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Semi- Analytical Study on Non-Isothermal Steady R-D Equation in a Spherical Catalyst and Biocatalyst

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ABSTRACT

The Lane-Emden Boundary Value Problem as it appears in chemical applications, science, and biochemical applications are employed. Two specific models are solved by applying the Ananthaswamy-Sivasankari method (ASM). The model in first problem is a reaction–diffusion equation of a spherical catalyst and the model in second problem is the reaction–diffusion process of a spherical biocatalyst. Obtain a reliable semi-analytical expression of the effectiveness factors and the concentrations. A graph is constructed for the obtained semi-analytical solutions.The effects of several parameters like dimensionless activation energy, Thiele modulus and dimensionless heat of reaction are shown in graphical representation. Our semi-analytical solution is compared with numerical simulation by using MATLAB and finds good fit in all parameters. The new analytical method ASM is helpful to solve many non-linear problems mainly Reaction-Diffusion equation.

1. Introduction

The Lane–Emden equation in Ref. [1-6] for shape factor α has the form

$$y''(x) + \frac{\alpha}{x} y'(x) + f\big(y(x)\big) = 0, \alpha > 0 \quad (1)$$

The Lane–Emden equation is a basic equation in the theory of stellar structure of the shape factor for $\alpha = 2$, i.e., signifying spherical bodies. These equation also describes the changes in temperature or concentration in physics, chemistry, biology, biochemistry, and many others domains in Ref. [1-9]. The singular behaviour that occurs at $x = 0$ gives the main difficulty for solving Eq. (1). Several numerical and analytical methods are utilized to solve Lane Emden equations with boundary conditions referred in Ref. [7, 10-20].

The Lane–Emden equation was first designed by astrophysicists Robert Emden and Jonathan Homer Lane and examined the behaviour of thermal in the spherical gas clouds dealing under the

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common attractiveness of molecules and obeying the classical laws of thermodynamics of Richardson *et al.,* [21]. The well–known Lane–Emden Equation is used to design more phenomena in astrophysics and mathematical physics, such as stellar structure theory, thermal behaviour of spherical gas clouds, isothermal gas spheres, and thermionic current theory. A significant amount of work has been done on these types of problems for various structures by previous studies [20-28]. A system of the Lane–Emden Equations emerge in the several model of phenomena in physical like the population evolution, pattern formation, chemical reactions, and others [29-32]. Lane-Emden Equations were solved numerically by Allayed *et al.,* [44] and analytically by Omaha [45].

In this article, we will first consider a single reaction that occurs non-isothermally on a spherical catalysts pellet. The concentration of chemical particles in a spherical catalyst is characterized by the Boundary Value Problem to the Lane–Emden Equation in Ref. [20, 33-35]. In addition, we will study reactions that proceed non-isothermally in spherical biocatalyst. Most chemical reactions take place inside the catalyst pellet, which are porous material by Saadatmandi [36]. We assume that the biochemical reaction follow Michaelis-Menten kinetics. Appropriate plots are used to illustrate the surprisingly high accuracy and fast convergence of our approximate solutions, while noting that an exact solution is almost always impossible to obtain. The recommended analysis indicates the reliability and is easily extended to more common in reaction-diffusion models in chemical reactor technology by Rach *et al.,* [4]. Also, Skrzypacz *et al.,* [50] and Golman *et al.,*[51] solved reaction-diffusion problems by using semi-analytical methods. The brief explanation of analytical and numerical simulation was explained by Crasta *et al.,* [52].

The major purpose of this study is to use ASM to obtain a semi- analytical solution of the non-isothermal steady Reaction-Diffusion (R-D) equation in a spherical catalyst and spherical biocatalyst. Now a days, researchers are solving this types of problems and provides solutions in implicit manner but our analytical findings offers solution in explicit form. As compared to several numerical methods and semi-analytical methods, this method requires a single iteration. The solution of semi-analytical and numerical simulation are then compared and showed in graphically. To interlined the effects of several parameters such as dimensionless activation energy, Thiele modulus and dimensionless heat of reaction. Also, the effectiveness factor is calculated by using ASM.

2. Mathematical Formulation of the Problems

2.1 Mathematical Model of Spherical Catalyst Equation

Several industrial reactors incorporate the heterogeneous reaction kinetics of packed catalyst pellets. A single catalytic pellet of radius R can be treated as a porous medium through which reactants occur simultaneously.

The species and energy balances for diffusive transport inside the pellet can be written as follows (Rach *et al.,* [46]):

$$D\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}C_A\right)+r_A=0 \tag{2}$$

$$K\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}T\right)+r_A\Delta H=0 \tag{3}$$

where the Arrhenius reaction rate of the temperature internal to the catalytic pellet is

$$r_A=k_{ref}C_A\,exp\left[\frac{-E}{R_g}\left(\frac{1}{T}-\frac{1}{T_S}\right)\right] \tag{4}$$

The boundary conditions are:

$$C_A|_{r=R} = C_{AS} \tag{5}$$

$$T|_{r=R} = T_S \tag{6}$$

$$\frac{d}{dr}C_A\bigg|_{r=R} = \frac{d}{dr}T\bigg|_{r=0} = 0 \tag{7}$$

i.e., physically the steady-state values at the surface are known and there is zero flux at the origin.

We assume that a first-order irreversible reaction takes place within the catalytic particle, the Heat of reaction ΔH, effective diffusivity D and effective thermal conductivity K are constants, no external resistance to mass and heat transfer exists and the reaction is exothermic.

From Eq. (2) and (3):

$$K\frac{d}{dr}\left(r^2\frac{d}{dr}T\right) = \Delta HD\frac{d}{dr}\left(r^2\frac{d}{dr}C_A\right) \tag{8}$$

Applying the integral operator $\int_R^r\int_0^r(.)drdr$ to both sides of Eq. (8) and using the boundary conditions in Eq. (5)-(7) provides:

$$T - T_S = \frac{(-\Delta H)D}{K}(C_{AS} - C_A) \tag{9}$$

Substituting Eq. (9) into Eq. (4), we obtain the Arrhenius reaction rate in terms of the reactant concentration internal to the catalytic pellet.

$$r_A = -k_{ref}C_A\,exp\left[\frac{E}{R_gT_S}\frac{\frac{(-\Delta H)DC_{AS}}{KT_S}\left(1-\frac{C_A}{C_{AS}}\right)}{1+\frac{(-\Delta H)DC_{AS}}{KT_S}\left(1-\frac{C_A}{C_{AS}}\right)}\right] \tag{10}$$

Substituting Eq. (10) into Eq. (2) and introducing the following dimensionless variables and parameters [3, 36, 38-41, 47].

$$x = \frac{r}{R}, u = \frac{C_A}{C_{AS}}, \gamma = \frac{E}{R_gT_a}, \beta = \frac{(-\Delta H)DC_{AS}}{KT_S}, \varphi^2 = \frac{k_{ref}R^2}{D}, \tag{11}$$

where x represents the dimensionless concentration along radial direction of catalytic pellet, r denotes the distance along radial direction of catalytic pellet, R indicates the Radius of the spherical catalytic pellet, u represents the dimensionless concentration along radial direction of catalytic pellet, C_A is the concentration of reactant Ainside the catalytic pellet, C_{AS} is the concentration of reactant Aat the surface of catalytic pellet, $\gamma$is the dimensionless activation energy, E represents the activation energy, R_g indicates the universal gas constant, T_a is the temperature inside the catalytic pellet, T_S is the temperature at the surface of catalytic pellet, β is the dimensionless heat of reaction, ΔH is the heat of reaction, D is the effective diffusivity inside the catalytic pellet, K is the effective thermal conductivity inside the catalytic pellet, φ is the Thiele modulus, k_{ref} is the reference reaction constant.

Rach *et al.,* [3], and in some of the references therein, we obtained the Lane-Emden BVP for the normalized concentration $u(x)$ as (Rancher Singh [37]). Which models the dimensionless concentration of chemical species within a spherical catalyst. In Eq. (12), γ indicates the Dimensionless activation energy, β denotes the Dimensionless heat of reaction, and φ represents the Thiele modulus as calculated at the surface of the spherical catalytic pellet by Rich *et al.,* [3]

$$u"(x)+\frac{2}{x}u'(x)-\varphi^2u(x)\,exp\left[\frac{\gamma\beta\big(1-u(x)\big)}{1+\beta\big(1-u(x)\big)}\right]=0 \tag{12}$$

$$u'(0)=0, u(1)=1 \tag{13}$$

The effectiveness factor η for a spherical pellet [20, 33-35] is defined as:

$$\eta=\frac{3}{\varphi^2}\frac{du}{dx}\bigg|_{x=1} \tag{14}$$

2.2 Mathematical Model Spherical Biocatalyst Equation

We consider an inert permeable spherical solid particle. Bacterial immobilization has been made inside the permeable particle by usual methods and this permeable particle with bacteria immobilized inside it is termed as biocatalyst. This biocatalyst is immersed in a pool of liquid containing the substrate. It is further assumed that the resistance due to film surrounding the biocatalyst is negligibly small under the operating conditions in the bioreactor. Thus, the substrate (A) diffuses inside biocatalyst and the biochemical reaction occurs therein simultaneously. The application of mass balance for the substrate (A) over a thin spherical shell inside the biocatalyst yields the following model equation along with the coupled BCs (boundary conditions) (Danish *et al.,* [48]):

$$D_e\left(\frac{d^2C_A}{dr^2}+\frac{2}{r}\frac{dC_A}{dr}\right)=\frac{r_mC_A}{K_m+C_A} \tag{15}$$

The respective boundary conditions are given as:

$C_A = C_{AS}$ at $r = R$ (catalyst surface) (16)

$\frac{dC_A}{dr} = 0$ at $r = 0$ (centre of the catalyst) (17)

While deriving this model equation it is assumed that the biochemical reaction follows Michaelis-Menten kinetics. The details of derivation of Eq. (15) to Eq. (17) may be found in an excellent text book by Fouler [49]. Introducing the following dimensionless variables [3, 38-41].

$$x=\frac{r}{R}, u=\frac{C_A}{C_{AS}}, \beta=\frac{C_{AS}}{K_m}, \varphi^2=\frac{-r_{AS}R^2}{D_eC_{AS}}=\frac{R^2r_m}{D_eK_m(1+\beta)}, \tag{18}$$

Where φ denotes the Thiele modulus and φ^2 signifies the ratio of the intrinsic chemical reaction rate in the absence of mass transfer limitation to the rate of diffusion through the catalyst, i.e

$$\varphi^2 = \frac{reaction\,rate\,at\,the\,catalyst\,surface}{diffusion\,rate\,through\,the\,catalyst\,pores} \tag{19}$$

Saadatmandi [36] was demonstrated that the spherical biocatalyst equation was organized by the Lane- Emden Boundary Value Problem for the normalized concentration $u(x)$ as (Randhir Singh [37]). Eq. (15) to Eq. (17) becomes:

$$u"(x) + \frac{2}{x}u'(x) - \varphi^2 \frac{(1+\beta)u(x)}{1+\beta u(x)} = 0, \tag{20}$$

$u'(0) = 0$, (Center of the catalyst) (21)

$u(1) = 1$, (Catalyst surface) (22)

which models the dimensionless concentration of chemical species in the spherical biocatalyst.

In Eq. (20), β defines the dimensionless heat of reaction, and φ notes the Thiele modulus as estimated at the surface of the spherical catalytic pellet in Ref. [3, 38-41].

The effectiveness factor η for a spherical pellet in Ref. [20, 33-36, 38-41] is defined as:

$$\eta = \frac{3}{\varphi^2}\frac{du}{dx}\bigg|_{x=1} \tag{23}$$

As mentioned above, we use the new approximate analytical method called Ananthaswamy-Sivasankari method (ASM) is utilized to solve the two models of Eqs. (12) and (20).

3. Approximate Analytical Expression for Eqs. (12) and (20) by using ASM

A new approach called Ananthaswamy-Sivasankari method (ASM) is demonstrated for the calculation of the second order non-linear ordinary differential equations. It also used to the solve differential equations for both linear and non-linear. This technique is also easily elongated to address several additional non-linear problems in the chemical, biological sciences and physical. However, the presented new technique is applicable to boundary value issues. Additional boundary conditions for the differential equation and its derivatives can be constructed. Section (a) explains the basic concept of ASM.

(a) Basic concept of Ananthaswamy-Sivasankari Method (ASM) (Chitra *et al.,* [42]; Sivasankari *et al.,* [43])

Consider first the non-linear boundary value problem

$$q: f(y, y', y'') = 0 \tag{24}$$

Where q represents the second order non-linear differential equation such that $y = y(x, c, d, \ldots)$ in which c, d are given parameters and at $x = x_0$ can be finite with the following boundary conditions:

$$At x = x_0 y(x) = y_\alpha (or) y'(x) = y_\beta \tag{25}$$

Assume that the approximate analytical solution for the non-linear equation is an exponential function of the form:

$$y(x) = le^{ax} + me^{-ax} \tag{26}$$

The unknown coefficients l and m are obtained by solving the non-linear differential equations as follows:

$$y(x) = le^{a\alpha} + me^{-a\alpha} = y_\alpha \tag{27}$$
$$y'(x) = le^{a\beta} - me^{-a\beta} = y_\beta \tag{28}$$

Eqs. (27) and (28) may be used to get the unknown parameters l and m.

The following non-linear differential equations are obtained by substituting an Eq. (26) into the Eq. (24)

$$q: f(y(x, l, m, a, c, d), y'(x, l, m, a, c, d), y''(x, l, m, a, c, d)) = 0 \tag{29}$$

This equation is valid at $x = x_0$. Solving the Eq. (29), the unknown parameter a can be obtained in terms of given parameters c and d.

3.1 Approximate Analytical Solution of the Spherical Catalyst by using ASM

The approximate analytical solution for the dimensionless concentration of chemical species in a spherical catalyst is given:

$$u(x) = \frac{e^{mx} + e^{-mx}}{e^{m} + e^{-m}} \tag{30}$$

The effectiveness factor η for a spherical pellet

$$\eta = \frac{3}{\varphi^2} tanh(m) \tag{31}$$

Where:

$$m = 1.0723 \tag{32}$$

3.2 Approximate Analytical Solution of the Spherical Biocatalyst by using ASM

The approximate analytical solution for the dimensionless concentration of chemical species in a spherical biocatalyst is given as:

$$u(x) = \frac{e^{ax} + e^{-ax}}{e^{a} + e^{-a}} \tag{33}$$

The effectiveness factor η for a spherical pellet

$$\eta = \frac{3}{\varphi^2} tanh(a) \tag{34}$$

Where

$$a = -1.1220 \tag{35}$$

4. Numerical Simulation

The approximate analytical solutions for the dimensionless concentration of the spherical catalyst and spherical biocatalyst for the steady state have been derived using Ananthaswamy-Sivasankari method (ASM). The solutions are derived using ASM for steady-state is explained in section (a) and (b), respectively. The derived semi-analytical expression along with the numerical simulation obtained using MATLAB have been plotted in Figure 1 and Figure 2. The MATLAB programming is given in section (c) for Eq. (12) and Section (d) for Eq. (20). The specific application of numerical simulation was described in depth by Jamal Ghoulish *et al.,* [53].

a) Approximate analytical solution of the Eq. (12) for the dimensionless concentration of the spherical catalyst by using Ananthaswamy-Sivasankari method (ASM)

However, the proposed new method is applicable to boundary value issues. For the differential equation and its derivatives, additional boundary conditions can be generated. The semi-analytical solution of the dimensionless concentration of the spherical catalyst is obtained approximately by using ASM is described below.

The following is an approximate analytical solution to Eq. (12) that satisfies the boundary condition as follows:

$$u(x) = Ae^{mx} + Be^{-mx} \tag{36}$$

$$\frac{du}{dx} = mAe^{mx} - mBe^{-mx} \tag{37}$$

Utilizing the boundary conditions in Eq. (13), we obtain the value of the parameters A & B as follows:

$$A = B, B = \frac{1}{e^m + e^{-m}} \tag{38}$$

Thus, Eq. (36), becomes:

$$u(x) = \frac{e^{mx} + e^{-mx}}{e^m + e^{-m}} \tag{39}$$

Now by using Eq. (39) in Eq. (12) and on simplification, we get:

$$m^2\left(\frac{e^{mx} + e^{-mx}}{e^m + e^{-m}}\right) + \frac{2}{x}m\left(\frac{e^{mx} - e^{-mx}}{e^m + e^{-m}}\right)$$
$$-\sigma^2\left(\frac{e^{mx} + e^{-mx}}{e^m + e^{-m}}\right)exp\left[\frac{\gamma\beta\left(1 - \left(\frac{e^{mx}+e^{-mx}}{e^m+e^{-m}}\right)\right)}{1 + \beta\left(1 - \left(\frac{e^{mx}+e^{-mx}}{e^m+e^{-m}}\right)\right)}\right] = 0 \tag{40}$$

Now taking $x = 0$, Eq. (40) becomes:

$$m^2\left(\frac{2}{e^m + e^{-m}}\right) - \sigma^2\left(\frac{2}{e^m + e^{-m}}\right)exp\left[\frac{\gamma\beta\big(1 - sec\,h\,(m)\big)}{1 + \beta\big(1 - sec\,h\,(m)\big)}\right] = 0 \tag{41}$$

$$m^2 - \sigma^2\, exp\left[\frac{\gamma\beta\big(1 - sec\,h\,(m)\big)}{1 + \beta\big(1 - sec\,h\,(m)\big)}\right] = 0 \tag{42}$$

On solving Eq. (42) by substituting the values of $\gamma = 0.5, \beta = 1, \sigma = 1$, we get:

$$m^2 - e^{\left(0.5000\frac{cosh(m)-1}{2\,cosh(m)-1}\right)} = 0 \tag{43}$$

By solving the above Eq. (43), we get the value of the parameter m as follows:

$$m = 1.0723 \tag{44}$$

Similarly, we get the values of m by changing the values of γ, β and σ.

Hence an approximate analytical solution of the dimensionless substrate concentration of the spherical catalyst $u(x)$ Eq. (12) is obtained as:

$$u(x) = \frac{e^{mx} + e^{-mx}}{e^m + e^{-m}} \tag{45}$$

(b) Approximate analytical solution of the Eq. (20) for the dimensionless concentration of the spherical biocatalyst by using Ananthaswamy-Sivasankari method (ASM)

However, the proposed new method is applicable to boundary value issues. For the differential equation and its derivatives, additional boundary conditions can be generated. The semi-analytical solution of the dimensionless concentration of the spherical biocatalyst is obtained approximately by using ASM is described below.

The following is an approximate analytical solution to Eq. (20) that satisfies the boundary condition as follows:

$$u(x) = le^{ax} + me^{-ax} \tag{46}$$

$$\frac{du}{dx} = ale^{ax} - ame^{-ax} \tag{47}$$

Utilizing the boundary conditions in Eq. (21), we obtain the value of the parameters l & m as follows:

$$l = m, l = \frac{1}{e^a + e^{-a}} \tag{48}$$

Thus, Eq. (46), becomes:

$$u(x) = \frac{e^{ax} + e^{-ax}}{e^a + e^{-a}} \tag{49}$$

Now by using Eq. (49) in Eq. (20) and on simplification, we get

$$a^2\left(\frac{e^{ax}+e^{-ax}}{e^a+e^{-a}}\right)+\frac{2}{x}a\left(\frac{e^{ax}-e^{-ax}}{e^a+e^{-a}}\right)-\sigma^2\left[\frac{(1+\beta)\left(\frac{e^{ax}-e^{-ax}}{e^a+e^{-a}}\right)}{1+\beta\left(1-\left(\frac{e^{ax}+e^{-ax}}{e^a+e^{-a}}\right)\right)}\right]=0 \tag{50}$$

Now taking $x=0$, Eq. (50) becomes:

$$a^2\left(\frac{2}{e^a+e^{-a}}\right)-\sigma^2\left[\frac{(1+\beta)\left(\frac{2}{e^a+e^{-a}}\right)}{1+\beta\left(\frac{2}{e^a+e^{-a}}\right)}\right]=0 \tag{51}$$

$$a^2-\sigma^2\,exp\left[\frac{(1+\beta)}{1+\beta\,sec\,h\,(a)}\right]=0 \tag{52}$$

On solving Eq. (52) by substituting the values of $\beta=1,\sigma=1$, we get:

$$\frac{a^2\,cosh(a)+m^2-2.\,cosh(a)}{cosh(a)+1}=0 \tag{53}$$

By solving the above Eq. (53), we get the value of the parameter m as follows:

$$a=-1.1220 \tag{54}$$

Similarly, we get the values of a by changing the values of β and σ.

Hence an approximate analytical solution of the dimensionless substrate concentration of the spherical biocatalyst $u(x)$ Eq. (20) is obtained as

$$u(x)=\frac{e^{ax}+e^{-ax}}{e^a+e^{-a}} \tag{55}$$

(c) Matlab Programming for Eq. (12)

```
function pdex4
m = 0;
x = linspace(0,1);
t=linspace(0,10000);
sol= pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
u1 = sol(:,:,1);
figure
plot(x,u1(end,:))
title('u1(x,t)')
xlabel('Distance x')
ylabel('u1(x,2)')
%--------------------------------------------------------------
function [c,f,s] = pdex4pde(x,t,u,DuDx)
c = 1;
f = DuDx;
```

```
sigma=1;
beta=1;
gamma=0.5;
F =(-sigma^2*u(1))*exp((gamma*beta*(1-u(1)))/(1+beta*(1-u(1))));
s = F;
%--------------------------------------------------------------
function u0 = pdex4ic(x);
%create a initial conditions
u0 = 1;
%--------------------------------------------------------------
function[pl,ql,pr,qr]=pdex4bc(xl,ul,xr,ur,t)
%create a boundary conditions
pl = 0;
ql = 1;
pr = ur-1;
qr = 0;
```

(d) Matlab Programming for Eq. (20)

```
function pdex4
m = 0;
x = linspace(0,1);
t=linspace(0,10000);
sol= pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
u1 = sol(:,:,1);
figure
plot(x,u1(end,:))
title('u1(x,t)')
xlabel('Distance x')
ylabel('u1(x,2)')
%--------------------------------------------------------------
function [c,f,s] = pdex4pde(x,t,u,DuDx)
c = 1;
f = DuDx;
sigma=1;
beta=1;
F =((-sigma^2)*(1+beta)*u(1))/(1+beta*u(1));
s = F;
%--------------------------------------------------------------
function u0 = pdex4ic(x);
%create a initial conditions
u0 = 1;
%--------------------------------------------------------------
function[pl,ql,pr,qr]=pdex4bc(xl,ul,xr,ur,t)
%create a boundary conditions
pl = 0;
ql = 1;
```

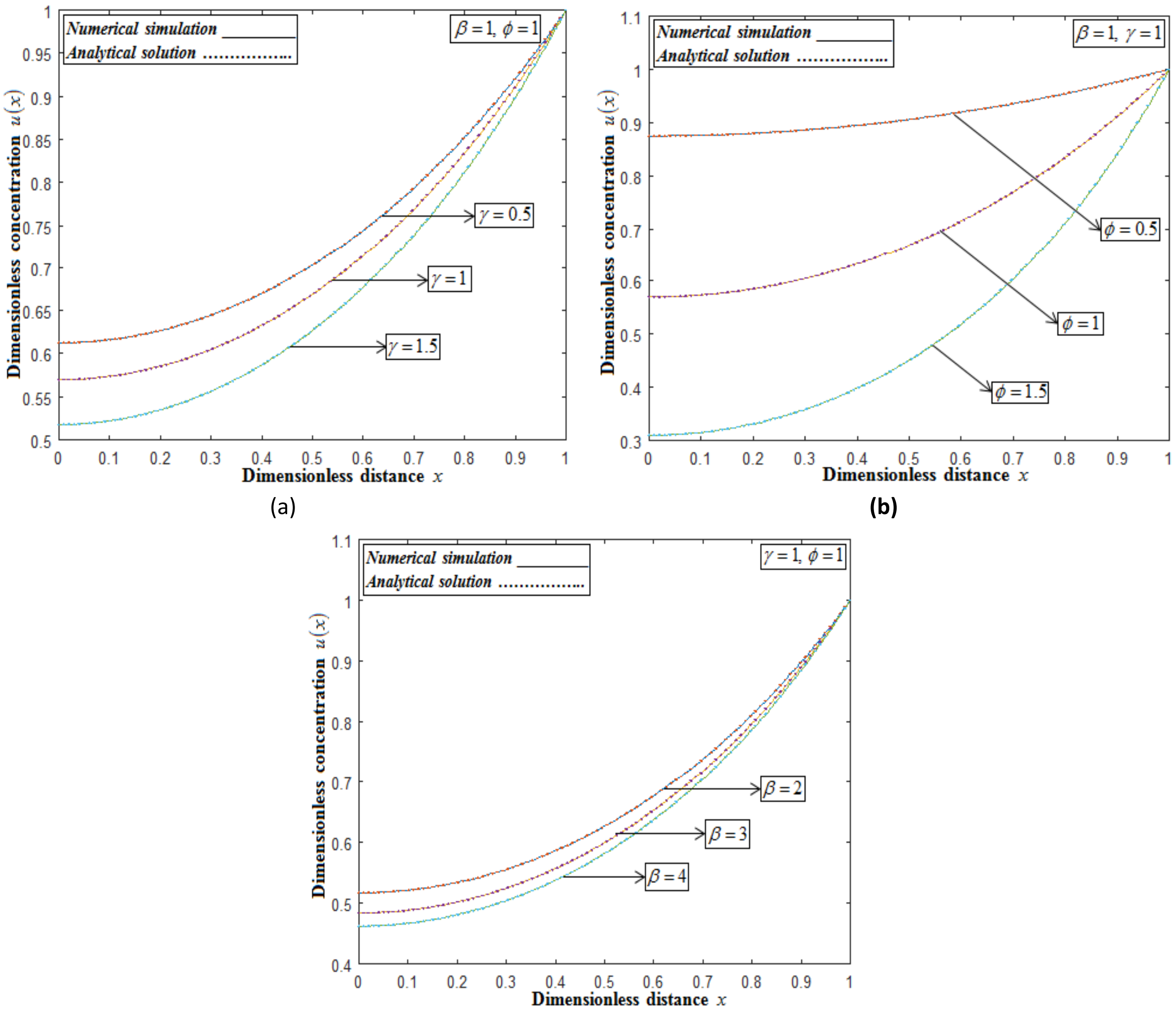
pr = ur-1;
qr = 0;

5. Results and Discussion

The semi-analytical solutions for the dimensionless concentration of the spherical catalyst and spherical biocatalyst for the steady state are given in the section (a) and section (b). Figure 1 shows that dimensionless concentration $u(x)$ versus the dimensionless distance x by utilizing Eq. (30). From the Figures 1(a) - 1(c) it is noted that, when the values of dimensionless activation energy γ, Thiele modulus ϕ and dimensionless heat of reaction β increases, the corresponding dimensionless concentration of spherical catalyst $u(x)$ decreases respectively for certain values of the remaining dimensionless parameters. Figure 2 shows that dimensionless concentration $u(x)$ versus the dimensionless distance x by using Eq. (33). From the Figures 2(a) - 2(b) it depicts that, when Thiele modulus ϕ and dimensionless heat of reaction β increases, the corresponding dimensionless concentration of spherical biocatalyst $u(x)$ decreases respectively for certain values of the further dimensionless parameters.

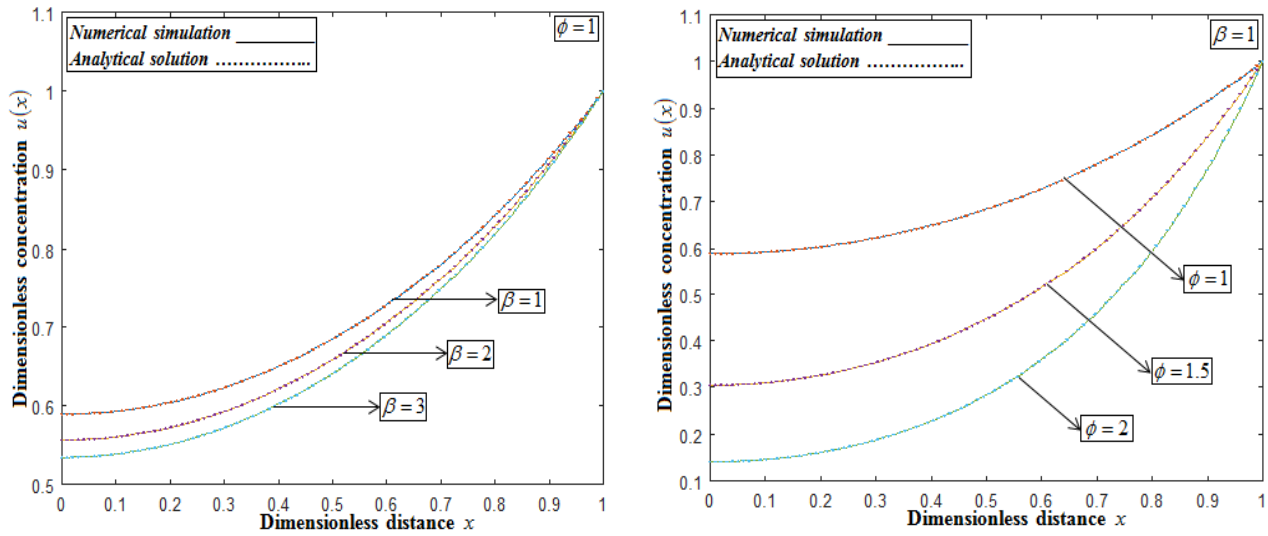
Figures 3(a)-3(b) shows that the effectiveness factor η versus Thiele modulus ϕ by using Eq. (31). From this Figures, it cleared that, when the values of dimensionless heat of reaction β and dimensionless activation energy γ increases, their corresponding effectiveness factor η also increases respectively. Figures 3(c)-3(d) shows that the effectiveness factor η versus dimensionless activation energy γ by using Eq. (12). From the Figure 3(c) it noted that, when the value of dimensionless heat of reaction β increases, their corresponding effectiveness factor η decreases. Also, from the Figure 3(d) it depicts that when the value of dimensionless heat of reaction β increases, their corresponding effectiveness factor η also increases respectively. Figures 3(e)-3(f) shows that the effectiveness factor η versus dimensionless heat of reaction β by using Eq. (31). From the Figure 3(e) it observed that, when the values of Thiele modulus ϕ increases, their corresponding effectiveness factor η decreases. Also, from the Figure 3(f) it interprets that when the values of dimensionless activation energy γ increases, their corresponding effectiveness factor η also increases respectively.

Figure 4(a) shows that the effectiveness factor η versus Thiele modulus ϕ by using Eq. (34). From the Figure 4(a) it is evident that, when the values of dimensionless heat of reaction β increases, their corresponding effectiveness factor η increases respectively. Also, Figure 4(b) shows that the effectiveness factor η versus dimensionless heat of reaction β . From the Figure 4(b) it represents that when the values of Thiele modulus ϕ increases, their corresponding effectiveness factor η decreases.



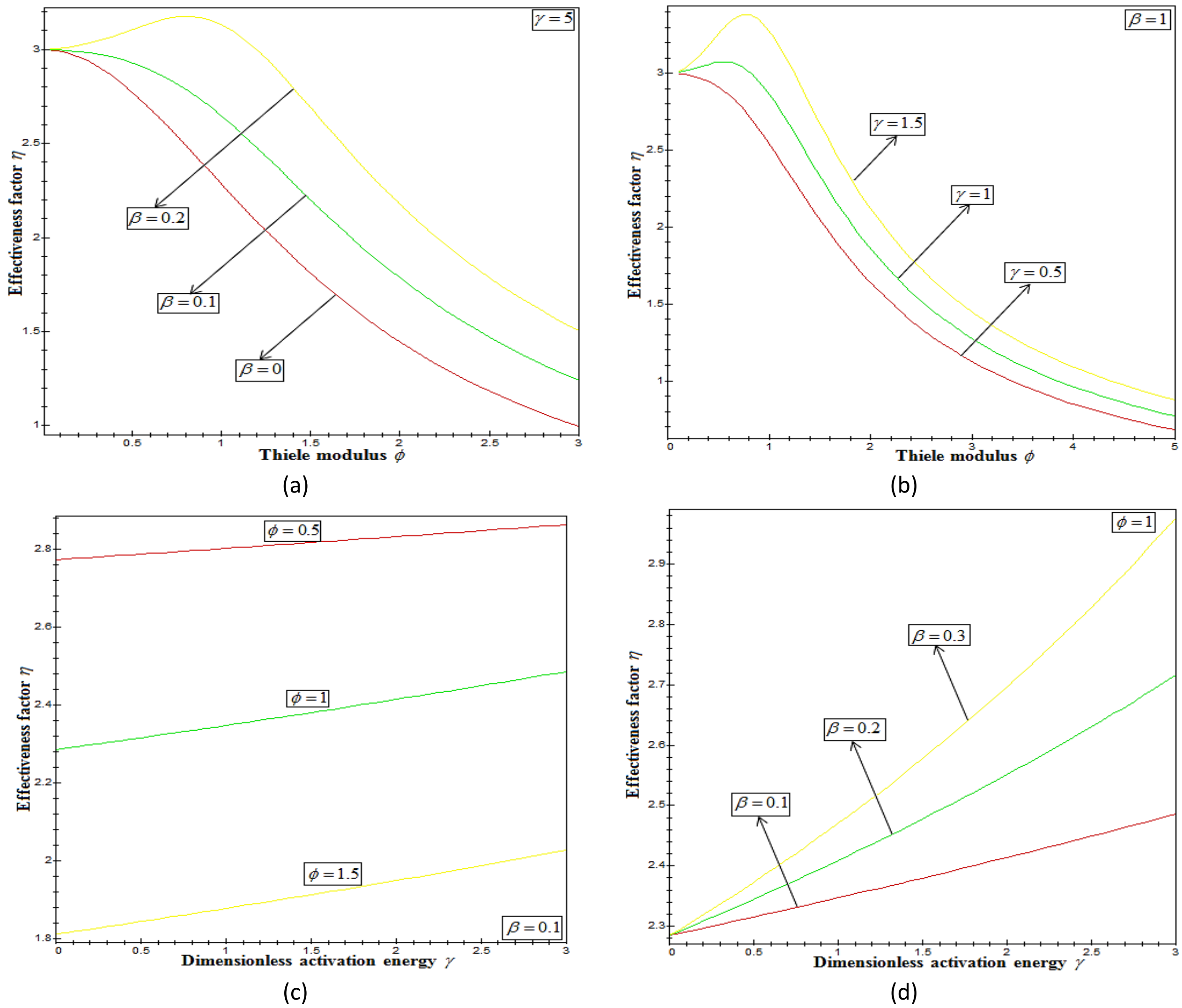
(a) (b) (c)

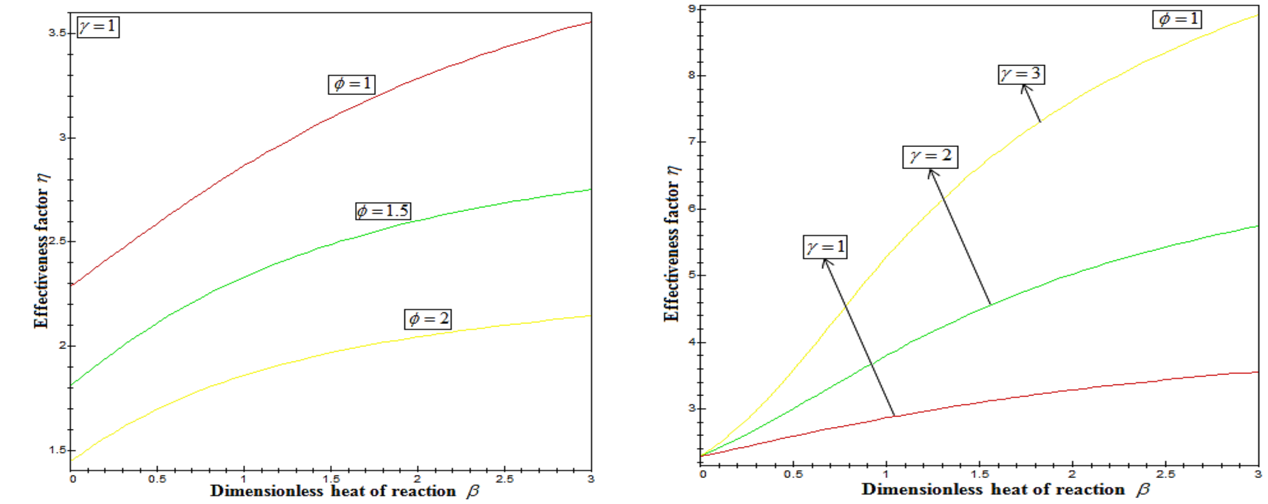
Fig. 1. (a) Dimensionless concentration $u(x)$ versus dimensionless distance x by utilizing Eq. (30) for distinct values of dimensionless activation energy γ and certain value of Thiele modulus ϕ and dimensionless heat of reaction β, (b) Dimensionless concentration $u(x)$ versus dimensionless distance x by utilizing Eq. (30) for distinct values of Thiele modulus ϕ and certain value of dimensionless activation energy γ and dimensionless heat of reaction β, (c) Dimensionless concentration versus $u(x)$ dimensionless distance x by utilizing Eq. (30) for distinct values of dimensionless heat of reaction β and certain value of dimensionless activation energy γ and Thiele modulus ϕ.



(a) (b)

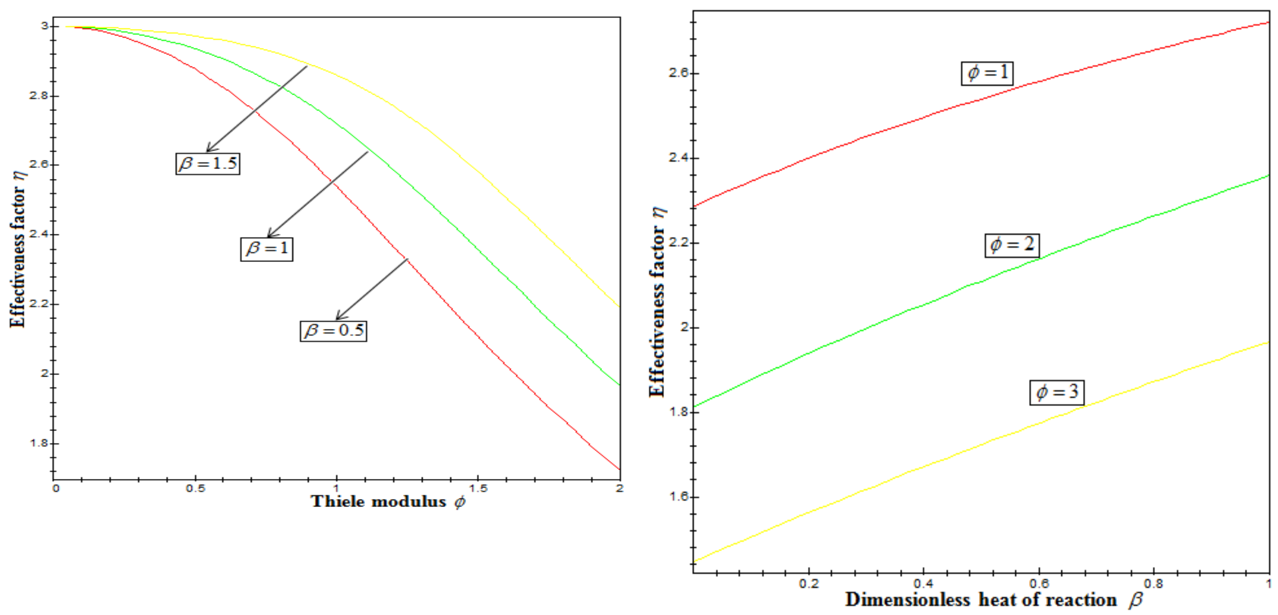
Fig. 2. (a) Dimensionless concentration $u(x)$ versus dimensionless distance x by utilizing Eq. (33) for distinct values of dimensionless heat of reaction β and certain value of Thiele modulus ϕ. (b) Dimensionless concentration $u(x)$ versus dimensionless distance x by utilizing Eq. (33) for distinct values of Thiele modulus ϕ and certain value of dimensionless heat of reaction β.





(e) (f)

Fig. 3. (a) Effectiveness factor versus Thiele modulus by utilizing Eq. (31) for distinct values of dimensionless heat of reaction and certain value of dimensionless activation energy. (b) Effectiveness factor versus Thiele modulus by utilizing Eq. (31) for distinct values of dimensionless activation energy and certain value of dimensionless heat of reaction. (c) Effectiveness factor versus dimensionless activation energy by utilizing Eq. (31) for distinct values of Thiele modulus and certain value of dimensionless heat of reaction. (d) Effectiveness factor versus dimensionless activation energy by utilizing Eq. (31) for distinct values of dimensionless heat of reaction and certain value of Thiele modulus. (e) Effectiveness factor versus dimensionless heat of reaction by utilizing Eq. (31) for distinct values of Thiele modulus and certain value of dimensionless activation energy. **(f)** Effectiveness factor versus dimensionless heat of reaction by utilizing Eq. (31) for distinct values of dimensionless activation energy and certain value of Thiele modulus.



(a) (b)

Fig. 4. (a) Effectiveness factor η versus Thiele modulus ϕ by utilizing Eq. (34) for distinct values of dimensionless heat of reaction β, (b) Effectiveness factor η versus dimensionless heat of reaction β by utilizing Eq. (34) for distinct values of Thiele modulus ϕ.

6. Conclusions

The two Lane-Emden type Boundary Value Problems that describes the diffusion of reactants in an idealized spherical porous catalytic pellet and in spherical porous biocatalyst pellet was examined. The new analytical method called Ananthaswamy- Sivasankari method (ASM) employed for both models, which greatly enhances the computational efficiency while overcoming the difficulty of the singular behaviour at the spherical origin $x=0$ of the pellets. For all parameter values, the results are in good agreement with the numerical simulation.

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