



## Casson Fluid Convective Flow in an Accelerated Microchannel with Thermal Radiation using the Caputo Fractional Derivative

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### ARTICLE INFO

#### Article history:

Received 14 March 2022

Received in revised form 12 July 2022

Accepted 27 July 2022

Available online 31 August 2022

#### Keywords:

Caputo fractional derivative; thermal radiation; Laplace transform

### ABSTRACT

The effect of the Caputo fractional derivative in unsteady boundary layer Casson fluid flow in an accelerated microchannel is investigated. In the presence of thermal radiation, the partial differential equations that governed the problem are studied. Using appropriate dimensionless variables, fractional partial differential equations are translated into dimensionless governing equations. The equations are then transformed into linear ordinary differential equations and solved analytically using the Laplace transform technique. These modified equations are then solved using the proper method, and the result is obtained in the form of velocity and temperature profiles using the Zakian's explicit formula approach. The influence of essential physical parameters on velocity and temperature profiles is investigated using graphical diagrams created with Mathcad software. It is found that the velocity and temperature profile increase as fractional parameter, and thermal radiation parameter increase. As Prandtl number increase, both profiles are decreasing. This result is crucial for understanding the fractional system of Casson fluid in microchannel.

## 1. Introduction

Fractional calculus has existed for long time, even longer than calculus as according to Khalil *et al.*, [1] where it can be divided into four phases. The first period is from 165 to 1975. In compliance with Delkhosh [2], the notation  $d^n y/dx^n$  has been invented by Leibnitz. Subsequently, after seeing the notation, L'Hospital raise an issue if the notation is  $1/2$ . Hence, Leibnitz did reply that this appears to be a paradox from which a beneficial conclusion can be reached one day, as reported by Katugampola [3]. The use of fractional derivatives has offered more general and accurate models of a system than classical models. Ray *et al.*, [4] have emphasized many studies on fractional derivatives have been produced to clarify their importance and diverse applications including De Oliveira and Vaz [5], Yang [6] and Mu'lla [7]. Caputo was the first to use the Laplace transform with the typical initial condition and create a fractional operator from the convolution product [8]. The difficulty of

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<https://doi.org/10.37934/cfdl.14.8.1219>

the unusual initial condition with no physical meaning and difficult to compute was solved with this first step since its kernel is singular. Many prior studies have looked into the Caputo fractional derivative operator as conducted by Reyaz *et al.*, [9] Saqib *et al.*, [10], Ramzan *et al.*, [11], Sheikh *et al.*, [12] and Shahrim *et al.*, [13].

Non-Newtonian fluid is a type of fluid that does not obey Newton's law. Casson fluid is the most popular fluid among all type of non-Newtonian fluid. It is said to be a shear thinning liquid, exhibiting infinite viscosity at zero shear stress. Moreover, when the yield stress imposed to the fluid is greater than the shear stress, the fluid behaved as a solid. Whereas, when the shear stress is greater than the applied yield stress, the fluid starts to flow. Toothpaste, blood, and jelly are examples of Casson fluid. Casson fluid have a significant application in broad area such as pharmaceutical, biomechanics, and cosmetics as reported by previous researchers. Khalid *et al.*, [14]. Saqib *et al.*, [15], Mahantesh *et al.*, [16], and Gudekote and Choudhari [17] have conduct some research that has been done investigating on Casson fluid.

Many academics are currently interested in the use of fluid problems in microdevices because of their importance in real-world challenges. Microfluidics is defined as the flow of fluids and gases in single or multiple phases through microdevices created using Micro Electromechanical Systems (MEMS) technology in an engineering environment as claim by Tabeling [18]. Microducts, micropumps, and microvalves are examples of MEMS devices that include fluid flows, according to Gal-el-Hak [19]. The application of microchannel can be seen has been utilised in the automotive, robotics, and telecommunications industries. Some microchannel investigations are also described in research carried out by Cao *et al.*, [20], Phu *et al.*, [21], and Joonabi and Kotnukar [22].

The convection flow of Casson fluid through an accelerated microchannel with Caputo fractional derivative has not been addressed yet, according to literature surveys. Thermal radiation is also being considered in order to increase the understanding of Casson fluid convective flow. As a result, the goal of this paper is to use the Laplace transform method to establish the exact solutions of convective flow of fractional Casson fluid through an accelerated microchannel. The effect of numerous parameters on fluid flow is analysed and discussed, including the Casson fluid parameter, fractional parameters, and Prandtl number.

## 2. Problem Definition

An unsteady free convection flow of an unidirectional and incompressible Casson fluid through an accelerated microchannel is studied. Both the fluid and the microchannel are initially at rest with a uniform temperature  $T_o$  (ambient temperature). The microchannel begins to accelerate in its plane at time  $t > 0$ , along with a velocity  $At$ , where the constant  $A$  is the plate's acceleration. The plate temperature is also raised to  $T_w$  (wall temperature), and then kept constant. The momentum and energy equations take the following forms:

$$\frac{\partial u(y,t)}{\partial t} = \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u(y,t)}{\partial y^2} + g\beta_r (T - T_o) \quad (1)$$

$$\rho c_p \frac{\partial T(y,t)}{\partial t} = k \frac{\partial^2 T(y,t)}{\partial y^2} - \frac{\partial q_r}{\partial y} \quad (2)$$

subject to suitable initial and boundary conditions.

$$\begin{aligned} u(y,0) = 0, \quad u(0,t) = 0, \quad u(d,t) = At, \\ T(y,0) = T_o, \quad T(0,t) = T_o, \quad T(d,t) = T_w, \end{aligned} \tag{3}$$

where  $u(y,t)$ ,  $T(y,t)$ ,  $\nu$ ,  $\beta$ ,  $g$ ,  $\beta_T$ ,  $\rho$ ,  $c_p$ ,  $k$ ,  $q_r$ , are the velocity of fluid, temperature of fluid, kinematic viscosity, parameter of Casson fluid, gravitational acceleration, thermal expansion coefficient, density of fluid, specific heat at constant pressure, thermal conductivity and thermal radiation parameter, respectively. Thermal radiation is optically thick fluid in one space coordinate with  $y$  in this research. It is written in the Rosseland approximation [23]:

$$q_r = -\frac{4\sigma}{3k_1} \frac{\partial T^4}{\partial y} \tag{4}$$

where  $\sigma$  is the Stefan-Boltzmann constant and  $k_1$  is the mean absorption coefficient.  $T^4$  is linearized about  $T_o$  and the other higher-order terms are ditched. Hence,  $T^4$  can be reduced to

$$T^4 = 4TT_o^3 - 3T_o^4 \tag{5}$$

Therefore, the energy equation becomes

$$\rho c_p \frac{\partial T(y,t)}{\partial t} = k \left( 1 + \frac{16\sigma T_o^3}{3kk_1} \right) \frac{\partial^2 T(y,t)}{\partial y^2} \tag{6}$$

Introducing the dimensionless variables as used by Khan *et al.*, [13],

$$v = \frac{u}{U}, \quad \tau = \frac{t\nu}{d^2}, \quad \xi = \frac{y}{d}, \quad \theta = \frac{T - T_o}{T_w - T_o}, \tag{7}$$

By replacing variables in Eq. (7) in Eqs. (1), (3) and (6), the governing equations as well as the boundary and initial conditions, reduce to non-dimensional equations. Then, by using Caputo fractional, where the Caputo derivative operator is

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \left( -\frac{1}{(t-\tau)^\alpha} \right) \frac{\partial f(\tau)}{\partial \tau} \partial \tau; 0 < \alpha < 1 \\ \frac{\partial f(t)}{\partial \tau}; \alpha = 1 \end{cases} \tag{8}$$

The fractional dimensionless momentum and energy equations can be written to

$$D_\tau^\alpha v = \frac{1}{\beta_o} \frac{\partial^2 v}{\partial \xi^2} + Gr\theta \tag{9}$$

$$Pr_{eff} D_\tau^\alpha \theta = \frac{\partial^2 \theta}{\partial \xi^2} \tag{10}$$

where the fractional parameter is denoted by  $\alpha$ . The equations are subjected to

$$\begin{aligned} v(\xi, 0) = 0, \quad v(0, \tau) = 0, \quad v(1, \tau) = \gamma\tau, \\ \theta(\xi, 0) = 0, \quad \theta(0, \tau) = 0, \quad \theta(1, \tau) = 1, \end{aligned} \tag{11}$$

Here,  $\beta_o$ ,  $Gr$ ,  $R$ , and  $Pr_{eff}$  are the parameter of dimensionless Casson fluid, thermal Grashof number, parameter of radiation and effective Prandtl number, respectively. For the simplicity,  $\gamma$  is set at  $\gamma = 1$ .

### 3. Solution of the Problem

The dimensionless fractional Eqs. (9) and (10) are solved using Laplace transform method and are transform into linear ordinary differential equations. Hence, it yields

$$\frac{\partial^2 \bar{\theta}}{\partial \xi^2} - Pr_{eff} q^\alpha \bar{\theta} = 0 \tag{12}$$

$$\frac{\partial^2 \bar{v}}{\partial \xi^2} - \beta_o q^\alpha \bar{v} = -\beta_o Gr \frac{1}{q \left[ e^{\sqrt{Pr_{eff} q^\alpha}} - e^{-\sqrt{Pr_{eff} q^\alpha}} \right]} \left[ e^{\xi \sqrt{Pr_{eff} q^\alpha}} - e^{-\xi \sqrt{Pr_{eff} q^\alpha}} \right] \tag{13}$$

Whereas the boundary conditions after applying the Laplace transform method are as follows:

$$\theta(0, q) = 0, \quad \theta(1, q) = \frac{1}{q}, \quad \bar{v}(0, q) = 0, \quad \bar{v}(1, q) = \frac{1}{q^2}, \tag{14}$$

Solving Eqs. (12) and (13) with boundary conditions Eq. (14) using the appropriate approach, obtain

$$\bar{\theta} = \frac{1}{q \left[ e^{\sqrt{Pr_{eff} q^\alpha}} - e^{-\sqrt{Pr_{eff} q^\alpha}} \right]} \left[ e^{\xi \sqrt{Pr_{eff} q^\alpha}} - e^{-\xi \sqrt{Pr_{eff} q^\alpha}} \right] \tag{15}$$

$$\bar{v} = \frac{\left( e^{-\xi \sqrt{\beta_o q^\alpha}} - e^{\xi \sqrt{\beta_o q^\alpha}} \right)}{\left( e^{-\sqrt{\beta_o q^\alpha}} - e^{\sqrt{\beta_o q^\alpha}} \right)} \left( \frac{Gr}{q \left( Pr_{eff} q^\alpha - \beta_o q^\alpha \right)} + \frac{1}{q^2} \right) - \frac{Gr_o \left( e^{\xi \sqrt{Pr_{eff} q^\alpha}} - e^{-\xi \sqrt{Pr_{eff} q^\alpha}} \right)}{q \left( e^{\sqrt{Pr_{eff} q^\alpha}} - e^{-\sqrt{Pr_{eff} q^\alpha}} \right) \left( Pr_{eff} q^\alpha - \beta_o q^\alpha \right)} \tag{16}$$

### 4. Inverse Laplace Transform

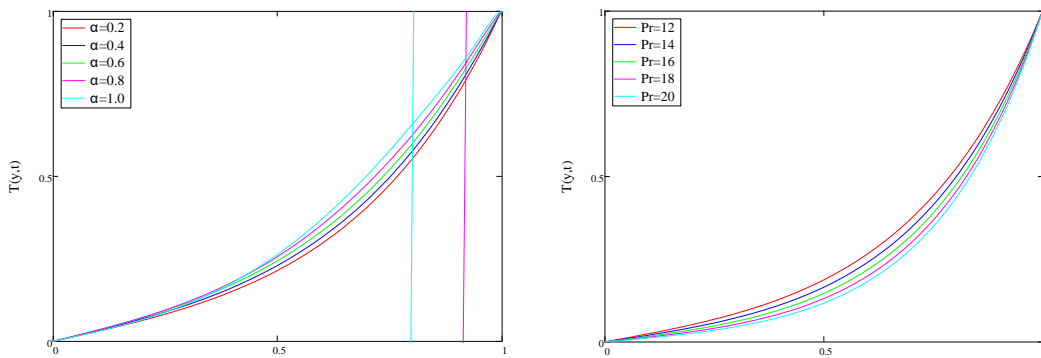
The inverse Laplace transform is computed numerically using Zakian's explicit formula approach since the obtain functions are explicit functions. By Zakian and Littlewood [24], the algorithm-based technique of Zakian is outlined by

$$f(t) = \frac{t}{2} \sum_{j=1}^N \operatorname{Re} \left\{ K_j F \left( \frac{\alpha_j}{t} \right) \right\}, \quad (17)$$

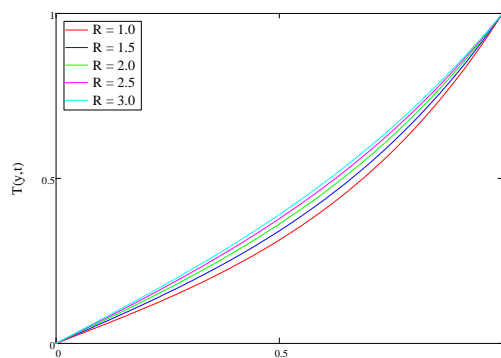
where  $K_j$  and  $\alpha_j$  are constants that can be either real or complex conjugate pairs. Referring to Halsted and Brown [25] for the lists values for  $K_j$  and  $\alpha_j$ .

### 5. Graphical Results

The effects of  $\alpha$  on the fluid temperature are depicted in Figure 1(a). As  $\alpha$  increases, the fluid temperature increases, and maximum curves obtained at  $\alpha=1$ . In comparison to the fractional Casson fluid model, the properties of the classical have a greater temperature. This occurred since a higher value of  $\alpha$  will aids in the strengthening of buoyant forces, which raises the thermal boundary layer, which enhances the temperature profile. The temperature profiles with varied values of  $Pr$  and  $R$  are depicted in Figures 1(b) and 1(c), respectively. As shown, an increase in  $Pr$  has resulted in a drop in the temperature profile. Increases in  $R$ , on the other hand, have an opposing influence on temperature profiles, which show a rising trend.



(a) Variation of temperature with different  $\alpha$       (b) Variation of temperature with different  $Pr$



(c) Variation of temperature with different  $R$

**Fig. 1.** Behaviour of  $T$  for different values of pertinent parameters

In Figure 2(a), as  $\alpha$  rises, so does the velocity distribution. The classical velocity distribution is at its maximum velocity distribution when the thickness of the momentum boundary layer is smaller than the thickness of the thermal boundary layer at  $\alpha=1$ . It is seen in Figure 2(b) that the fluid velocity has decreased due to the incremental values of  $Pr$ . Because it quantifies the ratio of viscous and thermal forces,  $Pr$  has a significant impact on fluid velocity. While Figure 2(c) demonstrate that

when  $\beta_o$  gets larger, the fluid's velocity tends to decrease. Due to the physical effect of  $\beta_o$ , in which a greater  $\beta_o$  value increases viscous forces while decreasing thermal forces. Then, the velocity of fluid will drop. The velocity profiles rise as  $Gr$  increases, as shown in Figure 2(d). The buoyant force is influenced positively by the value of  $Gr$ . It has a considerable impact on fluid velocity. The velocity of fluid increases as  $R$  increases in Figure 2(e). The larger the convection effect, which results in an increase in the velocity profile, the higher the value of  $R$ .

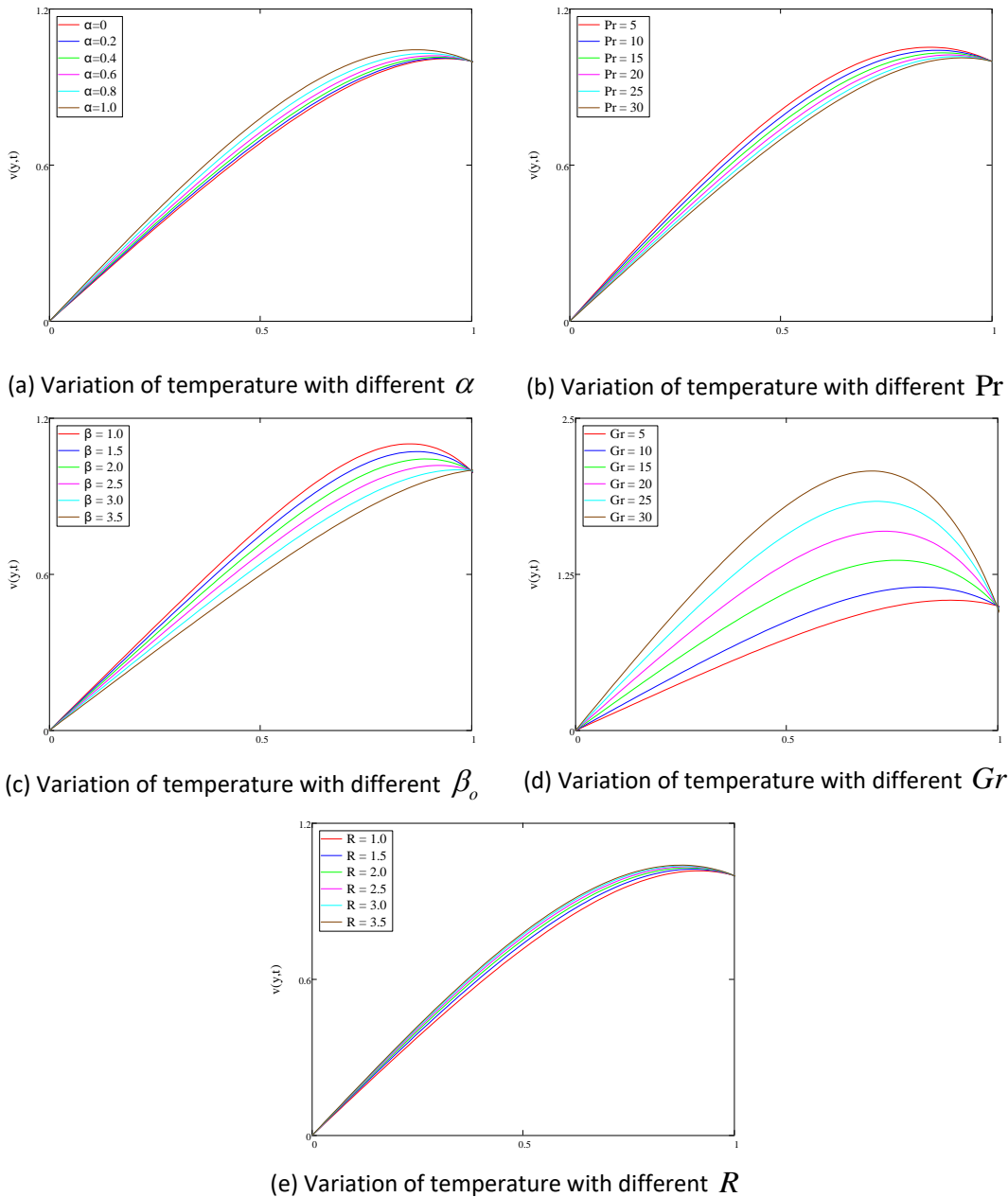


Fig. 2. Behaviour of  $v$  for different values of pertinent parameters

## 6. Conclusion

The application of Caputo fractional derivative to the convective flow of Casson fluid in an accelerated microchannel with thermal radiation is investigated in this article. For velocity and temperature profiles, the exact solution has been derived. The results from this research are:

- i. When the values of  $\alpha$ , and  $R$  increase, the velocity and temperature profile increase.
- ii. As the  $Pr$  value increases, the viscosity force increases, lowering the temperature and velocity profiles.
- iii. The velocity profile has shown a declining behaviour as  $\beta_0$  has been increased whereas the velocity profile increases as  $Gr$  increases.

### Acknowledgement

The authors would like to acknowledge the Ministry of Higher Education (MOHE) for the financial support through vote numbers, FRGS/1/2019/STG06/UNIKL/03/1.

### References

- [1] Khalil, Roshdi, Mohammed Al Horani, Abdelrahman Yousef, and Mohammad Sababheh. "A new definition of fractional derivative." *Journal of computational and applied mathematics* 264 (2014): 65-70. <https://doi.org/10.1016/j.cam.2014.01.002>
- [2] Delkhosh, Mehdi. "Introduction of derivatives and integrals of fractional order and its applications." *Appl. Math. Phys* 1, no. 4 (2013): 103-119. <https://doi.org/10.12691/amp-1-4-3>
- [3] Katugampola, U. N. (2011). A New Approach to Generalized Fractional Derivatives. *Bulletin of Mathematical Analysis and Applications*, 6(4), 1–15.
- [4] Ray, Santanu Saha, Abdou Atangana, S. C. Noutchie, Muhammet Kurulay, Necdet Bildik, and Adem Kilicman. "Fractional calculus and its applications in applied mathematics and other sciences." *Mathematical Problems in Engineering* 2014 (2014). <https://doi.org/10.1155/2014/849395>
- [5] de Oliveira, E. Capelas, and Jayme Vaz. "Tunneling in fractional quantum mechanics." *Journal of Physics A: Mathematical and Theoretical* 44, no. 18 (2011): 185303. <https://doi.org/10.1088/1751-8113/44/18/185303>
- [6] Yang, Xiao-Jun. "Advanced local fractional calculus and its applications." (2012): 1.
- [7] Mu'lla, Mariam Almahdi Mohammed. "Fractional Calculus, Fractional Differential Equations and Applications." *Open Access Library Journal* 7, no. 6 (2020): 1-9. <https://doi.org/10.4236/oalib.1106244>
- [8] Baleanu, Dumitru, Arran Fernandez, and Ali Akgül. "On a fractional operator combining proportional and classical differintegrals." *Mathematics* 8, no. 3 (2020): 360. <https://doi.org/10.3390/math8030360>
- [9] Reyaz, Ridhwan, Yeou Jiann Lim, Ahmad Qushairi Mohamad, Muhammad Saqib, and Sharidan Shafie. "Caputo fractional MHD Casson fluid flow over an oscillating plate with thermal radiation." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 85, no. 2 (2021): 145-158. <https://doi.org/10.37934/arfmts.85.2.145158>
- [10] Saqib, Muhammad, Hanifa Hanif, T. Abdeljawad, Ilyas Khan, Sharidan Shafie, and K. Soopy Nisar. "Heat transfer in mhd flow of maxwell fluid via fractional cattaneo-friedrich model: A finite difference approach." *Comput. Mater. Contin* 65, no. 3 (2020): 1959-1973. <https://doi.org/10.32604/cmc.2020.011339>
- [11] Ramzan, Muhammad, Zaib Un Nisa, Ahmad Shafique, and Mudassar Nazar. "Slip and Thermo Diffusion Effects on the Flow Over an Inclined Plate." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 94, no. 2 (2022): 13-28. <https://doi.org/10.37934/arfmts.94.2.1328>
- [12] Sheikh, Nadeem Ahmad, Dennis Ling Chuan Ching, Hamzah Sakidin, and Ilyas Khan. "Fractional Model for the Flow of Brinkman-Type Fluid with Mass Transfer." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 93, no. 2 (2022): 76-85. <https://doi.org/10.37934/arfmts.93.2.7685>
- [13] Shahrim, Muhammad Nazirul, Ahmad Qushairi Mohamad, Lim Yeou Jiann, Muhamad Najib Zakaria, Sharidan Shafie, Zulkhibri Ismail, and Abdul Rahman Mohd Kasim. "Exact solution of fractional convective Casson fluid through an accelerated plate." *CFD Letters* 13, no. 6 (2021): 15-25. <https://doi.org/10.37934/cfdl.13.6.1525>.
- [14] Gudekote, Manjunatha, and Rajashekhar Choudhari. "Slip effects on peristaltic transport of Casson fluid in an inclined elastic tube with porous walls." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 43, no. 1 (2018): 67-80.
- [15] Khalid, Asma, Ilyas Khan, Arshad Khan, and Sharidan Shafie. "Unsteady MHD free convection flow of Casson fluid past over an oscillating vertical plate embedded in a porous medium." *Engineering Science and Technology, an International Journal* 18, no. 3 (2015): 309-317. <https://doi.org/10.1016/j.jestch.2014.12.006>
- [16] Saqib, Muhammad, Farhad Ali, Ilyas Khan, and Nadeem Ahmad Sheikh. "Heat and mass transfer phenomena in the flow of Casson fluid over an infinite oscillating plate in the presence of first-order chemical reaction and slip

- effect." *Neural Computing and Applications* 30, no. 7 (2018): 2159-2172. <https://doi.org/10.1007/s00521-016-2810-x>
- [17] Mahanthesh, B., T. Brizlyn, SabirAli Shehzad, and B. J. Gireesha. "Nonlinear thermo-solutal convective flow of Casson fluid over an oscillating plate due to non-coaxial rotation with quadratic density fluctuation: Exact solutions." *Multidiscipline Modeling in Materials and Structures* 15, no. 4 (2019): 818-842. <https://doi.org/10.1108/MMMS-06-2018-0124>
- [18] Tabeling, P. "Some basic problems of microfluidics." (2001).
- [19] Gad-el-Hak, Mohamed. "The fluid mechanics of microdevices—the Freeman scholar lecture." (1999): 5-33. <https://doi.org/10.1115/1.2822013>
- [20] Cao, Limei, Peipei Zhang, Botong Li, Jing Zhu, and Xinhui Si. "Numerical study of rotating electro-osmotic flow of double layers with a layer of fractional second-order fluid in a microchannel." *Applied Mathematics Letters* 111 (2021): 106633. <https://doi.org/10.1016/j.aml.2020.106633>
- [21] Phu, Nguyen Minh, Pham Ba Thao, and Duong Cong Truyen. "Heat and fluid flow characteristics of nanofluid in a channel baffled opposite to the heated wall." *CFD Letters* 13, no. 1 (2021): 33-44. <https://doi.org/10.37934/cfdl.13.1.3344>
- [22] Beleri, Joonabi, and Asha S. Kotnurkar. "Peristaltic Transport of Ellis Fluid under the Influence of Viscous Dissipation Through a Non-Uniform Channel by Multi-Step Differential Transformation Method." *Journal of Advanced Research in Numerical Heat Transfer* 9, no. 1 (2022): 1-18.
- [23] Darzi, M., M. Vatani, S. E. Ghasemi, and D. D. Ganji. "Effect of thermal radiation on velocity and temperature fields of a thin liquid film over a stretching sheet in a porous medium." *The European Physical Journal Plus* 130, no. 5 (2015): 1-11. <https://doi.org/10.1140/epjp/i2015-15100-y>
- [24] Zakian, V., and R. K. Littlewood. "Numerical inversion of Laplace transforms by weighted least-squares approximation." *The Computer Journal* 16, no. 1 (1973): 66-68. <https://doi.org/10.1093/comjnl/16.1.66>
- [25] Halsted, D. J., and D. E. Brown. "Zakian's technique for inverting Laplace transforms." *The Chemical Engineering Journal* 3 (1972): 312-313. [https://doi.org/10.1016/0300-9467\(72\)85037-8](https://doi.org/10.1016/0300-9467(72)85037-8)