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Variational Analysis: Collapse of the Static Soliton Wave Beams in a One-Dimensional Discrete System

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ABSTRACT

A system that experiences sudden state changes at specific times is said to be discrete. The majority of systems that are studied in operations research and management science, such as transportation or communication studies, are under the application of discrete systems. This study investigates the analytical study of the static soliton for Cubic-Quintic Discrete Nonlinear Schrödinger Equation (DNLSE) in discrete system. Subsequently, static soliton, that is often used to characterize specific self-action regime in a continuous one-dimensional problem, is defined as a self-reinforcing wave packet that keeps its form and velocity while it travels in a medium. Moreover, it is well-known that the NLSE is a known integrable equation of partial differential equation. Therefore, the variational approximation method is applied to transform the partial differential equation of the main equation into ordinary differential equations, thus, to derive the equations for soliton parameters evolution during the interaction process. The method is used to qualitatively study the Discrete NLSE and characterize self-action modes. It is shown that in discrete media, both wide and narrow wave beams (relative to the grating scale) experience weakened diffraction, resulting in the "collapse" of the one-dimensional wave field when the power is greater than the critical threshold. As a result, the central fiber is able to self-channel radiation.

Keywords:
Soliton; discrete nonlinear Schrödinger equation; nonlinear equations; discrete system; partial differential equation; variational approximation method

1. Introduction

The Nonlinear Schrödinger Equation (NLSE) is a mathematical framework employed for the purpose of explaining the dynamics of specific wave phenomena observed in many physical systems, with particular emphasis on nonlinear optics and condensed matter physics [1]. It was first formulated by Benney & Newell in 1967 where this equation is widely employed in the analysis of light pulse propagation within optical fibers [2]. The equation exhibits a strong connection with wave particles, especially soliton wave, which is first introduced by Scottish scientist and engineer, John

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Scott Russel in 1834 [3]. The discrete NLSE then is an extension form of the NLSE under the discrete system where it is a framework of the nonlinear lattice dynamics model and becoming attracting researcher interest on its possession of a distinctive solution referred to as a soliton [4]. This solution propagates with a constant pattern and velocity. The equation has a wide range of practical applications, including denaturation, circuits for electricity and phase transformation occurrences within double-stranded Deoxyribonucleic Acid (DNA), biomolecular chain dynamics, optical beam propagation on nonlinear waveguides, and material formation processes, particularly Bose-Einstein Condensation (BEC) on optical lattices [5]. Kevrekidis under his study mentioned that discrete NLSE is derived from the continuum NLSE for the wave function, ψ in the presence of a periodic potential [6]. Etten mentioned in his work [7] on how a discrete system of nonlinear equations can be represented as a nonlinear discrete-time system.

A discrete system exists in which each point is precisely isolated from the others. At a given moment in time, the characteristics of each site serve to denote a particular state of the system. The values of other sites may impact the way in which variables alter on a given site [8]. In order to solve the DNLSE problem, the variational approximation method (VA) method is utilized to study the propagation of the solitons. This approach is used to obtain an approximate inference based on the solitons' parameters in existing complicated models.

The Discrete NLSE can be generalized by adding several terms into the equation. Discrete soliton in Cubic NLSE have been investigated several times and gave numerous incidents on the soliton wave beams [1,4,9]. This paper focuses on the discrete soliton solution in Cubic-Quintic NLSE in self-action mode, where this equation is derived from the inclusion of cubic and quintic nonlinear elements into the NLSE.

Our objective in this paper consists of studying the static soliton interactions in discrete NLSE and analyzing the static soliton scattering behavior in discrete cubic-quintic NLSE using a variational approximation approach. The study of finding an analytical solution to a partial differential equation has been widely investigated and has become theoretically important for several research fields other than soliton propagation. Omar *et. al.,* [10] have studied the analytical solution of unsteady MHD casson fluid with thermal radiation and chemical reactions in a porous medium.

Generally, the manuscript is arranged as follows. The Introduction section provides a literature overview, research background, and statement of the problem. Section 2 introduces the model and governing equations for the research. Section 3 describes the variational approximation method employed throughout the investigation. Section 4 presents the analytical and numerical simulations of the entire work, which explain the results and discussions of the collapse of soliton wave beams in discrete NLSE. Lastly, the manuscript concludes at Section 5 and summarizes the result of the study.

2. The Model and Governing Equation

The Discrete Nonlinear Schrödinger Equation (DNLSE) applied to a one-dimensional system serves as the basis for our main equation model while incorporating cubic and higher-order quintic nonlinearity terms into the system. This framework enables us to analyze the system's response under various conditions as well as investigate its stability and evolution dynamic of the wave beams throughout the propagation path. The following equation demonstrates the Cubic-Quintic DNLSE for a set of evenly spaced ideal waveguide in its simplest form, elucidating its formulation and significance in describing the system's behaviour:

$$i\frac{\partial\psi_n}{\partial z}+\psi_{n+1}+\psi_{n-1}+\left|\psi_n\right|^2\psi_n+\left|\psi_n\right|^4\psi_n=0 \tag{1}$$

where $\psi_n(z)$ is the complex wave function characterized at the nth site and z is the propagation direction. The evolution of the wave packets in the nth light guide across the z-axis is considered to be driven by the interplay of the third and fifth-order nonlinearities of the medium along with its interaction solely with neighbouring light guides. Similar to the case of continuous NLSE, the system in Eq. (1) maintains its Hamiltonian or the energy structure in the form of

$$H = \sum_{n=-\infty}^{\infty} \left[\psi_{n+1}\psi_n^* + \psi_{n+1}^*\psi_n + \frac{1}{2}|\psi_n|^4 + \frac{1}{3}|\psi_n|^6 \right] \tag{2}$$

while also upholding the wave-field power conservation, represented by

$$P = \sum_{n=-\infty}^{+\infty} |\psi_n|^2 \quad , \tag{3}$$

typically referred to as the norm of the system. In particular, the behaviour of the wave beams is greatly influenced by the value of P which acts as a controlling parameter for the entire system. Through deliberate manipulation of P, one can modulate the characteristics of the radiation to meet the desired outcomes for different applications effectively.

Utilizing Eq. (1) as our foundation, the investigation on the potential scattering behaviour for a wave beam with a collimated pattern introduced into a periodic arrangement of the light guides in the z-axis orientation will be conducted thoroughly, specifically by means of variational approach. This method is outlined in detail in the following section.

3. Methodology: Variational Approximation Method

The Variational Approximation (VA) Method serves as the primary tool for examining the behavior and interaction of the soliton wave beams scattering process, including their scattering dynamics within the cubic-quintic discrete NLSE. This investigation proceeds in the case of static states of the one-dimensional system Discrete NLSE. Initially, the main equation of the Eq. (1) is tackled using the VA method to derive analytical solutions for the evolution of soliton parameters, essential for characterizing the soliton scattering phenomenon.

This method stands out as a key theoretical approach for studying non-integrable equations with soliton characteristics, dating back to its initial application by Anderson (1983) [11]. Anderson first employed this method to examine soliton behavior within a significantly perturbed NLSE, particularly in nonlinear optics.

Essentially, the VA method offers approximate solutions based on certain assumptions. It is undeniable that NLSE typically considered a non-integrable equation, leading to the absence of an analytical solution. VA method then enables the simplification of the Partial Differential Equation (PDE) governing the primary equation into an Ordinary Differential Equation (ODE) [12], facilitating the approximation of coupled equations governing soliton width and center of mass position. These approximations are then interpretable through numerical simulations, crucial for analyzing the incidents phenomena, given the significance of these parameters in wave propagation.

The effectiveness of this method hinges on selecting an appropriate trial function, which is subsequently substituted into the Lagrangian density for further analysis [13]. In our case, we employed the Gaussian ansatz outlined in Eq. (7). A critical aspect here is that the evolution equations governing soliton parameters should be analytically evaluable through the averaged/effective Lagrangian. For instance, the Gaussian profile and the hyperbolic secant function with time dependent parameters, are commonly used as trial functions in articles since both functions satisfy the objective in many circumstances [14-17]. To emphasize, the utilization of these approximate solutions is of great value in acquiring a deeper understanding of the physical event elucidated by the NLSE [1].

4. Results

4.1 Variational Analysis of PDE

In the framework of nonlinear wave equations, it is beneficial to employ the variational approach to provide an approximate depiction of the evolution of the wave beams, particularly in scenarios where traditional analytical techniques fall short. Below is the Lagrangian of the system for the Cubic-Quintic DNLSE in Eq. (1) as the initial parameter for this approach,

$$L=\sum_{n=-\infty}^{\infty}\mathrm{L}_n=\sum_{n=-\infty}^{\infty}\left[\frac{i}{2}\left(\psi_n\frac{\partial\psi_n^*}{\partial z}-\psi_n^*\frac{\partial\psi_n}{\partial z}\right)-\psi_{n+1}\psi_n^*-\psi_{n+1}^*\psi_n-\frac{1}{2}|\psi_n|^4-\frac{1}{3}|\psi_n|^6\right] \tag{4}$$

where L_n denotes the Lagrangian density. By the application of the Poisson summation formula on the continuous argument function $F(x)$, i.e.,

$$\sum_{n=-\infty}^{\infty}F(n)=\int_{-\infty}^{\infty}F(x)\sum_{n=-\infty}^{\infty}\exp(2\pi inx)dx \tag{5}$$

the Lagrangian presented in Eq. (4) is restructured into a more practical form given by

$$L=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{i}{2}\left(\psi\frac{\partial\psi^*}{\partial z}-\psi^*\frac{\partial\psi}{\partial z}\right)-\psi(x+1)\psi^*-\psi^*(x+1)\psi-\frac{1}{2}|\psi_n|^4-\frac{1}{3}|\psi_n|^6\right]e^{2\pi inx}dx \tag{6}$$

which further facilitates calculation process within this continuous approximation approach, considering the discrete nature of the system. Following this, a Gaussian wave packet characterized by a parabolic phase front (Balakin *et al.,* [4]) is adopted as the trial function:

$$\psi(x,t)=\sqrt{\frac{P}{a\sqrt{\pi}}}\exp\left(-\frac{x^2}{2a^2}+i\beta x^2\right) \tag{7}$$

where $a(z)$ and $\beta(z)$ are the wave-packet parameters of the width and phase-front curvature, respectively, while P signifies the wave-field power. Then, the chosen ansatz above is substituted into Eq. (6), followed by the integration of the Lagrangian density over the continuous domain of x which then results into the effective Lagrangian as depicted below.

$$\begin{aligned}L=&\frac{Pa^2}{2}\frac{d\beta}{dz}\sum_{n=-\infty}^{\infty}(1-2\pi^2n^2a^2)e^{-\pi^2n^2a^2}-2Pe^{-\frac{1}{4a^2}-\beta^2a^2}\sum_{n=-\infty}^{\infty}\cos(\pi n)e^{-\pi^2a^2n^2-2\pi n\beta a^2}\\&-\frac{P^2}{2\sqrt{2\pi}a}\sum_{n=-\infty}^{\infty}e^{-\pi^2a^2n^2/2}-\frac{P^3}{3\sqrt{3}a^2\pi}\sum_{n=-\infty}^{\infty}e^{-\pi^2a^2n^2/3}.\end{aligned} \tag{8}$$

To adequately describe the process, it is sufficient to concentrate solely on the term pertaining to $n = 0$ in our analysis when dealing with the wave beams that have a width of $a >> 1/\pi$. Consequently, a reduced Lagrangian for the system is obtained as follows.

$$L_c = \frac{Pa^2}{2}\frac{d\beta}{dz} - 2Pe^{-\frac{1}{4a^2}-\beta^2a^2} - \frac{P^2}{2\sqrt{2\pi}a} - \frac{P^3}{3\sqrt{3}\pi a^2}. \tag{9}$$

Next, the Euler-Lagrange equation is formulated for each ansatz parameters a, β such that

$$\frac{d}{dz}\left(\frac{\partial L_c}{\partial a_z}\right) - \frac{\partial L_c}{\partial a} = 0,\ \frac{d}{dz}\left(\frac{\partial L_c}{\partial \beta_z}\right) - \frac{\partial L_c}{\partial \beta} = 0 \tag{10}$$

and eventually we arrive at the approximate system of ODE for the width a and phase-front curvature β as in the following equations.

$$\frac{da}{dz} = 4\beta a e^{-\frac{1}{4a^2}-\beta^2a^2} \tag{11}$$

$$\frac{d\beta}{dz} = -\frac{P}{2\sqrt{2\pi}a^3} - \frac{2P^2}{3\sqrt{3}a^4\pi} + \frac{1-4\beta^2a^4}{a^4}e^{-\frac{1}{4a^2}-\beta^2a^2} \tag{12}$$

These two coupled equations manifest the main outcome of this variational analysis, elucidating the dynamics of soliton within the context of Cubic-Quintic Discrete NLSE. The next step involves numerical simulations of the coupled equations, Eq. (11) and Eq. (12), to examine the behaviour of the discrete soliton in self-action mode. By initially setting $\beta = 0$ and $\frac{d\beta}{dz} = 0$ for a collimated wave beam, the correlation between power, P and the width, a is established as below.

$$\begin{aligned} P_1 &= \frac{1}{8}\left(-3a\sqrt{\frac{3\pi}{2}} - \sqrt{9a^2 + 64\sqrt{3}\mathrm{e}^{-\frac{1}{4a^2}}}\sqrt{\frac{3\pi}{2}}\right) \\ P_2 &= \frac{1}{8}\left(-3a\sqrt{\frac{3\pi}{2}} + \sqrt{9a^2 + 64\sqrt{3}\mathrm{e}^{-\frac{1}{4a^2}}}\sqrt{\frac{3\pi}{2}}\right) \end{aligned} \tag{13}$$

From the above equation, P_2 is taken into account to maintain a non-negative value of power which is essential for our analysis. With different value of P, the numerical simulations are conducted for coupled equation a and β, revealing the scattering process of static soliton wave in discrete cubic-quintic NLSE in the following section.

4.2 Numerical Simulation of ODE

According to the study by Balakin *et al.,* [4], two distinct self-action regimes are apparent in the one-dimensional continuum case, with one characterized by spatial solitons exhibiting wider wave beam widths, and another one with narrower widths as compared to a homogeneous wave channel size. However, the study suggested that analytical investigations are only feasible for the former case in a discrete system.

Following this, numerical simulations herein are made based on two different sizes of initial widths for the wave beams such that we consider $a_0 = 10$ for a wide beam and $a_0 = 2$ for a narrow beam. For both cases, the value of the corresponding initial powers are calculated according to the correlation in the below part of Eq. (13) whereby $P_0 = 0.485591$ and $P_0 = 1.58411$ for $a_0 = 10$ and $a_0 = 2$, respectively. When $P = P_0$, a spatial soliton is formed in a single light guide channel where the width remains constant and the stability of the soliton is maintained along z direction. Nevertheless, a small deviation in P compared to P_0 leads to a shift in the width of the wave beam, notably exhibiting oscillation throughout the propagation path. These phenomena can be referred to Figures 1 and 3 where the evolutions of the wave beam width are shown for the case of wide and narrow beams with varying the values of power P.

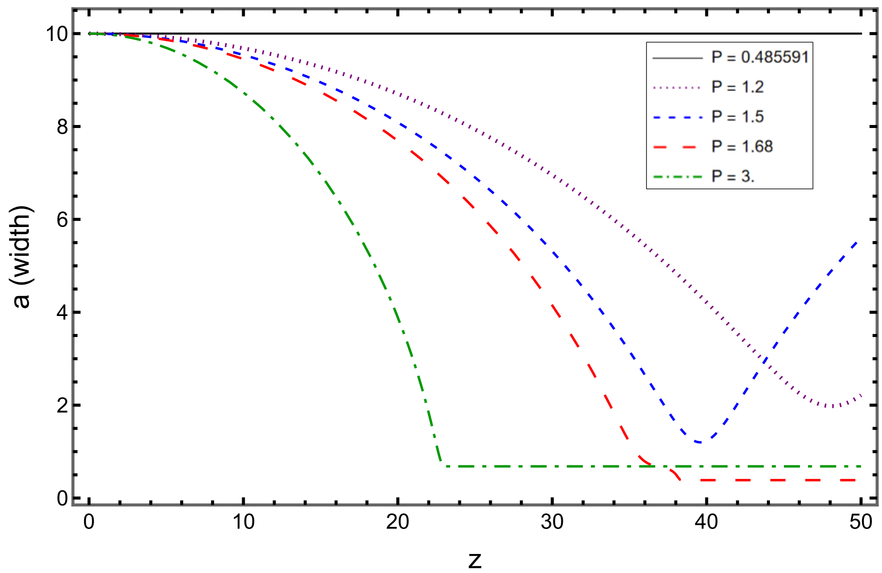
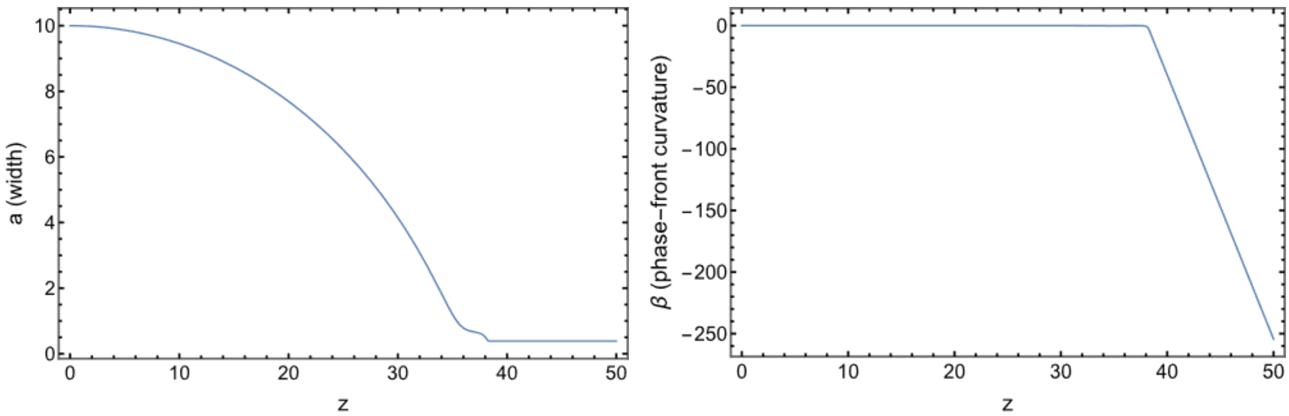


Fig. 1. The dependence of a with respect to z across different values of P for initial width of $a_0 = 10$

During the initial phase of the periodic process, the width of the wide wave beam decreases as depicted in Figure 1. On the other hand, the narrow wave beam widens when $P < P_0$ and subsequently, becomes narrower periodically as P is slightly larger than P_0. At a critical power threshold, P_{cr}, the behaviour of the wave beam width undergoes a sudden change. In particular, a qualitative shift occurs in the self-action regime when $P > P_{cr}$, resulting in the collapse of the wave beam into a unified wave channel. Figures 2 and 4 illustrate specifically the phenomena of the collapse for both initially wide and narrow beams, respectively, when the power exceeds the critical value. Accordingly, the critical power value for the wide wave beam is found to be $P \approx 1.65$, whereas for the narrow wave beam, it stands at $P_{cr} \approx 1.8$. Figure 3 displays the dependence of a with respect to z across different values of P for initial width of $a_0 = 2$.



(a) (b)

Fig. 2. The collapse of the wave beam into a single wave channel for an initially wide beam, when (a) Gradual decrease in wave beam width (b) Sudden decrease in phase-front curvature

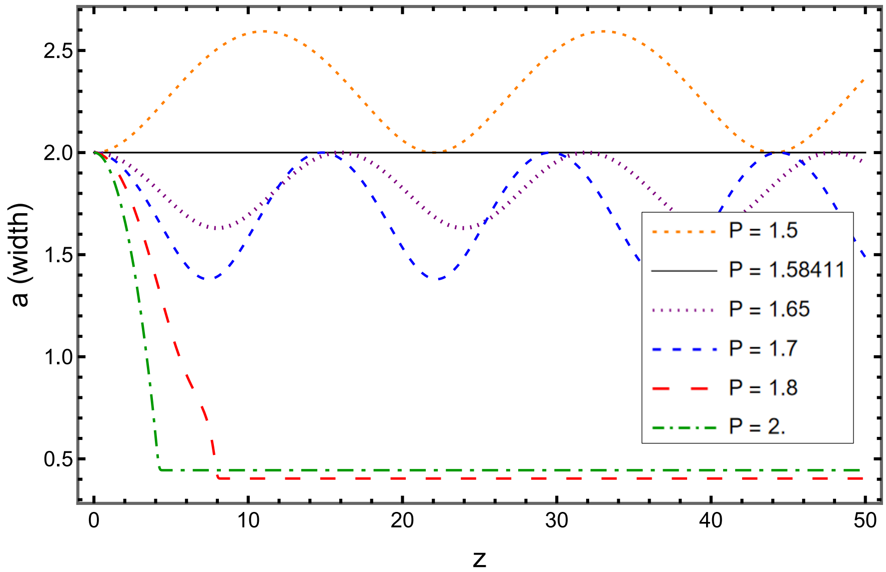
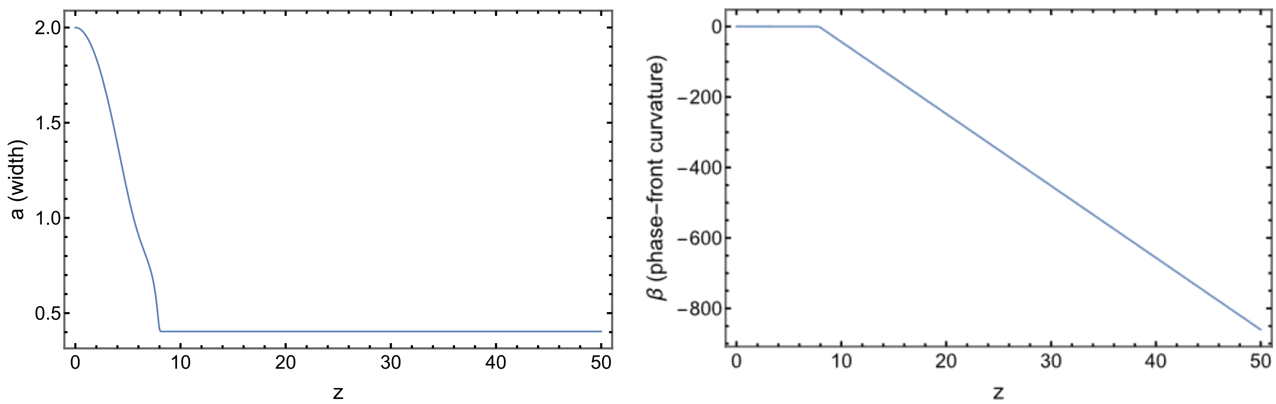


Fig. 3. The dependence of a with respect to z across different values of P for initial width of $a_0 = 2$



(a) (b)

Fig. 4. The collapse of the wave beam into a single wave channel: sudden decrease in wave beam width (a) and phase-front curvature (b) for an initially narrow beam, $a_0 = 2$ when $P = 1.8$.

5. Conclusions

The research developed a variational analysis for a qualitative study mainly on the discrete system of the cubic-quintic nonlinear Schrödinger equation (NLSE) and classify self-action mode. In this paper, the partial differential equation (PDE) of the discrete cubic-quintic NLSE is solved by converting it into ordinary differential equations (ODEs) of the coupled equations, Eq. (11) and Eq. (12), by using the VA method. It is proven that an approximate solution can provide results for the scattering process of discrete soliton in a nonlinear system. Sabdin *et. al.,* [18] in their study on nonlinear telegraph equations with source terms and time-fractional nonlinear telegraph equations, proved that an approximate solution works for their work using the Multistep Modified Reduced Differential Transform Method (MMRDTM) [18,19].

It is observed that the collapse of the wave beam in one-dimensional discrete system takes place when the power surpasses its critical value. This phenomenon results in periodic alterations in the width of the beam throughout the propagation process. The study indicates that during the self-action regime, the width of the wave beam is localized within a region which is approximately equivalence to the characteristic size of the grating. Such localization of the beam width highlights the intricate dynamics governing the discrete systems where the properties of the wave beam and the structure of the medium plays significant roles in characterizing the scattering process as a whole.

The discrete system of nonlinear equations is particularly significant to our understanding and modelling of wide range of physical phenomena. Notably, it provides a valuable framework for analyzing systems featuring discrete structures such as lattices, networks, and arrays. These systems are typically found in fields like condensed matter physics, optics, and biology. Moreover, discrete nonlinear equations often support localized solutions such as solitons, breathers, and discrete breathers. These solutions are of great importance in describing wave propagation, facilitating information transmission, and enabling efficient energy transfer across various physical systems where discrete nature of interactions is paramount.

Furthermore, the study on the numerical simulations of discrete systems utilizing the discrete nonlinear equations forms the basis for numerical simulations of complex systems. In particular, researchers are able to explore behaviour of nonlinear systems, study bifurcations, and analyze the stability of solutions in a computationally efficient manner, leading to advancements in technology and scientific understanding. For instance, Lazar *et al.,* [20] had investigated the Pacemaker and ICD Troubleshooting advanced technology where the different formation of continuous and discrete system is presented.

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