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Relationship between Mesh Number and Singular Perturbation Parameter for the Solution of Singularly Perturbed Two-Point Boundary Value Problem

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ABSTRACT

There is a considerable discussion in computational fluid dynamics over the mesh structure problems for the numerical computation. Due to wide range of mesh schemes proposed by fluid dynamists, there is sometimes confusion over the correct scheme for certain problem. Furthermore, some schemes, if improperly used, can lead to nonphysical solution. We emphasize in this paper the importance of the mesh structure and singular perturbation parameter relationship in numerical solution of a singularly perturbed two-point boundary value problem. Based on the perturbation parameter, we particularly suggest a systematic technique in setting the mesh number. This is done by adopting mesh of Shishkin type. It becomes clear that the parameters of interest are linearly related. Since it is necessary to have a decision-making that is more structured, and reduce heuristic error in mesh of computational domain determination, such relationship serves as a guideline for the numerical solution of a singularly perturbed problem that is physically correct.

1. Introduction

1.1 Singularly Perturbed Two-Point Boundary Value Problem

Singular perturbation problems occur very frequently in computational fluid dynamics (CFD) applications including geophysical fluid dynamics as well as oceanic and atmospheric circulation. In an interested domain which is the slim region referred as boundary layer, there is rapid growth in solution in many physical situations. The numerical solution with boundary layer appearance is the reason why the treatment of singular perturbation problems is very difficult. Initially there is a slow change in the solution in some parts, followed by a sharp rise or fall, depending on the boundary

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conditions, in some other parts. Thus a multi scale character is an important aspect of the solution of a singularly perturbed boundary value problem [1].

General problem is defined in differential form as

$$Lu := -\epsilon\varphi'' + b(x)\varphi' + c(x)\varphi = f(x), \text{ for } x \in (0,1), \quad (1)$$

with the boundary conditions

$$\varphi(0) = 0, \varphi(1) = 1, \quad (2)$$

and the assumptions

$\epsilon > 0$,
$b(x) > 0$ for all $x \in [0,1]$,
$c(x) \geq 0 \in [0,1]$,

where $-\epsilon\varphi''$is diffusive, $b(x)\varphi'$ is convective, $c(x)\varphi$ is reactive and $f(x)$ is source/sink terms. The functions *b*(*x*), *c*(*x*), and *f*(*x*) are sufficiently smooth. Applying the transform of variable $x \to (1-x)$, we still have a problem of form in Eq. (1) when $b(x) < 0$ in $[0,1]$. Note that only when $b(x) \neq 0$ for all $x \in [0,1]$ does Eq. (1) become important with regard to convection. When $\epsilon \ll \|b\|_{L^\infty(\Omega)}$, the parameter ϵ is defined as singular perturbation parameter, and the problem is said to be singularly perturbed. In the case where there are no convection, reaction and source, Eq. (1) represents the pure diffusion process where the solution is linear in space, which is not our interest here.

The consistency and stability of the finite difference technique are compromised when ϵ is small, due to the influence of a boundary layer which appears generally in the solution of Eq. (1) and Eq. (2) at *x* = 1. The consistency of the technique could be improved provided that one chooses the boundary values in such a way that no boundary layer exists. This, however, does not guarantee the stability of the technique [2].

In this paper, we consider *c*(*x*) = *f*(*x*) = 0 where Eq. (1) reduces to

$$Lu := -\epsilon\varphi'' + b(x)\varphi' = 0, \text{ for } x \in (0,1), \quad (3)$$

involving an unknown parameter φ. In the case of convection-diffusion problem, for instance, diffusion causes φ as scalar concentration to spread, while convection carries it along with the moving fluid element [3-4]. The sharp change of φ in space occurs after it initially grows slowly over a defined distance when ϵ is small, given appropriate boundary conditions. The sudden rise of φ serves two numerical purposes; to test severely the method of discretization, and the selection of compatible computational domain mesh structure.

1.2 Mesh Issues

For decades, mesh type issues and their selection have been seriously discussed in computational fluid dynamics (CFD) community due to its importance in numerically solving a governing equation. Mesh scheme effectiveness in the computation time reduction and its influence on numerical accuracy are among the major concerns [5-15]. Popular mesh concepts may involve two-mesh schemes [16-17], refinement or un-refinement of mesh [18], mesh number minimization [19], methods of multimesh [20-26], and spurious oscillation [27].

Lattice Boltzmann scheme (LBS) performs at a similar level as Lax-Wendroff scheme of the one-step second-order with respect to simulation time decrease [5,10]. The scheme is even better than that of projection method based finite difference.

In Ma *et al.,* [13]'s study, the employment of an operator interpolation scheme, Richardson extrapolation technique usage, and the application of a fourth-order compact difference scheme were involved in an extensive numerical procedure to solve convection-diffusion equation on the fine mesh with sixth-order accuracy. Other high-order-accuracy schemes corresponding to such equation were also proposed in Xu [9]'s and Ge *et al.*, [14]'s studies.

The improper decrease of mesh number may cause spurious oscillation. One of the successful methods is that of component-wise splitting [7]. The method which is an absolutely stable finite difference scheme greatly decreases mesh number of the analysis system by introducing the irregular mesh size. In addition to clear illustration of the problem of oscillation, it was proposed in Jeon *et al.*, [27]'s study that severe oscillations in the solution of convection dominated diffusion equation could be decreased by means of the upwind hybrid difference method which has been improved. It is also worth to note that Superbee, MINMOD, and SMART were shown to be effective in producing physically realistic calculation results. These second-order discretization schemes were evaluated in Guenther *et al.*, [12]'s study where 'unphysical solution' term was coined to portray a more general problem.

In this paper, the model problem in Eq. (3) which may be considered as a 'special' convection-diffusion problem is discretized by finite difference technique on a mesh of Shishkin type; a piecewise equidistant mesh is defined in such a way that it is sufficiently fine in a neighborhood of $x = 1$. There is a need to prevent φ profile from being nonphysical by determining a valid minimum mesh number. In order to achieve physically realistic solution of the flow problem with less pre-computation time, we examine the flow parameter of interest ϵ in Eq. (3) in connection with mesh number N. The work is an extension to that of mesh and low Peclet numbers' relationship that was discussed in Abdullah [28-29]'s studies. Since the solutions are at the risk of being nonphysical, the reduction of mesh number needs to be done with care.

Notwithstanding the fact that various mesh schemes have been widely studied among researchers, the contribution of singular perturbation parameter ϵ and the mesh number N in preventing spurious oscillatory solutions remains open. It is important to study such contribution in order to increase the efficiency in selecting the computational domain mesh, and in eradicating some heuristic parts in numerical solution of a singularly perturbed two-point boundary value problem. This research aims at formulating the relationship between ϵ and N to avoid nonphysical solutions.

2. Methodology

2.1 Governing Equation Discretization

We begin with the model problem in Eq. (3);

$$-\epsilon\varphi'' + b\varphi' = 0 \text{ in } (0,1)$$

Defining the boundary conditions as

$$\varphi(0) = 0$$
$$\varphi(1) = 1 \qquad (4)$$

The illustration in Figure 1 shows the φ profiles for three ranges of b/ϵ.

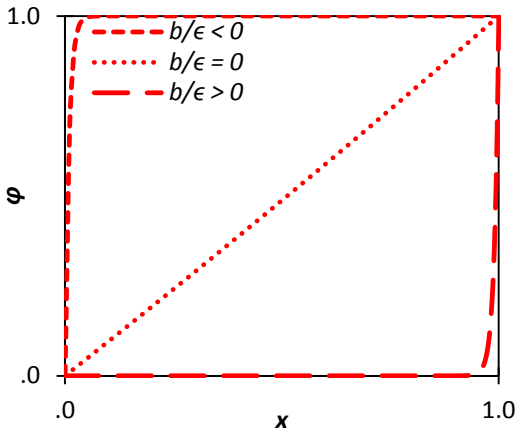

Fig. 1. Solution profiles for different ranges of b/ϵ where boundary conditions are fixed

The corresponding solution domain is covered by a mesh of Shishkin type. The numerical analysis results support the use of this mesh type. Given that the mesh number N to be an odd integer. The number of interval is thus given by $(N-1)$. Defining x to be the independent variable whose domain is discretized (i.e. $x = [0,1]$), we defined the node

$$x_{i+1} = x_i + \frac{2}{N-1} - 2r_e \frac{\ln(N-1)}{N-1},$$

for the nodes $x_0, \dots, x_{(N-1)/2}$, and

$$x_{i+1} = x_i + 2r_e \frac{\ln(N-1)}{N-1},$$

for the nodes $x_{(N-1)/2}, \dots, x_{N-1}$,

where $0 \leq i \leq (N-1)$, $i \in \mathbb{Z}$, and the parameter $r_e > 0$. Thus we have two sizes of interval Δx (i.e. Δx_A and Δx_B) depending on the nodes. The meeting point between the coarse and the very fine mesh is known as the transition point. Its location is given by

$$x = x_{(N-1)/2} = r_e \ln(N-1)$$

Clearly $\sum \Delta x_i = 1$. The mesh is illustrated in the Figure 2 below.

Fig. 2. Computational molecules in Shishkin mesh

The idea is to discretize the governing equation as represented by Eq. (3) such that the partial derivatives are approximated by nodal algebraic expression. Thus for every single node, there is an algebraic equation version of Eq. (3), where the variables to be determined are those at that and instant nodes. The system of algebraic equations is given by

$$C_P\varphi_P + \sum_m C_m\varphi_m = Q_p \tag{5}$$

where P denotes the nodes at which the algebraic equations are allocated, while m index runs over the immediate left and right nodes. The elements of the corresponding matrix C in Eq. (5) are stored as three $n \times n$ array. They are non-zeros only on the matrix's main diagonal (represented by C_{ii}) as well as the diagonals immediately below and above it (represented by C_L and C_R, respectively). Using the three-point computational molecules, Eq. (5) becomes

$$C_P\varphi_P + C_R\varphi_{i+1} + C_L\varphi_{i-1} = Q_P \tag{6}$$

Discretization of all terms in Eq. (3) is done by using central difference scheme (CDS), both for the outer derivative in the first term

$$-[(\epsilon\varphi')']_i \approx \frac{(\epsilon\varphi')_{i+\frac{1}{2}} - (\epsilon\varphi')_{i-\frac{1}{2}}}{\frac{1}{2}(x_{i-1} - x_{i+1})} \tag{7}$$

and the inner derivative

$$\left.\begin{aligned}(\epsilon\varphi')_{i+\frac{1}{2}} &\approx \epsilon\frac{\varphi_{i+1} - \varphi_i}{x_{i+1} - x_i}\\ -(\epsilon\varphi')_{i-\frac{1}{2}} &\approx \epsilon\frac{\varphi_i - \varphi_{i-1}}{x_{i-1} - x_i}\end{aligned}\right\}, \tag{8}$$

as well as the second term in Eq. (3)

$$-[(b\varphi)']_i \approx b\frac{\varphi_{i+1} - \varphi_{i-1}}{x_{i-1} - x_{i+1}} \tag{9}$$

The contributions of the first and second terms to the coefficients of the algebraic Eq. (6) are therefore;

$$C_R = C_R^{second} + C_R^{first}$$

$$= \frac{b}{x_{i+1} - x_{i-1}} - \frac{2\epsilon}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)};$$

$$C_L = C_L^{second} + C_L^{first}$$

$$= -\frac{b}{x_{i+1} - x_{i-1}} - \frac{2\epsilon}{(x_{i+1} - x_{i-1})(x_i - x_{i-1})};$$

$$C_P = C_P^{second} + C_P^{first}$$

$$= -\left(C_R^{first} + C_L^{first}\right)$$

Linearization is not required by the numerical solution of Eq. (3). This is due to the linearity of the differential equation, thus the approximate equation contains only terms which are linear. The linear

system of the algebraic Eq. (6) is solved by applying Thomas' algorithm (i.e. tridiagonal matrix algorithm). We set

$$b = 1.0, r_e = .03 \tag{10}$$

Illustration in Figure 3 shows the nonphysical solution of Eq. (3) where spurious oscillation occurs due to improper minimization of the number of mesh *N*. Note that the negative values in Figure 3 are magnified 10^{32} times for enhanced visibility; physically correct profile is not supposed to have negative values. It is therefore necessary to find a technique that is systematic rather than heuristic to use in the determination of *N*.

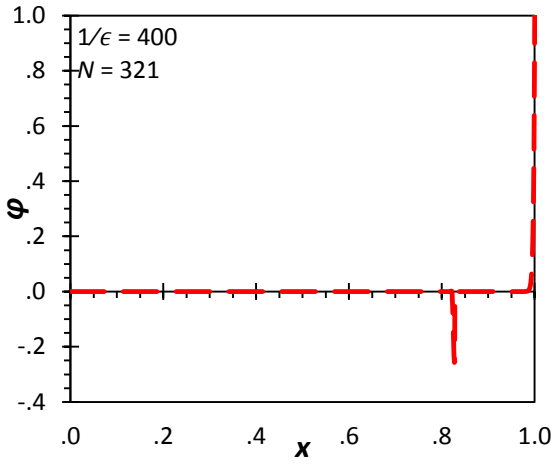

Fig. 3. The insufficient mesh number *N* leads to nonphysical behaviour of φ profile over computational domain

2.2 Sequences of $1/\epsilon$ *and the Mesh Number*

The range of $1/\epsilon$ of interests is $[50,1600]$. The relationship between $1/\epsilon$ and mesh number N is represented by a set of ordered pairs $((1/\epsilon)_i, N_i), i = 1,2, \dots, \mathrm{n}$.

We define a sequence of $1/\epsilon$ by

$$\begin{aligned} &(1/\epsilon)_i, \\ &(1/\epsilon)_{i+1} = (1/\epsilon)_i/p, \\ &(1/\epsilon)_{i+2} = (1/\epsilon)_{i+1}/p, \\ &\vdots \\ &(1/\epsilon)_n = (1/\epsilon)_{n-1}/p, \end{aligned} \tag{11}$$

where the constants $i, p \in \mathbb{Z}^+$.

Next, defining a sequence of N by

$$N_i\ ,$$

$$N_{i+1} = \frac{N_i + 1}{q} - \frac{(-1)^{\left(\frac{N_i+1}{q}+1\right)} - 1}{2},$$

$$N_{i+2} = \frac{N_{i+1}+1}{q} - \frac{(-1)^{\left(\frac{N_{i+1}+1}{q}+1\right)}-1}{2}, \quad (12)$$

:

$$N_n = \frac{N_{n-1}+1}{q} - \frac{(-1)^{\left(\frac{N_{n-1}+1}{q}+1\right)}-1}{2},$$

where the constants $i, q \in \mathbb{Z}^+$.

Let $1/\epsilon$ is in *W*, *N* is in *X*, and g be a function from its domain *W* to its target *X* that serves as a collection of ordered pairs of the form $(1/\epsilon, N)$. Such collection must satisfy the conditions below;

Condition 1 Every element $1/\epsilon$ in the domain of g (i.e. *W*) has a value $g(1/\epsilon) = N$ in the target of g (i.e. *X*) such that $(1/\epsilon, N)$ is one of the ordered pairs.

Condition 2 Each element of the domain *X* has a unique value under g such that if both $(1/\epsilon, N)$ and $(1/\epsilon, N')$ are among the ordered pairs that make up the function, then $N = N'$. Thus, the function g is a mechanism which assigns at most one element $g(1/\epsilon)$ of the target to each element $1/\epsilon$ of the domain.

We write

$$N = g(1/\epsilon),$$

signifying that the ordered pair $(1/\epsilon, N)$ is in the collection of ordered pairs which defines the function g. Thus the set $\{g(1/\epsilon)$: $1/\epsilon$ is a positive real number in *W*$\}$ of values of g is the image of g.

Let

$$i = 1, n = 6, (1/\epsilon)_1 = 1600, N_1 = 2561, \text{and}$$

$$p = q = 2, \quad (13)$$

such that the sequence in Eq. (11) and Eq. (12) become

$$1600, 800, 400, 200, 100, 50$$

and

$$2561, 1281, 641, 321, 161, 81$$

respectively.

Proposition The ordered pairs $(1/\epsilon, N)$ are generated from the sequences' elements in Eq. (11) and Eq. (12) whose independent variables and boundary values are set in Eq. (13), and satisfy Condition 1 and 2 such that;

$$\{((1/\epsilon)_1, N_1), ((1/\epsilon)_2, N_2), \ldots\ldots, ((1/\epsilon)_6, N_6)\}$$

$$= \{(1600, 2561), (800,1281), (400,641),(200,321),$$

$$(100, 161), (50, 81)\}$$

3. Results and Discussion

The setting of parameters as well as the boundary conditions for problem in Eq. (3) are given in Eq. (4), and Eq. (10) to Eq. (13). Plots in Figure 4 show the numerically calculated values of φ for $1/\epsilon$ of interests, where the profiles change exponentially with respect to *x*-direction, and the area below the curve represented by the integral $\int_0^1 \varphi(x)dx$ is inversely proportional to $1/\epsilon$. The plots show correct physical behaviors of the solutions, and prove the Proposition

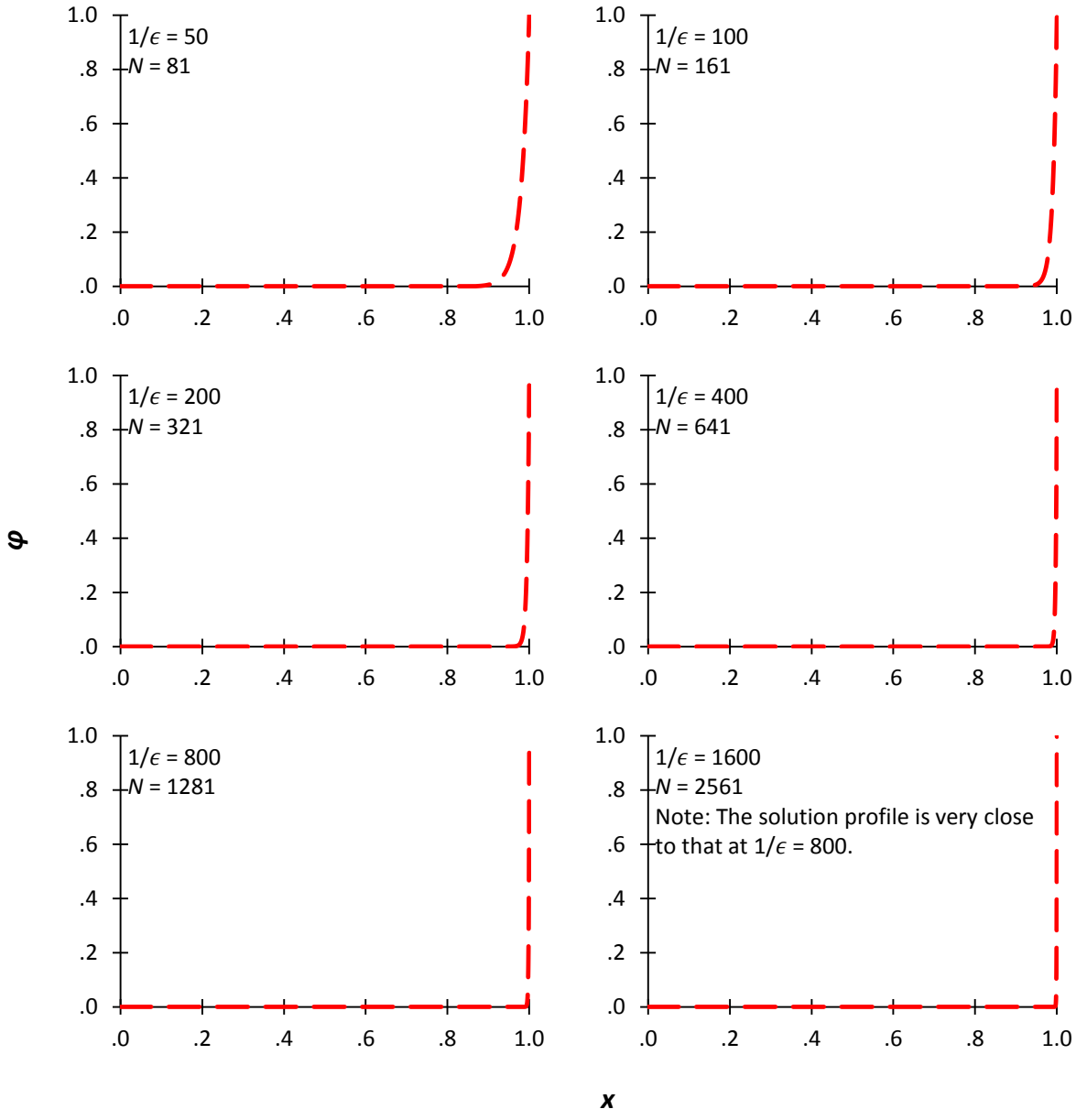

Fig. 4. The solution profile at ϵ << 1, numerically calculated

It is found in this numerical calculation of a singularly perturbed two-point boundary value problem that the minimum mesh number N_{min} is linearly proportional to $1/\epsilon$. This is given in Figure 5 where N_{min} must be sufficient for the φ profile to behave physically correctly. In other words, if $N < N_{min}$ for the corresponding $1/\epsilon$, where N in an element in Eq. (12), than the solution would oscillate as, for instance, shown previously in Figure 3. If $N \geq N_{min}$, the solution is physically correct.

Thus, for $50 \leq 1/\epsilon \leq 1600$, the minimum mesh number N_{min} for solving problem in Eq. (3), with the conditions in Eq. (4) and Eq. (10), is expressed as a linear function of $1/\epsilon$;

$$N_{min} = m/\epsilon + d, \tag{14}$$

where m and d are curve slope and a constant, respectively.

4. Conclusions

We propose a new technique to obtain an appropriate mesh based on the understanding of the influence of singular perturbation parameter ϵ on the number of mesh *N* (see Figure 5). The key aspect of this work is the formulation of the special function as a collection of ordered pairs of the form $((1/\epsilon)_i, N_i), i = 1,2, \ldots, n$ for the given singularly perturbed two-point boundary value problem, where both parameters of interest (i.e. $1/\epsilon$ and *N*) are linearly related as shown in Eq. (14).

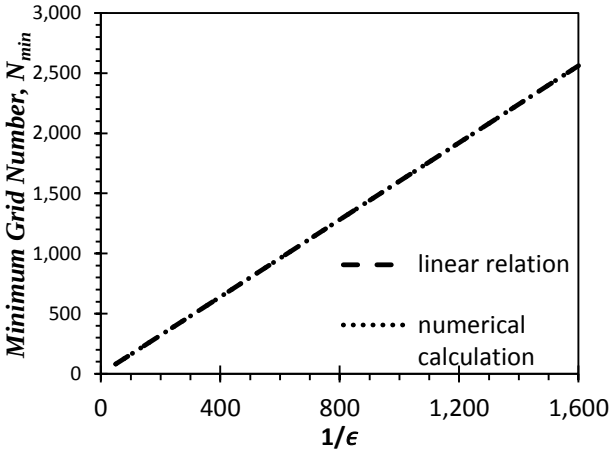

Fig. 5. Minimum mesh number N_{min} as a linear function of $1/\epsilon$

Such quantitative guideline to solve a singularly perturbed problem forms a basis for a more effective approach in the mesh selection for CFD procedures. It sheds light on the possibility of a more general framework on the selection of mesh type, the influence of the flow parameter/s and the mesh quality in finite difference numerical scheme, and of singular perturbation parameter ϵ (e.g. low, transition, and high) on types of solution error.

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