow nanofluid etc., which does not depend on small parameters. Ananthaswamy *et al.,* [29,30] describes the approximate analytical solution for steady hydromagnetic permeable channel flow of a conducting fluid with variable electrical conductivity fluid with variable electrical conductivity and asymmetric navier slip at channel wall are derived analytically.

Yanala Dharmendar reddy *et al.,* [31] investigated the effects of radiation and convective boundary conditions. Shankar Goud Bejawada *et al.,* [32] influenced that the relevant factors on non dimensional fluid flowing areas, heat and mass transmission rates is investigated, additionally the opposite direction is noted for opposing flow occurrences. Dharmendar Reddy *et al.,* [33] examine that thermal radiation impact of MHD boundary layer flow of William nanofluid along a stretching surface with porous medium in account of thermal slips and velocity can be discussed numerically. Pramod kumar *et al.,* [34] discussed about the study effect of soret number on MHD free convection flow of heat and mass transfer of an electricity conducting non-Newtonian fluid through a vertical moving porous plate. Bejawada Shankar Goud *et al.,* [35] explored that the flow phenomena of hydromagnetic nanofluid thermal stratified through permeable medium due to the influence of the radiative heat energy. Umair Khan *et al.,* [36] objective of this study is to develop a methodology for identification of flow regime using dynamic pressure signals and deep learning techniques. Rifky Ismail *et al.,* [37] studied that the deacetylation temperature impact towards the crystalline index, chemical bond and morphology of Chitosan synthesized. Khan *et al.,* [38] analyse the influence of Brownian motion and thermophoresis on a nonlinearly permeable stretching sheet in a nanofluid was solved numerically.

The above mentioned articles point out that the convective heat transfer of nanofluid is advanced topic of academic and industrial research due to its various applications. The purpose of this research is to examine the combined impacts of the MHD nanofluid convective boundary circumstances. The corresponding equations are analytically solved via Mq-HAM. The obtained solution is compared with the numerical solution (previous work). Also, the effects of physical factors involved on dimensionless velocity, temperature, concentration profiles are explained using graphical results.

2. Mathematical Formulation

Consider the steady two-dimensional boundary layer flow of the nanofluid past a stretching surface with linear velocity u_w and x is the coordinate measured along the stretching surface as shown in Figure 1. A steady uniform stress leading to equal and opposite force along the x-axis, the ambient values attained as y tends to infinity of $T$and C are denoted by T_∞ and C_∞. The uniform magnetic field of force B_0 is imposed in the y-direction according to Ref. [8].

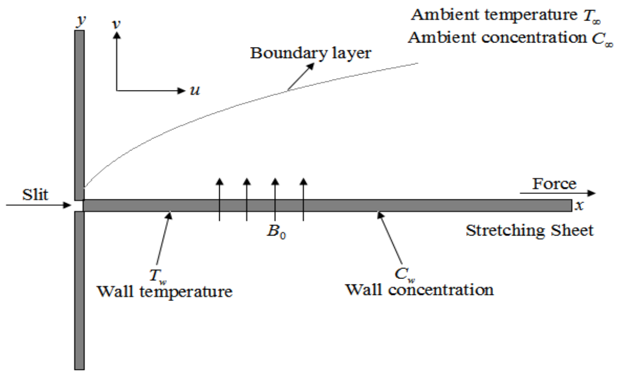

Fig. 1. Physical diagram of the flow geometry

For the current study, the governing basic steady conservation of mass, momentum, thermal energy and nanoparticles equations described in Ref. [8] are given as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho f}\frac{\partial p}{\partial y}+v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma^* B^2{}_0}{\rho f}u \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho f}\frac{\partial p}{\partial y}+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+$$
$$T\left\{D_B\left(\frac{\partial C}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}\right)+\left(\frac{D_T}{T_\infty}\right)\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right]\right\}+\frac{\mu f}{(\rho Cp)f}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma^* B^2{}_0}{(\rho Cp)f}u^2 \tag{4}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\left(u\frac{\partial^2 C}{\partial x^2}+v\frac{\partial^2 C}{\partial y^2}\right)+\left(\frac{D_T}{T_\infty}\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{5}$$

Subject to the boundary conditions reported in Ref. [38]:

$$u=u_w(x)=ax, v=s, T=T_w, C=C_w aty=0,$$
$$u=v=0, T=T_\infty, C=C_\infty asy\to\infty \tag{6}$$

where u and v are the velocity components along the axes x and y(ms^{-1}), respectively, ρf is the density of the base fluid $(kg.m^{-3})$, v is the kinematic viscosity $(m^2.s^{-1})$, σ^* is the electrical conductivity $(\Omega m)^{-1}$, $p$is the fluid pressure, $\alpha=\frac{k_f}{(\rho c)_f}$ is the thermal diffusivity , D_B is the Brownian diffusion coefficient, D_T is the thermophoretic diffusion coefficient (m^2/s),$\tau = \frac{(\rho c)p}{(\rho c)_f}$ is the ratio between the effective heat capacity of the nanoparticle material and heat capacity of the fluid with ρbeing the density, c is the volumetric volume expansion coefficient and ρp is the density of the particles, c_p is the fluid specific heat at constant pressure, μ_f is the viscosity of the base fluid and s is suction (or injection) parameter, respectively, T is the temperature of the fluid (k) , C is the fraction of the volume of nanoparticles $(kg.m^{-3})$, T_w is the temperature of the stretching surface (k), C_w is the fraction of the volume of nanoparticles on the stretching surface,T_∞ is the ambient temperature (k) and C_∞ is the fraction of the volume of ambient nanoparticles, under the related work [10].

$$f(\eta) = \frac{\psi}{\alpha {Ra_x}^{1/4}}, \theta(\eta) = \frac{T-T_\infty}{T_w-T_\infty}, \varphi(\eta) = \frac{C-C_\infty}{C_w-C_\infty}, \tag{7}$$

where ψ is a stream function provided by

$$u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x} \tag{8}$$

Well then Eq. (1) identically satisfied. For converting Eq. (1) – (5) with the boundary conditions Eq. (6) into the following non-linear ordinary differential equations. a similarity solution in Ref. [10] was implemented.

$$f''' + \left(\frac{1}{4\,Pr}\right)[3ff'' - 2(f')^2] - Mf' = 0 \tag{9}$$

$$\theta'' + \frac{3}{4}f\theta' + Nb\varphi'\theta' + Nt(\theta')^2 + Ec(f'')^2 + MEc \rightleftarrows\rightleftarrows (f')^2 = 0 \tag{10}$$

$$\varphi'' + \frac{3}{4}Lef\varphi' + \frac{Nt}{Nb}\theta'' = 0 \tag{11}$$

With the boundary conditions:

$$\begin{gathered} f(0) = s, f'(0) = 1, \theta(0) = 1, \varphi(0) = 1, \\ f'(\infty) = 0, \theta(\infty) = 0, \varphi(\infty) = 0. \end{gathered} \tag{12}$$

Where primes indicate differentiation for η and $Pr, N\,b, Nt, M, Ec, Le$ are Prandtl number, Brownian motion parameter, thermophoresis parameter, Eckert number, and magnetic parameter and Lewis number, respectively. The physical parameters below are described by:

$$\begin{gathered} \eta = \frac{y}{x}{Ra_x}^{1/4}, Pr = \frac{v}{a}, Le = \frac{\alpha}{D_B}, Nb = \frac{(\rho c)p(\varphi w - \varphi\infty)}{(\rho c)_f\alpha}, Nt = \frac{(\rho c)D_T(T_w - T\infty)}{(\rho c)_f\alpha T_\infty}, \\ M = \frac{\sigma^* {B_0}^2}{\mu_f}L^2, L = \sqrt[3]{\frac{v\alpha {Ra_x}^{\frac{1}{4}}}{(1-\varphi_\infty)\beta g(T_w - T\infty)}}, Ec = \sqrt[3]{\frac{[g\beta(1-\varphi_\infty)]^2\left[\frac{v\alpha Ra_x}{T_w - T_\infty}\right]}{(C_p)_f}}, \\ Ra_x = \frac{(1-\varphi_\infty)\beta g(T_w - T_\infty)x^3}{v\alpha} \end{gathered} \tag{13}$$

Here gravitational acceleration, volumetric expansion co-efficient of the fluid, nanoparticle volume fraction at the surface, ambient nanoparticle volume fraction attained as $y$tends to be infinite and local Rayleigh number, respectively, are also the symbols $g, \beta, \varphi_w, \varphi_\infty andRa_x$. In addition, $f, \theta, and\varphi$ are the dimensionless of the stream function, temperature, and volume of nanoparticles respectively. It distinguishes the local skin frictionc_f, the reduced Nusselt number Nur and the reduced Sherwood number Shr

$$\frac{1}{2}\left(\frac{\Pr \mathrm{Re}_x{}^2}{Ra_x{}^{\frac{3}{4}}}\right)c_f = f''(0), \qquad Nur = -\theta'(0), \quad Shr = -\phi'(0) \tag{14}$$

where, Re_x is the local Reynolds number based on the stretching velocity$u_w(x)$.

3. Approximate Analytical Solution of the Non-Linear Differential Equation. Using the Modified q-Homotopy Analysis Method [25-28]:

Homotopy analysis method is a non-perturbative analytical method for obtaining series solutions to non-linear equations. and has been successfully applied to various problems in science and engineering. In comparison with other perturbative and non perturbative analytical methods, HAM offers the ability to adjust and control the convergence of a solution via the so-called convergence-control parameter. Because of this, HAM has proved to be the most effective method for obtaining analytical solutions to highly non-linear differential equations. Previous applications of HAM have mainly focused on non-linear differential equations in which the non-linearity is a polynomial in terms of the unknown function and its derivatives.

Liao [17-22] proposed a powerful analytical method for non-linear problems, namely the Homotopy analysis method. This method provides an analytical solution in terms of an infinite power series. However, there is a practical need to evaluate this solution and to obtain numerical values from the infinite power series. In order to investigate the accuracy of the Homotopy analysis method (HAM) solution with a finite number of terms, the system of differential equations were solved. The Homotopy analysis method is a good technique comparing to another perturbation method. The Homotopy analysis method contains the auxiliary parameter$h$, which provides us with a simple way to adjust and control the convergence region of solution series. The approximate analytic expression for the dimensionless velocity, dimensionless angular velocity, dimensionless temperature, dimensionless concentration profile, skin friction, Nusselt number and Sherwood number by using the modified q-Homotopy analysis method (Mq-HAM) is as follows:

$$f''(0) = a^2 - (a+1) + \left(\begin{array}{l} \frac{1}{4p}\left(\frac{3a^2 c_{10}}{c_2} - \frac{c_9}{M-1} - \frac{(-a-1)^2 c_8}{c_3} + \frac{4(a+1)^2}{2M-8} + \frac{4a^4}{c_6} \right) \\ -\frac{8a^4}{c_6} - \frac{8(a+1)^2}{2M-8} + \frac{4a(a+1)(-a-1)^2}{c_3} \\ +(n+h)\left(\frac{-a^5}{c_2} + \frac{(a+1)}{M-1} \right) - (n+h)M\left(\frac{-a^3}{c_2} + \frac{(a+1)}{M-1} \right) \\ - \left(\begin{array}{l} \frac{1}{4p}\left(\begin{array}{l} -\frac{3a^2 c_{10}}{c_2} + \frac{c_9}{M-1} - \frac{(-a-1)^2 c_8}{c_3} \\ -\frac{2(a+1)^2}{2M-8} - \frac{2a^4}{c_6} \end{array} \right) \\ +4\left(\frac{a^3}{c_6} + \frac{(a+1)^2}{2M-8} \right) + \frac{4a(a+1)(-a-1)}{c_3} \\ +(n+h)c_{11} - (n+h)M\left(\frac{-a^3}{c_2} + \frac{(a+1)}{M-1} \right) \end{array} \right) / \sqrt{M} \end{array} \right) / n \tag{15}$$

$$c_1 = \frac{e^{-\eta}(a+1)}{M-1}, c_2 = -a^3 + Ma, c_3 = -(a+1)^3 + M(a+1), c_4 = e^{-\eta(a+1)}, c_5 = \frac{e^{-2v(a+1)^2}}{-8+2M}$$
$$c_6 = -8a^3 + 2Ma, c_7 = e^{-2a\eta}, c_8 = a^2(a+1) + a + 1, c_9 = s(a+1) - a(a+1), c_{10} = sa^2 + a^3$$
$$c_{11} = \frac{a^4}{c_2} - \frac{a+1}{M-1}, a = Le * Nt * Nb * Ec, a1 = Pr * Le * s, \tag{16}$$

$$-\theta'(0) = a_1 + 1 + \left(\begin{array}{c} \frac{-3(s+a_1)(-a_1-1}{4(a_1+1)} - \frac{3(a_1+1)(-2a_1-1)}{4(2a_1+1)^2} + \frac{3(a_1+1)^2(-a_1-2)}{(a_1+2)^2} + \frac{Nt(-2a_1-2)}{4} \\ + \frac{NtLe(a_1+1)\left(-a_1-1-\frac{NtLe}{Nb}\right)}{\left(-a_1-1-\frac{NtLe}{Nb}\right)^2} + Ec\left(\frac{-a_1{}^3}{2} + \frac{(a_1+1)^2}{2} - \frac{2a_1{}^2(-a_1-1)}{(a_1+1)} \right) \\ +MEc\left(\frac{-a_1}{2} - \frac{(a_1+1)^2}{2} - \frac{2a_1(-a_1-1)}{a_1+1} \right) - a_1 - 1 \end{array} \right) \tag{17}$$

$$f(\eta) = e^{-a\eta} - e^{-\eta}(a+1) + s + a$$

$$+\left(\begin{array}{l} -\left(\begin{array}{l}\left(\frac{1}{4}\left(\frac{-3ae^{-a\eta}c_{10}}{c_2}+\frac{e^{-\eta}c_9}{M-1}-\frac{-(-a-1)c_4c_8}{c_3}-2c_5-\frac{2a^3c_7}{c_6}\right)\right)\\ +\frac{4a^3c_7}{c_6}+4c_5+\frac{4a(a+1)(-a-1)c_4}{c_3}\end{array}\right)/\left(\sqrt{M}e^{-\sqrt{M}\eta}\right)\\ -\left(\begin{array}{l}\frac{\frac{3c_{10}}{c_2}-\frac{c_9}{M-1}-\frac{c_8}{c_3}+\frac{(a+1)^2}{2M-8}+\frac{a^2}{c_6}}{4p}+\frac{2a^2}{c_6}+\frac{2(a+1)^2}{2M-8}\\ -\frac{4(a(a+1))}{c_3}-(n+h)\left(\frac{-a^3}{c_2}+\frac{a+1}{M-1}\right)\end{array}\right)\\ +(n+h)M\left(\frac{-a}{c_2}+\frac{a+1}{M-1}\right)\\ +\frac{1}{4p}\left(-\frac{3ae^{-a\eta}c_{10}}{c_2}+\frac{e^{-\eta}c_9}{M-1}-\frac{(-a-1)c_4c_8}{c_3}-2c_5-\frac{2c_7a^3}{c_6}\right)\\ +\frac{4a^3c_7}{c_6}+4c_5+\frac{4a(-a-1)(a+1)c_4}{c_3}\\ +(n+h)\left(\left(\frac{a^4e^{-a\eta}}{c_2}-c_1\right)-(n+h)a\left(\frac{a^2e^{-a\eta}}{c_2}-c_1\right)\right)/\sqrt{M}\\ +\frac{1}{4p}\left(\frac{3ae^{-a\eta}c_{10}}{c_2}-\frac{e^{-\eta}c_9}{M-1}-\frac{c_4c_8}{c_3}+c_5+\frac{c_7a^2}{c_6}\right)\\ -\frac{2a^2c_7}{c_6}-2c_5+\frac{4a(a+1)c_4}{c_3}\end{array}\right)/n \qquad (18)$$

$$\theta(\eta)=e^{-\eta(a_1+1)}-h\left(\begin{array}{l}\frac{3(s+a_1)}{4(a_1+1)}+\frac{3(a_1+1)}{4(2a_1+1)^2}-\frac{3(a_1+1)^2}{4(a_1+2)^2}-\frac{Nt\,Le(a_1+1)}{\left(a_1+1+\frac{Nt\,Le}{Nb}\right)^2}-\frac{Nt}{4}\\ -Ec\left(\frac{a_1{}^2}{4}-\frac{(a_1+1)^2}{4}-\frac{2a_1{}^2}{a_1+1}\right)-M\,Ec\left(\frac{1}{4}+\frac{1}{4}(a_1+1)^2-\frac{2a_1}{a_1+1}\right)-1\\ -\frac{3(s+a_1)e^{-\eta(a_1+1)}}{4(a_1+1)}-\frac{3(a_1+1)e^{-\eta(2a_1+1)}}{4(2a_1+1)^2}+\frac{3(1+a_1)^2e^{-\eta(a_1+2)}}{(a_1+2)^2}+\frac{Nt\,e^{-2\eta(+1)a_1}}{4}\\ +\frac{Nt\,Le(a_1+1)e^{-\eta\left(a_1+1+\frac{NtLe}{Nb}\right)}}{\left(a_1+1+\frac{NtLe}{Nb}\right)^2}+E\left(\frac{a_1{}^2e^{-2ax}}{4}-\frac{e^{-2\eta}(a_1+1)^2}{4}-\frac{2a_1{}^2e^{-\eta(a_1+1)}}{a_1+1}\right)\\ +MEc\left(\frac{e^{-2a_1\eta}}{4}+\frac{e^{-2\eta}(a_1+1)^2}{4}-\frac{2a_1e^{-\eta(a_1+1)}}{(a_1+1)}\right)+e^{-\eta(a_1+1)}\end{array}\right) \qquad (19)$$

$$\varphi(\eta)=e^{-\frac{LeNt\eta}{Nb}}-n-h-\frac{3}{4}hLe\left(-s-\frac{3a}{4}-\frac{1}{(a+1)^2}+\frac{1}{4}\right)-\frac{hNt}{Nb}+(n+h)e^{-\eta}$$
$$+\frac{3}{4}hLe\left(-e^{-\eta}(s+a)-\frac{e^{-\eta(a+1)}}{(a+1)^2}+\frac{1}{4}(a+1)e^{-2\eta}+hNt\frac{e^{-\eta(a_1+1)}}{Nb}\right) \qquad (20)$$

$$-\varphi'(0)=\frac{LeNt}{Nb}+n+h-\frac{3hLe}{4}\left(s+\frac{a}{2}-\frac{-a-1}{(a+1)^2}-\frac{1}{2}\right)-\frac{hNt(-a_1-1)}{Nb} \qquad (21)$$

$$\left.\begin{aligned}
A22 &= -\left((n+h)+\frac{3hLe}{4}\left((s+a_1)-\frac{1}{(a_1+1)^2}+\frac{(a_1+1)}{4}\right)+\frac{hNt}{Nb}\right), B22=0, B11=0\\
&c_2 = 0, c_1 = \frac{3(a_1+s)}{4(a+1)}+\frac{3(a_1+1)}{4(2a+1)^2}-\frac{3(a_1+1)^2}{4(a+2)^2}-\frac{NtLe(a_1+1)}{\left(a_1+1+\frac{NtLe}{Nb}\right)^2}-\frac{1}{4}Nt\\
&-Ec\left(\frac{{a_1}^2}{4}-\frac{(a_1+1)^2}{4}-2\frac{{a_1}^2}{a_1+1}\right)-MEc\left(\frac{1}{4}+\frac{1}{4}(a_1+1)^2-\frac{2a_1}{a_1+1}\right)+1
\end{aligned}\right\} \tag{22}$$

$$C11 = \left(\begin{array}{c}
\frac{1}{4Pr}\left(\begin{array}{c}\frac{-3ae^{-a\eta}(sa^2+a^3)}{-a^3+Ma}+\frac{e^{-\eta}(s(a+1)-a(a+1)}{M-1}-\\
\frac{(-a-1)e^{-\eta(a+1)}(a^2(a+1)+a+1)}{-(a+1)^3+M(a+!)}-2\frac{e^{-2\eta}(a+1)^2}{2M-8}-2\frac{a^3e^{-2a\eta}}{-8a^3+2Ma}\end{array}\right)\\
+4\frac{a^3e^{-a\eta}}{-8a^3+2aM}+4\frac{e^{-2\eta}(a+1)^2}{2M-8}+4\frac{a(a+1)(-a-1)e^{-\eta(a+1)}}{-(a+1)^3+M(a+1)}\\
+(n+h)\left(\frac{a^4e^{-a\eta}}{-a^3+Ma}-\frac{e^{-\eta}(a+1)}{M-1}\right)-(n+h)M\left(\frac{a^2e^{-a\eta}}{-a^3+Ma}\right)
\end{array}\right)/\sqrt{M}e^{-\sqrt{M}\eta} \tag{23}$$

$$A11 = -B11-C11-\left(\begin{array}{l}
\frac{1}{4\Pr}\left(\begin{array}{c}\frac{-3(sa^2+a^3)}{-a^3+Ma}-\frac{(s(a+1)-a(a+1)}{M-1}-\\
\frac{(a^2(a+1)+a+1)}{-(a+1)^3+M(a+1)}+\frac{(a+1)^2}{2M-8}+\frac{a^2}{-8a^3+2Ma}\end{array}\right)\\
+\frac{2a^2}{-8a^3+2aM}+\frac{2(a+1)^2}{2M-8}-4\frac{a(a+1)}{-(a+1)^3+M(a+1)}\\
-(n+h)\left(-\frac{a^3}{-a^3+Ma}+\frac{(a+1)}{M-1}\right)\\
+(n+h)M\left(\frac{a}{-a^3+Ma}+\frac{a+1}{M-1}\right)
\end{array}\right)/\sqrt{M}e^{-\sqrt{M}\eta} \tag{24}$$

4. Results and Discussion

The approximate analytical expression of the dimensionless stream function$f(\eta)$, dimensionless temperature$\theta(\eta)$, and dimensionless concentration profile $\varphi(\eta)$ were derived by using modified q-Homotopy analysis method (Mq-HAM). In Figure 1 represent the physical diagram of flow geometry. In Figure 2 depicts that the dimensionless stream function $f(\eta)$ versus dimensionless coordinate η .In dimensionless velocity by varying other parameters are shown in table 1, 2 and 3. From Figures 3 to 9 represents the dimensionless temperature $\theta(\eta)$ versus dimensionless space variableη. In Figures 3 to 4, it indicates that, when the thermophoresis parameterNt increases, the corresponding dimensionless coordinate also increases in some fixed value of the other dimensionless parameters. From Figures 5 to 7, it shows that, when parameter of Magnetic parameter M increases, the corresponding dimensionless coordinate also increases in some fixed value of the other dimensionless parameters. From Figures 8 to 9, it shows that, when increases the Prandtl number Pr increases, then the corresponding dimensionless coordinate decreases in some fixed values of the other dimensionless parameters.

From Figures 10 to 15 represents the effects of dimensionless concentration profile$\varphi(\eta)$versus dimensionless space variable η.From Figures 10 to 12 it is noted that, when the values of Lewis number Le increases, the corresponding dimensionless coordinate decreases in some other fixed values of dimensionless parameters. From Figures 13 to 15, it represents that the value of Brownian motion parameterNbincreases, the corresponding dimensionless parameter also increases for some fixed values of other dimensionless parameters.

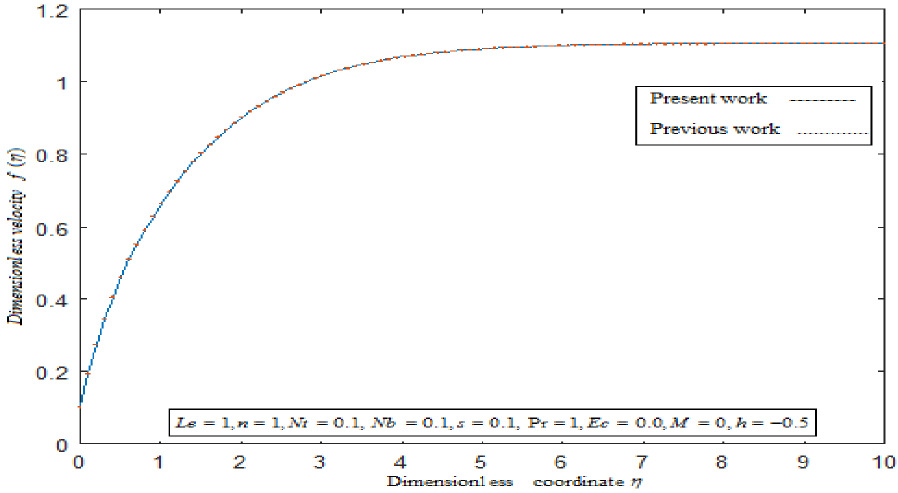

Fig. 2. Dimensionless space variable η versus the dimensionless stream function$f'(\eta)$.The curves are plotted using the Eq. (18) for various values of dimensionless parameters that can be compared with the previous study using the parameters$Le, Pr, N\,t, Nb, M, S, Ec$.

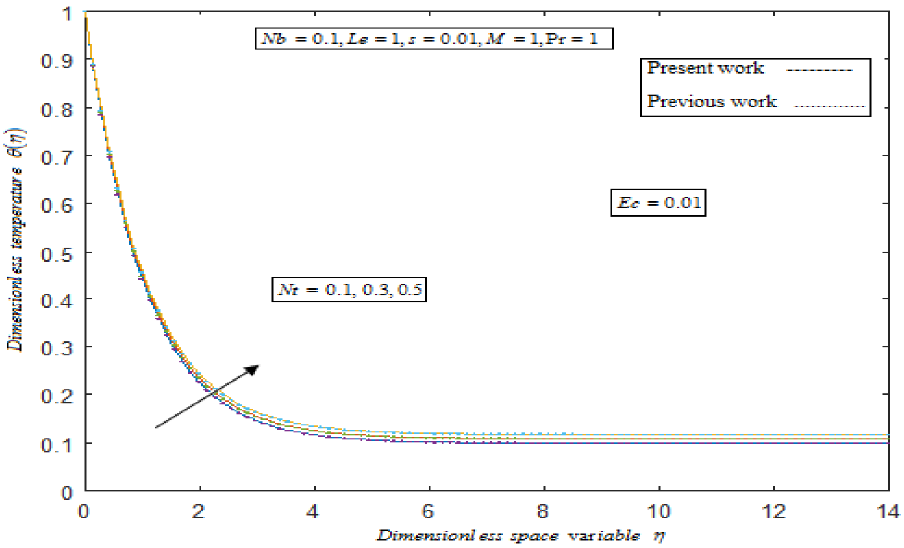

Fig. 3. Dimensionless space variable ηversus the dimensionless temperature$\theta(\eta)$.
The curves are plotted using the Eq. (19) for various values of thermophoresis parameter Nt and in some fixed values of $Le, Pr,\quad Nb, M, S, Ec$

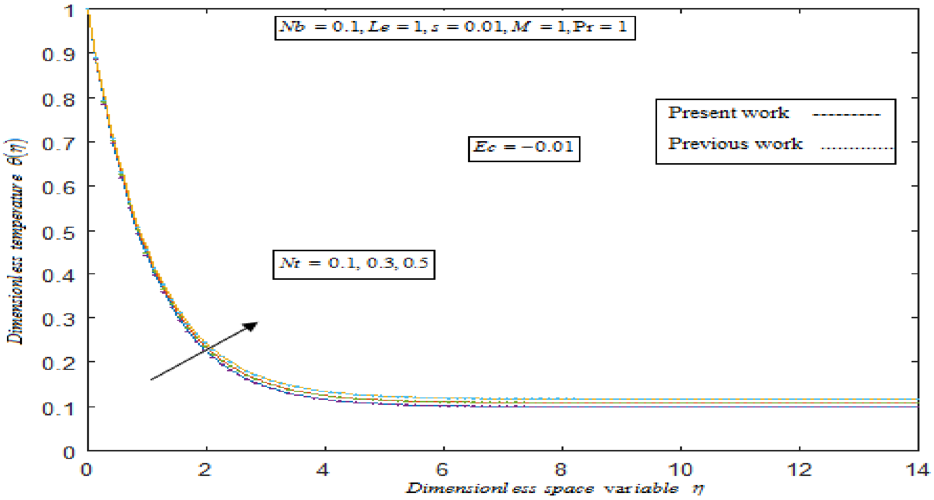

Fig. 4. Dimensionless space variable ηversus the dimensionless temperature$\theta(\eta)$. The curves are plotted using the Eq. (19) for various values of thermophoresis parameter Nt and in some fixed values of $Le, Pr, \quad Nb, M, S, Ec$

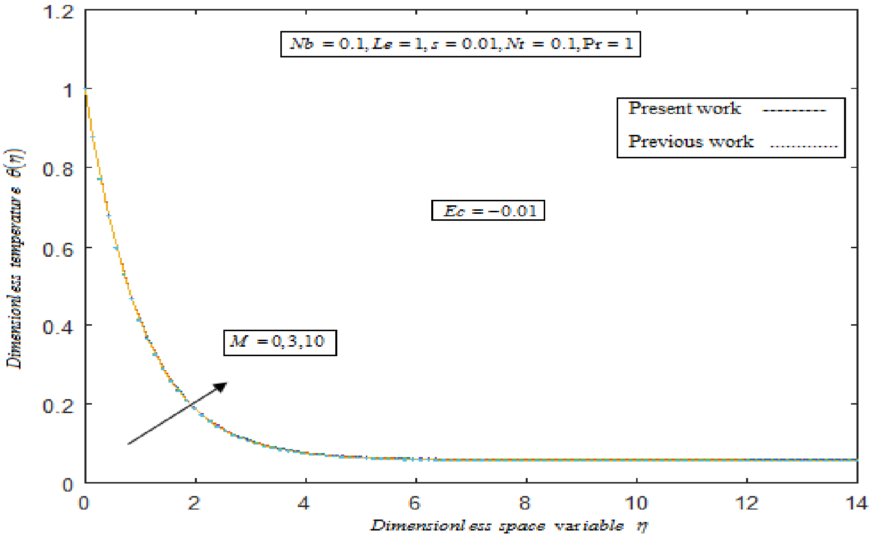

Fig. 5. Dimensionless space variable ηversus the dimensionless temperature$\theta(\eta)$ The curves are plotted using the Eq. (19) for various values of Magnetic parameter M and in some fixed values of $Le, Pr, \quad Nb, Nt, S, Ec$

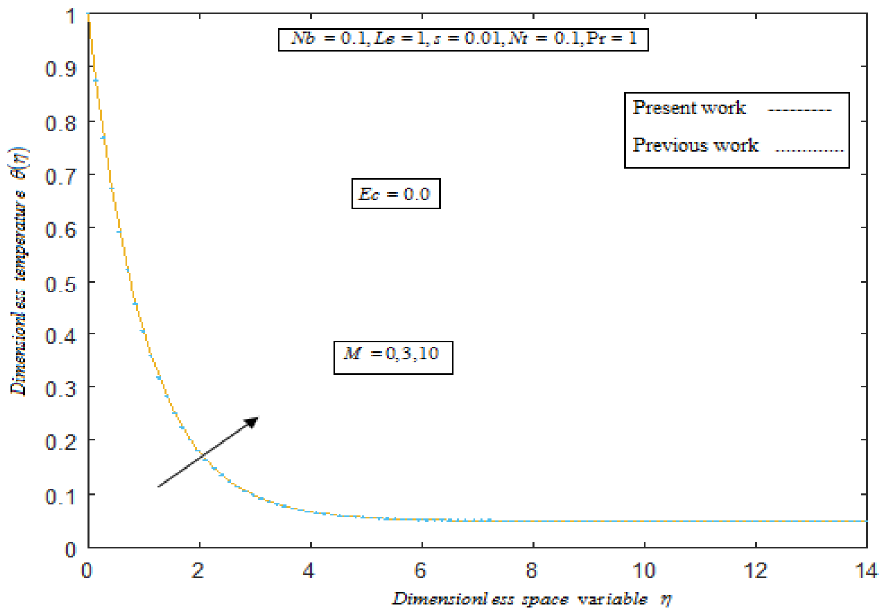

Fig. 6. Dimensionless space variable ηversus the dimensionless temperature $\theta(\eta)$. The curves are plotted using the Eq. (19) for various values of Magnetic parameter M and in some fixed values of $Le, Pr,\quad Nb, Nt, S, Ec$

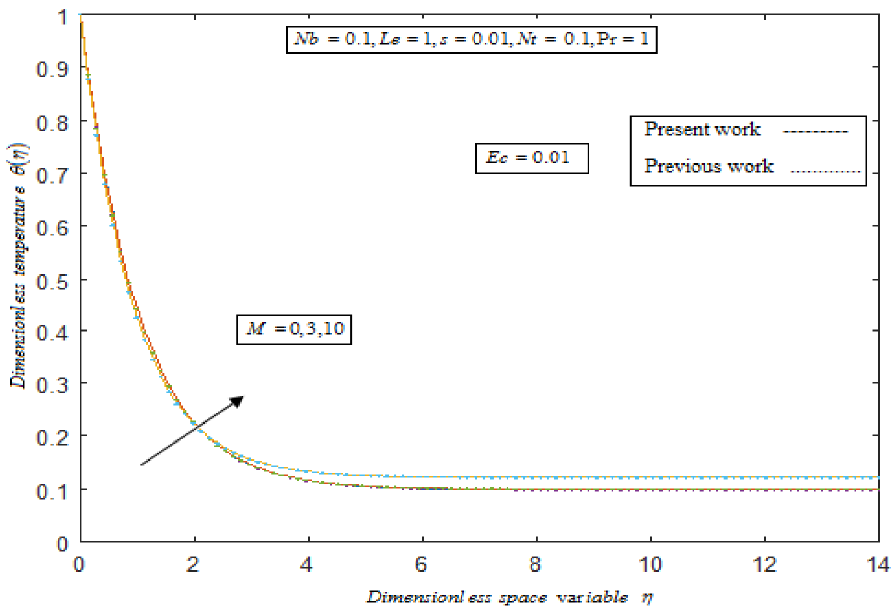

Fig. 7. Dimensionless space variable ηversus the dimensionless temperature $\theta(\eta)$. The curves are plotted using the Eq. (19) for various values of Magnetic parameter M and in some fixed values of $Le, Pr,\quad Nb, Nt, S, Ec$

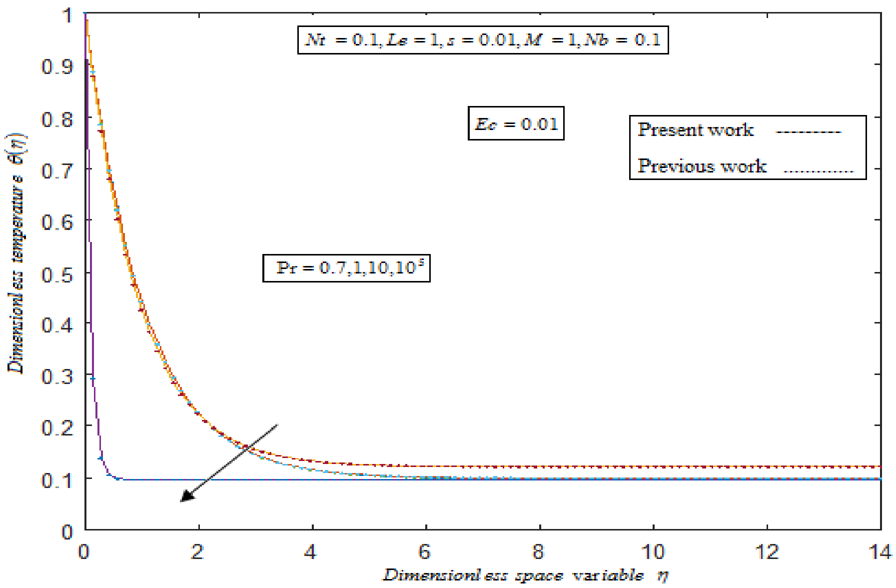

Fig. 8. Dimensionless space variable ηversus the dimensionless temperature $\theta(\eta)$ The curves are plotted using the Eq. (19) for various values of Prandtl number Pr and in some fixed values of Le, Nb, Nt, S, Ec

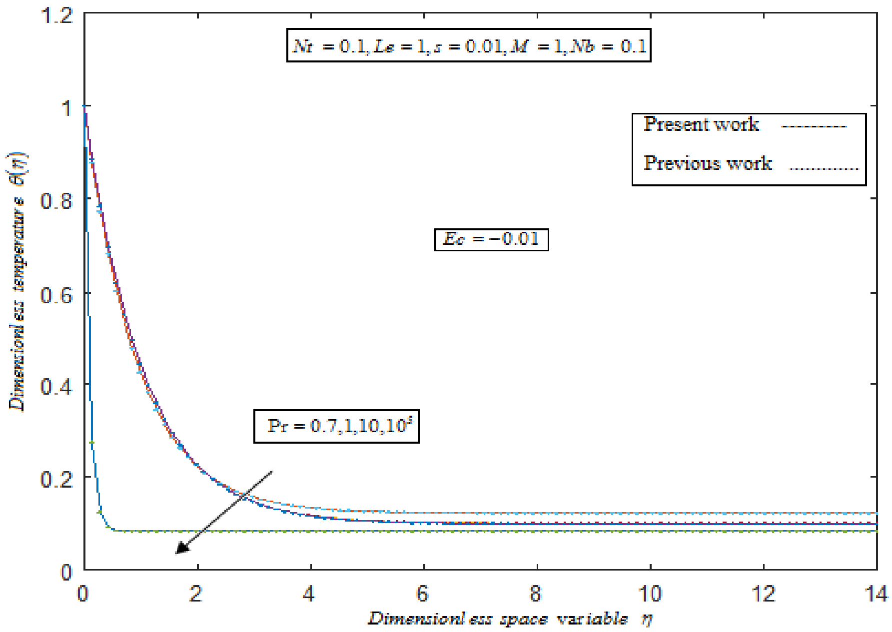

Fig. 9. Dimensionless space variable ηversus the dimensionless temperature $\theta(\eta)$The curves are plotted using the Eq. (19) for various values of Prandtl number Pr and in some fixed values of Le, Nb, Nt, S, Ec

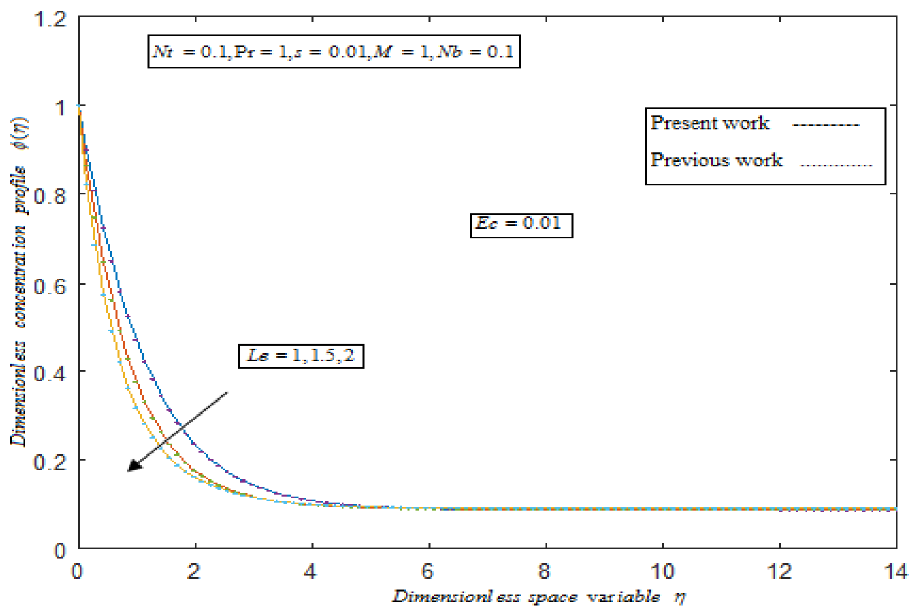

Fig. 10. Dimensionless space variable ηversus the dimensionless concentration profile $\varphi(\eta)$. The curves are plotted using the Eq. (20) for various values of Lewis number Le and in some fixed values of $Pr, N\,b, Nt, S, Ec$

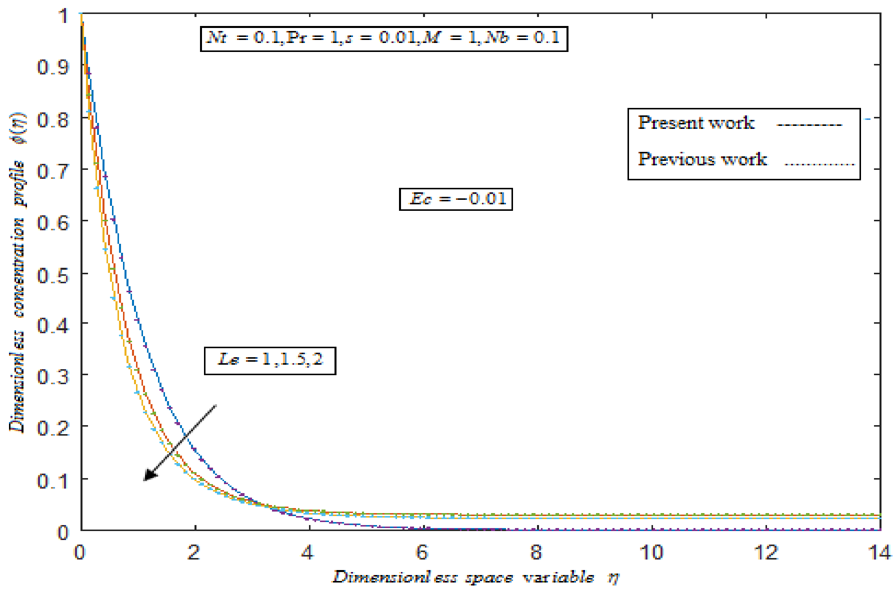

Fig. 11. Dimensionless space variable η versus the dimensionless concentration profile $\varphi(\eta)$. The curves are plotted using the Eq. (20) for various values of Lewis number Le and in some fixed values of $Pr, N\,b, Nt, S, Ec$

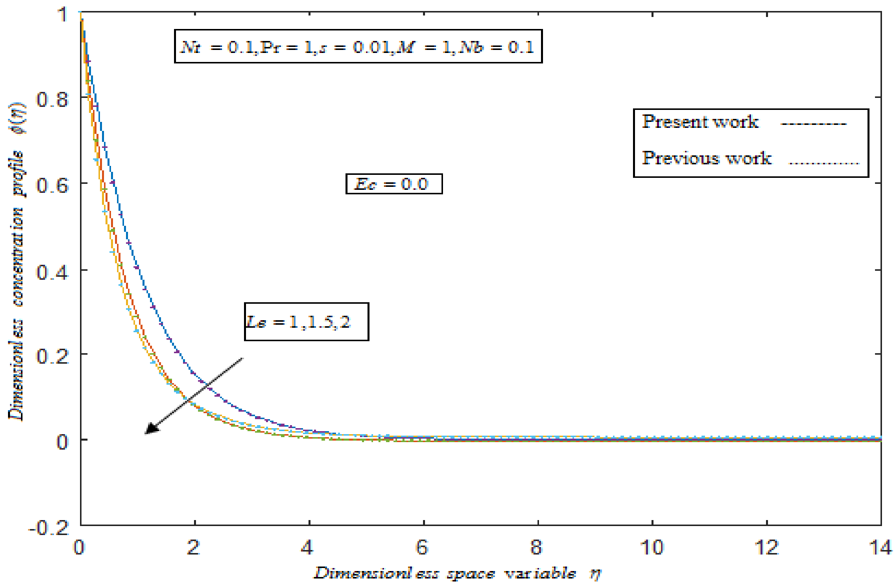

Fig. 12. Dimensionless space variable ηversus the dimensionless concentration profile $\varphi(\eta)$. The curves are plotted using the Eq. (20) for various values of Lewis number Le and in some fixed values of $Pr, N\,b, Nt, S, Ec$

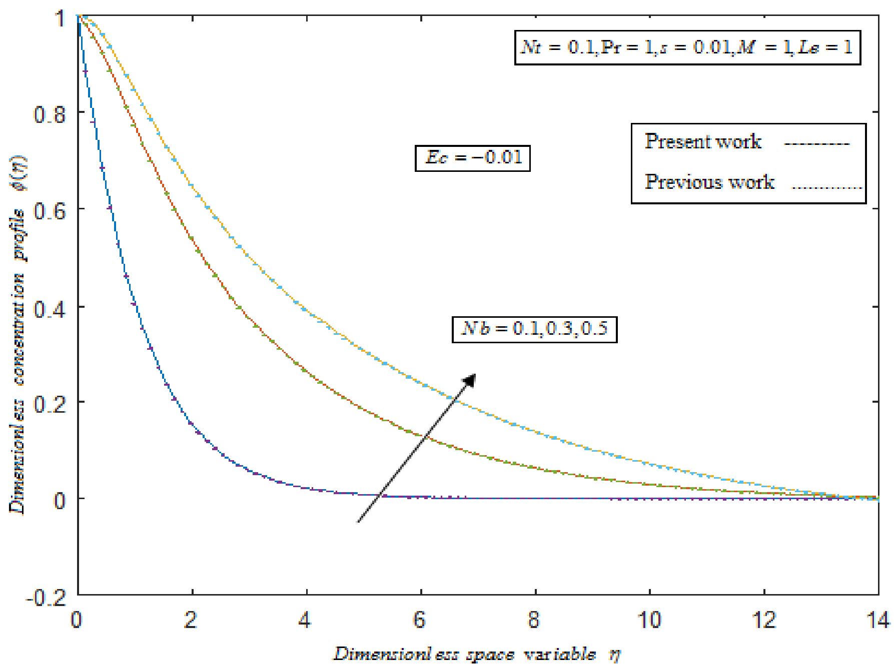

Fig. 13. Dimensionless space variable η versus the dimensionless concentration profile $\varphi(\eta)$. The curves are plotted using the Eq. (20) for various values of Brownian motion parameterNb and in some fixed values of $Pr, L\,e, Nt, S, Ec$

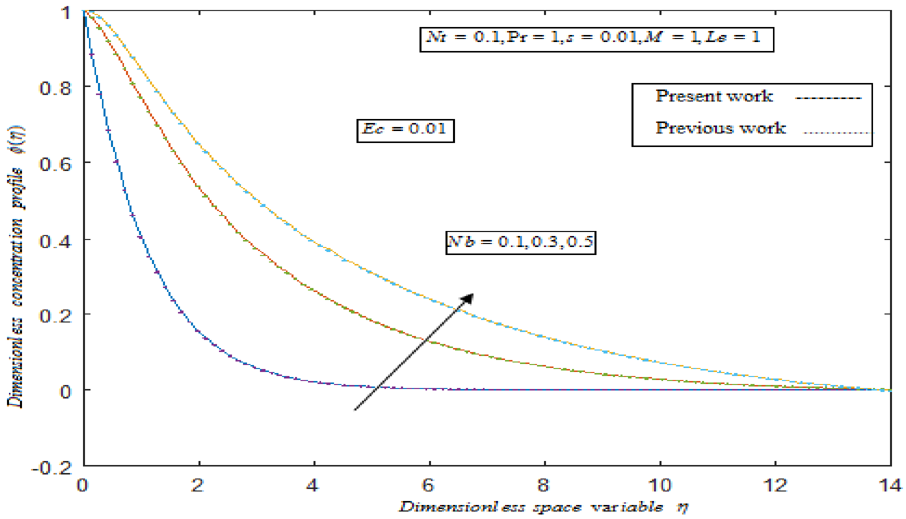

Fig. 14. Dimensionless space variable η versus the dimensionless concentration profile $\varphi(\eta)$. The curves are plotted using the Eq. (20) for various values of Brownian motion parameterNb and in some fixed values of $Pr, L\,e, Nt, S, Ec$

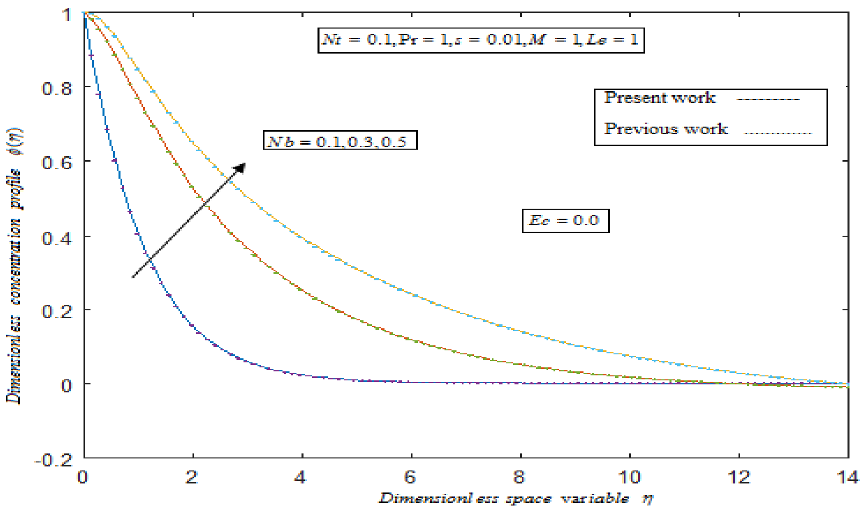

Fig. 15. Dimensionless space variable η versus the dimensionless concentration profile $\varphi(\eta)$. The curves are plotted using the Eq. (20) for various values of Brownian motion parameterNb and in some fixed values of $Pr, L\,e, Nt, S, Ec$

Table 1
Comparison test results for local skin friction$f''(0)$ using Eq. (15)
a). When $Nb = 0.1, Nt = 0.1, Le = 10, Ec = 0.0, s = 1, M = 1$ at different values of Pr
b). when $Nb = 0.1, Nt = 0.1, Le = 10, Pr = 1, Ec = 0.0, M = 1$ at different values of s

Pr	Numerical solution	Analytical solution	Error%	S	Numerical Solution	Analytical solution	Error %
1	-1.24162	-1.24192	0.000	-10	-0.06457	-0.06559	-0.014
3	-0.591478	-0.59106	0.001	-5	-0.1167	-0.11688	-0.0001
10	-0.291396	-0.29127	0.000	0.5	-0.2702	-0.27051	-0.0009
10^3	-0.10255	-0.10219	-0.003	5	-0.5004	-0.50164	-0.002
10^5	-0.100026	-0.10012	-0.002	10	-0.8254	-0.82303	-0.001
Average Error Percentage			-0.001				-0.001

Table 2
Comparison test results for reduced Nusselt number$Nur = -\theta'(0)$ with different values of Nt, Nbusing the Eq. (17). a). When $Le = 1, Pr = 1, Ec = 0.0, s = 1, M = 1$

Nb	$Nt = 0.1$			$Nt = 0.3$			$Nt = 0.5$		
	Numerical solution	Analytical solution	Error%	Numerical solution	Analytical solution	Error %	Numerical solution	Analytical solution	Error %
0.1	0.18658	0.18677	0.001	0.16625	0.14064	0.182	0.147136	0.11586	0.269
0.3	0.15068	0.15042	0.001	0.13212	0.13195	0.001	0.11468	0.11461	0.000
0.5	0.11767	0.10032	0.171	0.10078	0.10051	0.002	0.08494	0.08456	0.004
Average error %			0.056			0.061			0.091

Comparison test results for reduced Sherwood number $shr = -\varphi'(0)$ with different values of Nt, Nb using the Eq. (21), b). when $Le = 1, Pr = 1, Ec = 0.0, s = 1, M = 1$

Nb	$Nt = 0.1$			$Nt = 0.3$			$Nt = 0.5$		
	Numerical Solution	Analytical solution	Error%	Numerical solution	Analytical solution	Error %	Numerical solution	Analytical solution	Error %
0.1	0.32228	0.32220	0.00	0.15991	0.15906	0.005	0.05928	0.05923	0.00
0.3	0.40331	0.39592	0.186	0.36796	0.36658	0.003	0.35122	0.35028	0.002
0.5	0.41898	0.41834	0.001	0.40802	0.40882	-0.001	0.40711	0.40697	0.000
Average error %			0.062			0.002			0.002

Table 3
Comparison test results for reduced Nusselt number $Nur = -\theta'(0)$ with different values of Ec, M using the Eq. (17), a). When $Le = 1, Pr = 1, Nb = 0.1, s = 1, Nt = 0.1$

M	$Ec = -0.01$			$Ec = 0.0$			$Ec = 0.01$		
	Numerical solution	Analytical solution	Error%	Numerical solution	Analytical solution	Error %	Numerical solution	Analytical solution	Error %
0	0.59819	0.59535	0.004	0.48010	0.47800	0.004	0.36189	0.36151	0.001
1	0.60583	0.60562	0.0003	0.39629	0.39153	0.121	0.18658	0.18618	0.002
10	0.80447	0.80440	0.000	0.21698	0.21172	0.024	-0.37063	-0.37031	0.000
Average error %			0.001			0.049			0.001

Comparison test results for reduced Sherwood number $shr = -\varphi'(0)$ with different values ofEc, M, using the Eq. (21) b). when $Le = 1, Pr = 1, Nt = 0.1, Nb = 0.1, s = 1$

M	$Ec = -0.01$			$Ec = 0.0$			$Ec = 0.01$		
	Numerical solution	Analytical solution	Error%	Numerical solution	Analytical solution	Error %	Numerical solution	Analytical solution	Error %
0	0.09915	0.001	0.20295	0.20649	-0.017	0.30686	0.30195	0.016	0.001
1	-0.06471	-0.004	0.12870	0.12441	0.034	0.32228	0.32205	0.007	-0.004
10	-0.50287	-0.003	0.07547	0.07534	0.001	0.65395	0.65305	0.001	-0.003
Average error %			-0.002			0.006			0.008

5. Conclusions

In this work, the effects of various physical parameters on nanofluid that MHD flow past a stretching surface were explored. Analytically a system of non-linear ordinary differential equations was solved by using q-Homotopy analysis method. The graphical results show a good fit as compared to numerical solution. The following results were highlighted from the findings:

i) Nusselt number is a decreasing function for some fixed value of $Ec = 0.01$,
ii) Sherwood number is an increasing function for variation of Nt with Nb by varying the other dimensionless parameters.
iii) Eckert number and thermophoresis parameter improves the temperature field and increases the thermal boundary layer thickness.

iv) Prandtl number Pr reduces the temperature field and decreases the thermal boundary layer thickness, where Pr does not affect the temperature profile.

Furthermore, we may extend this method to solve the physical, chemical and engineering sciences problems. It is anticipated that the result of this research would be applicable to a wide variety of technical and industrial processes. This modified q-Homotopy analysis methods gives excellent flexibility to the expression of the solution and how the solution is explicitly obtained, and provides great freedom in choosing the base function of the desired solution in MHD nanofluid. This method can also be used to solve other physical problems like entropy generation, nanofluid, thermal boundary layer etc. The HAM provided a convenient way to control the convergence of approximation series which is a fundamental qualitative difference in analysis between HAM and other methods. This shows the validity and great potential of HAM for non-linear boundary value problems in science and engineering.

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