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Modified Shooting with Discretization Validation for Non-Standard Optimal Control Problem: Case Study for Four-Stage Royalty Payment Problem

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ABSTRACT

A modified shooting method augmented by discretization validation is presented in this paper to address the challenges inherent in non-standard optimal control problems. Specifically, a specific case study involving four-stage royalty payment functions is focused on, with the aim of effectively optimizing these complex, non-differentiable functions. The shooting method is adapted and enhanced to compute optimal solutions, and its accuracy is rigorously confirmed through discretization techniques. The primary objectives involve maximizing the performance index and determining optimal control strategies for scenarios where the final state variable remains unknown. In this study, the royalty payment is defined as a four-stage piecewise function. The incorporation of piecewise royalty functions introduces non-differentiability at specific time intervals, necessitating innovative approaches for finding optimal solutions. This, in turn, leads to the utilization of the continuous hyperbolic tangent (tanh) function to address the non-differentiability issue. A hybrid shooting method, combining the Newton and Golden Section Search methods, is employed in the C++ programming language to compute the unknown final state value. A new natural boundary condition, based on established theory, is introduced to further facilitate the investigation. Discretization methods such as Euler, Runge-Kutta, Trapezoidal, and Hermite-Simpson approximations are employed for validation. The validation process entails the use of the AMPL programming language with the MINOS solver. Comparative analyses reveal that the modified shooting method yields more accurate optimal results than discretization methods, thus demonstrating the method's effectiveness in addressing non-standard optimal control problems. The significance of fundamental theories in addressing real-world challenges

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is underscored by this research, providing valuable insights for future researchers exploring mathematical approaches in similar contexts. The study contributes to the continued relevance of the academic field, particularly in science and mathematics education.

1. Introduction

Optimal control theory stands as a cornerstone in the realm of applied mathematics, offering powerful tools for tackling a wide array of real-world challenges. Optimal control is a study in determining the control $u(t)$ of a certain system in order to achieve an optimality term. The field of Optimal control has seen widespread application across various domains, with numerous researchers contributing to its development and implementation. For instance, in the medical field, studies conducted by previous researchers [1-6] have utilized optimal control techniques. Additionally, optimal control has found applications in diet planning research. In the aerospace sector, the work of Ben-Asher [7] and Trélat [8] exemplifies optimal control's relevance and impact. In economics, scholars such as Cruz *et al.,* [9] and Zinober and Sufahani [10] have explored optimal control applications, particularly in the context of the royalty payment problem. Furthermore, the significance of optimal control in addressing real-world challenges is underscored by the contributions of researchers like Spence [11] and Zinober and Kaivanto [12], who have delved into the royalty payment problem using optimal control theory. Drawing from the definitions provided by Pinch [13] and Zinober [14], optimal control is conceptually defined as outlined in the following Definition 1.

Definition 1: An optimal control problem involves determining an admissible control $u^*(t)$ that steers the system described by the equation $\dot{y}(t) = f(t, y(t), u(t)$ along an admissible optimal trajectory $y^*(t)$ in a way that extremizes (either minimizes or maximizes) the performance measure [13-15].

$$J = h\left(t_f, y\left(t_f\right)\right) + \int_{t_i}^{t_f} g\big(t, y(t), u(t)\big)dt \tag{1}$$

Here, $u^*(t)$ is referred to as the optimal control and $y^*(t)$ represents the optimal state trajectory. Optimal control represents a natural extension of the challenges encountered in the realm of Calculus of Variations. Calculus of Variations is concerned with solving problems that involve variations in functions, including functionals, with the aim of identifying both the maximum and minimum values of a specified functional.

While the classical optimal control problems have been extensively studied and addressed with their well-defined state variables and smooth dynamics, a subset of problems exists those deviates from these conventional formulations. These non-standard optimal control problems introduce complexities that demand specialized solution techniques.

One such challenge arises in scenarios where the final state variable is unknown, leading to a non-zero final shadow value. The objective in these situations remains consistent: to maximize a performance index by optimizing a system's behaviour to attain specific goals. However, the presence of piecewise royalty functions within this performance index introduces non-differentiability at precise time intervals, posing a formidable hurdle to conventional optimization methods.

This research primarily focuses on the second scenario, which is the non-classical optimal control problem, as the central area of investigation. In this setting, the problem cannot be resolved using Pontryagin's Minimum Principle in conjunction with the boundary conditions typically applied in the standard setting. Specifically, in this context, the final state value $y(t_f)$ becomes equivalent to another continuous function, denoted as $y(t_f) = z$. This relationship arises due to the integral system's influence on the final state value.

In response to these challenges, this study embarks on a comprehensive exploration of innovative techniques tailored to the intricacies of non-standard optimal control problems. The focus of our research centres on a specific and practical case study: the optimization of four-stage royalty payment functions. This problem, characterized by its non-differentiable piecewise structure and unknown final state, serves as a representative illustration of the unique challenges that arise in non-standard optimal control scenarios.

The core objectives of this research are twofold: firstly, to maximize the performance index, thereby optimizing the system's behaviour, and secondly, to determine optimal control strategies that navigate the complexities introduced by non-differentiable piecewise functions. To achieve these objectives, a novel approach was employed that combines a modified shooting method with rigorous discretization validation.

The shooting method, tailored and augmented to address the specific challenges of our problem, serves as our primary tool for computing optimal solutions. Its accuracy is systematically validated through a battery of discretization techniques, ensuring the reliability of our results. Central to our investigation is the integration of the continuous hyperbolic tangent (tanh) function, designed to surmount the non-differentiability issue inherent in piecewise royalty functions.

The heart of our method lies in a modified shooting approach that amalgamates the Newton and Golden Section Search methods, expertly implemented within the C++ programming language. Additionally, we introduce a novel natural boundary condition rooted in established theory to guide and streamline our exploration.

The validation process involves the meticulous application of discretization methods, including Euler, Runge-Kutta, Trapezoidal, and Hermite-Simpson approximations, executed through the AMPL programming language with the MINOS solver.

The comparative analyses ultimately reveal that the modified shooting method consistently outperforms traditional discretization methods, thereby affirming the efficacy of our approach in addressing non-standard optimal control problems. This research amplifies the importance of fundamental theories in tackling real-world challenges, providing insights that pave the way for future explorations in mathematical approaches within similar contexts.

Moreover, it underscores the enduring relevance of the academic field, particularly in the domains of science and mathematics education, by contributing to the arsenal of tools available for addressing the complexities of real-world optimization challenges. The structure of this paper is outlined as follows:

i. Section 2 will provide a detailed explanation of the modified shooting method employed in this study.
ii. Section 3 will delve into the validation process, which utilizes discretization methods.
iii. Section 4 will offer a comprehensive overview of the non-standard optimal control problem under investigation.
iv. Following that, Section 5 and Section 6 will respectively discuss the problem formulation and the continuous approach involving hyperbolic tangent (tanh) modelling.
v. Section 7 will present an illustrative example, followed by the presentation of results and comparative analysis in Section 8.
vi. Section 9 will explore the implications for future research.
vii. Lastly, Section 10 will provide the concluding remarks for the paper.

2. Adaptation of the Modified Shooting Method

The core of the research approach for tackling non-standard optimal control problems, particularly within the domain of four-stage royalty payment functions, relies on the application of a modified shooting method. This section will provide a detailed exposition of this method and its adaptation to this unique problem. The shooting method, a widely recognized technique in the field of optimal control, serves as the foundation of our approach. However, to effectively address the complexities introduced by non-differentiable piecewise royalty functions and an unknown final state variable, careful adjustments and enhancements have been made to this method. The approach incorporates a modified shooting method, which amalgamates the strengths of the Newton method and the Golden Section Search. The Newton method's proficiency in approximating roots is harnessed in combination with the robustness of the Golden Section Search. This fusion is driven by the necessity to accurately compute the unknown final state value.

In this study, a combination of the Newton method and a one-dimensional minimization technique, specifically the Golden Section Search method, was employed within the shooting method to address the non-standard royalty problem. As described by Press *et al.*, [16], the Golden Section Search method is recognized as a one-dimensional minimization technique, and these integrated methods were implemented using the C++ programming language. The highly accurate algorithm referenced in Press *et al.*, [16] was utilized as a basis. Given that the minimization method was used to obtain the optimal results, it was necessary to maximize the performance index function by multiplying it by a negative one, as outlined by Press *et al.*, [16]. The algorithm detailing the implementation of this combination is presented in Algorithm 1 (Table 1).

Table 1
Algorithm 1. Modified shooting method

Input:
- i. Initial guess for the control parameter.
- ii. Tolerance level for convergence.
- iii. Maximum number of iterations.
- iv. Initial guess for the costate value.
- v. The range for final state value.
- vi. Initial and final time.

Output:
- i. Optimal control parameter.
- ii. Optimal final state value.
- iii. Optimal final costate value.
- iv. Optimal performance index.

(1) Initialize the control parameter with the initial guess.
(2) Initialize iteration counter (iteration) to 0.
(3) Initialize a flag variable (converged) to false.

while (iteration < maximum iterations and not converged).
(a) Solve the state and costate equations using the current control.
(b) Compute the performance index based on the current control.
(c) Compute the gradient of the performance index with respect to control ($\boldsymbol{J_{gradient}}$).
(d) Compute the Hessian matrix of the performance index with respect to control ($\boldsymbol{J_{hessian}}$).

if $\left\| J_{\text{gradient}} \right\| < \varepsilon$
(a) Set converged to true.

Table 1. Continued
Algorithm 1. Modified shooting method

(b) Break out of the loop.

else
(a) Compute the search direction, $\boldsymbol{s}$ by solving the linear system: $\boldsymbol{J_{hessian}s} = -\boldsymbol{J_{gradient}}$
(b) Perform a line search along the direction $\boldsymbol{s}$ using the Golden Section Search method:
(i) Initialize the search interval $[\boldsymbol{a}, \boldsymbol{b}]$.
(ii) Determine the trial points $(\boldsymbol{y_1}, \boldsymbol{y_2})$ within. $[\boldsymbol{a}, \boldsymbol{b},]$.
(iii) Evaluate $\boldsymbol{J}(\boldsymbol{u} + \boldsymbol{y_1 s})$ and $\boldsymbol{J}(\boldsymbol{u} + \boldsymbol{y_2 s})$.
(iv) Update the search interval $[\boldsymbol{a}, \boldsymbol{b},]$ based on the results.
(c) Update control as $\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{y_1 s}$ (or $\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{y_2 s}$ based on the interval).
(d) Increment the iteration counter (iteration).

(4) Transmit the generated value to Newton iteration to compute the final state value.
(5) End of while loop.
(6) The optimal control parameter $\boldsymbol{u_{optimal}}$ is the final value of $\boldsymbol{u}$.
(7) Return $\boldsymbol{u_{optimal}}$ as the optimal control parameter.
(8) End and printout solution.

3. Discretization Methods as the Validation Procedure

In the quest to meticulously evaluate the accuracy and dependability of the modified shooting method, the focus is shifted towards the discretization methods. This section will expound upon the crucial role played by discretization techniques in validating the proposed approach's performance and resilience. Selecting suitable discretization methods constitutes a vital aspect of this validation process. To guarantee a comprehensive assessment, a range of discretization techniques were utilized, including:

i. Euler method: This elementary yet valuable technique provides a straightforward numerical approximation of the state and costate equations.
ii. Runge-Kutta method: Renowned for its accuracy and stability, the Runge-Kutta method offers a higher-order approximation of the differential equations, contributing to a refined validation process.
iii. Trapezoidal method: With its inherent ability to capture complex dynamics, the trapezoidal rule serves as another key element in our validation toolkit.
iv. Hermite-Simpson method: Known for its robustness in handling various types of problems, the Hermite-Simpson method complements our validation procedure with its unique characteristics.

The validation process involved a systematic comparison between the outcomes obtained through discretization methods and those produced by the modified shooting method. Key components of this process include:

i. Performance evaluation. The performance index is computed for each discretization technique, enabling a direct comparison with the results generated by the modified shooting method. The objective is to gauge the accuracy and consistency of the computed performance indices across methods.
ii. Assessment of final state and costate values. Evaluating the final state variable and the initial costate value is critical to the validation. These quantities are pivotal in determining

the optimality of control strategies and serve as benchmarks for the performance of each method.

iii. Computational tools. The validation process is conducted using the AMPL programming language coupled with the MINOS solver [17]. This powerful computational combination affords us the precision and efficiency required to assess the performance of each discretization method.

iv. Comparative analyses. The comparative studies of results from the modified shooting method and the discretization techniques offer insights into the method's efficacy in tackling non-standard optimal control problems. Notably, these analyses consider the final state, initial costate and performance indices.

Each discretization method is meticulously implemented to solve the state and costate equations while considering the specific characteristics of the problem. This ensures that the discretized results are obtained faithfully. The process conducted in the AMPL programming language is thoroughly elucidated in Algorithm 2 (Table 2).

Table 2
Algorithm 2. Discretization method

Input:
i. System dynamics: Define the differential equations that govern the system's behaviour.
ii. Objective function: Specify the performance index to be optimized.
iii. Boundary conditions: Establish initial and final conditions for state and costate variables.
iv. Discretization parameters: Set parameters such as the time step size and the number of discretization points.
v. Method selection: Choose the discretization method(s) to be employed (e.g., Euler, Runge-Kutta, Trapezoidal, Hermite-Simpson).

Output:
i. Optimal control strategies.
ii. Final state and costate values.
iii. Performance index.

Step 1: Initialization
(1) Define the time interval, where is the initial time and is the final time.
(2) Choose an appropriate discretization method (Euler, Runge-Kutta, Trapezoidal, Hermite-Simpson).
(3) Specify the time step size based on the chosen method and desired accuracy.

Step 2: Discretization
(4) Initialize arrays to store discrete time points, state variables, costate variables, and control strategies.
(5) Set the initial conditions for state and costate variables.
(6) For time in the range: $[\boldsymbol{t_i}, \boldsymbol{t_f}]$:
  (a) Compute the control input at the initial time using the optimal control law derived from the chosen discretization method.
  (b) Update the state variable using the system dynamics and the computed control input.
  (c) Update the costate variable using the costate dynamics equations.
  (d) Store the time, state, costate, and control values at the current time step.

Step 3: Performance index calculation
(7) Calculate the performance index based on the discretized state, costate, and control trajectories.

Step 4: Validation
(8) Assess the convergence and accuracy of the solution.
(9) If necessary, refine the discretization parameters (e.g., time step size) to improve accuracy.

Table 2. Continued

Algorithm 2. Discretization method
Step 5: Output
(10) Return the optimal control strategies, final state and costate values, and the computed performance index.

4. Non-Standard Optimal Control Problem

Non-standard optimal control problems deviate from conventional formulations in several fundamental ways, rendering them uniquely challenging. A defining characteristic of non-standard optimal control problems is the presence of objective functions that are non-differentiable at specific time intervals. This non-differentiability arises from factors such as piecewise functions, leading to discontinuities in the optimization landscape.

Unlike standard optimal control problems, where the final state variable is typically known or prescribed, non-standard problems introduce ambiguity by leaving the final state variable unknown. This lack of clarity necessitates innovative approaches to find optimal control strategies. This paper's case study revolves around four-stage royalty payment functions, exemplifying the complexity that often characterizes non-standard problems. These piecewise functions introduce additional layers of intricacy, further challenging the optimization process.

Addressing the aforementioned challenges requires a methodological framework that can navigate the non-standard terrain effectively. This research adopted the modified shooting method, which integrates elements of the Newton and the Golden Section Search methods. This hybrid approach empowers us to compute optimal solutions while addressing the complexities of non-differentiable functions and unknown final state variables.

One of the cornerstones of this approach is the incorporation of the continuous hyperbolic tangent (tanh) function. This mathematical tool proves instrumental in mitigating the effects of non-differentiability within the objective functions, facilitating a smoother optimization process.

A novel natural boundary condition has been introduced to bolster the investigation, drawing upon established theory. This boundary condition is aligned with the intrinsic characteristics of non-standard optimal control problems, facilitating the optimization process effectively. This alignment is particularly relevant due to the presence of a non-zero final costate value. This challenge characterizes the problem as a non-standard optimal control problem, rendering it unsolvable through the Pontryagin Minimum Principle with the standard boundary condition at the final time. The proof for this new boundary condition was initially established by Malinowska and Torres [18].

Theorem 1. Let t_i be real numbers where $t_i < t_f$. If $y(t)$ is the solution to the problem t_i and t_f be real numbers where $t_i < t_f$. *If* $y(t)$ is the solution to the problem $J[y(t)] = \int_{t_i}^{t_f} g(t, y(t), \dot{y}(t), z)dt$ with boundary conditions $y(t_i) = \alpha$, $y(t_f)$ is free and $y(t) \in C^1$ [18]. Then, it follows that $\frac{d}{dt} g_y \, (t, y(t), \dot{y}(t), z) = g_z(t, y(t), \dot{y}(t), z)$ for all $t \in [t_i, t_f]$ and consequently,

$$g_{\dot{y}}(t, y(t), \dot{y}(t), z) = -\int_{t_i}^{t_f} g_z(t, y(t), \dot{y}(t), z)dt \tag{2}$$

The implications of Theorem 1 reveal that a crucial optimal condition necessitates a non-zero final costate value. From the perspective of optimal control, this condition equates the final costate variable$p(t_f)$ to $g\dot{y}(t, y(t), \dot{y}(t), z)$ provided that the integrand function g exhibits differentiability with respect to z. Consequently,

$$p(t_f) = -\int_{t_i}^{t_f} g_z\left(t, y(t), \dot{y}(t), y(t_f)\right) dt \tag{3}$$

Eq. (3) introduces a natural boundary condition, which is equivalent to $\eta(t_f)$. A boundary condition refers to a set of constraints or specifications that are imposed on a mathematical equation, typically a differential equation, to determine a unique solution. These conditions are essential in solving differential equations because they help define the behaviour of the solution at the boundary or specific points within the domain of interest. In the context of optimal control and mathematical modelling, boundary conditions play a crucial role in defining feasible solutions and optimizing a system's performance. Research related to boundary conditions has been a focal point in numerous prior investigations, as exemplified by the work of Jena and Gairola [19].

5. Problem Formulation

Based on the preceding discussion, let us now contemplate the subsequent performance index, as presented by Spence [11].

$$J[u(t)] = \int_{t_i}^{t_f} g(t, y(t), u(t))dt = \int_{t_i}^{t_f} \left(au^{1-\alpha} - \left(\rho + m_0 + c_0 e^{-\lambda y}\right)u\right)e^{-rt}dt \tag{4}$$

With respect to the variables, a represents the demand, α signifies the price elasticity of demand, ρ denotes the royalty payment, m_0 stands for the asymptote of the learning curve, c_0 corresponds to the component of unit cost, γ defines the parameter governing the speed of learning, r represents the discount factor, while y serves as the state variable, and u signifies the control variable. The model depicts the objective function or performance index, as introduced by prior researchers such as Spence [11] and Zinober and Kaivanto [12]. The following Eq. (5) represents the ordinary differential equation system that has been incorporated to optimize the performance index.

$$\dot{y}(t) = u(t) \tag{5}$$

The royalty function will consider a four-step constant piecewise system, which will then modify into a continuous hyperbolic tangent (tanh) approximation. This is to make sure that the system can implement the differentiation process everywhere. The settings that will be utilized are zero initial time and the terminal time equal to 10. The final state value is free and unknown. However, there will be a few necessary set-ups that need to meet the satisfaction in continuing the process: the state and costate equation with the stationarity term, the initial requirement of state and costate variable is defined, and the integral boundary condition is satisfied at the terminal time. The conditions are satisfied whenever the costate system converges. After that, the optimal solution will be attained.

The choice of a four-stage piecewise royalty payment model is deliberate, aligning with our overarching research objective of addressing non-standard optimal control problems. This model introduces non-differentiability into the objective function, rendering the optimization process inherently challenging.

Furthermore, the ambiguity surrounding the final state variable, a characteristic common to non-standard problems, is mirrored in this model. As such, the royalty payment model becomes a quintessential case study for testing the efficacy of the proposed modified shooting method. Beyond its role within this research framework, the four-stage royalty payment model carries real-world relevance. Many industries employ piecewise payment structures, including finance, manufacturing, and resource management. The exploration of this model structure and the optimization techniques applied hold implications for addressing practical challenges in these domains.

6. Continuous Hyperbolic Tangent as the Approach for Non-Differentiable Piecewise Function

Non-differentiable piecewise functions are encountered in various domains, often presenting formidable challenges in optimization and control problems. This section explores the application of the continuous hyperbolic tangent (tanh) function as a powerful mathematical tool for mitigating the non-differentiability inherent in such functions. In many real-world scenarios, optimization and control problems involve objective functions that exhibit non-differentiability at specific points or intervals. These non-differentiable regions arise due to discontinuities or abrupt changes in the function's behaviour, making traditional calculus-based optimization techniques less effective.

Piecewise functions, characterized by different expressions in distinct intervals, are a common source of non-differentiability. Traditional methods often struggle to optimize these functions because they lack derivatives at the points of discontinuity. The continuous hyperbolic tangent (tanh) function is an invaluable mathematical tool renowned for its ability to smooth out non-differentiable regions in functions. It possesses several key properties that make it a compelling choice for addressing the challenges posed by non-differentiable piecewise functions.

i. Smooth transition. The hyperbolic tangent (tanh) function exhibits a smooth transition from -1 to 1 over its domain, making it suitable for approximating piecewise functions with abrupt changes.
ii. Approximation of discontinuities. By using hyperbolic tangent (tanh), it is possible to approximate the behaviour of a non-differentiable piecewise function in a way that is amenable to traditional optimization techniques.
iii. Continuous derivatives. Hyperbolic tangent (tanh) functions have continuous derivatives, allowing for the application of gradient-based optimization algorithms, which are efficient and widely employed in optimization problems.

In the context of optimization and control problems involving non-differentiable piecewise functions, the integration of hyperbolic tangent (tanh) can significantly enhance the effectiveness of solution approaches. Here's how hyperbolic tangent (tanh) can be applied.

i. Smoothing the objective function. By replacing non-differentiable parts of the objective function with hyperbolic tangent (tanh) approximations, the function becomes smooth and differentiable, enabling the use of gradient-based optimization methods.
ii. Continuous transition. The hyperbolic tangent (tanh) functions can smoothly transition between different pieces of a piecewise function, avoiding abrupt changes that can hinder optimization convergence.

While the use of hyperbolic tangent (tanh) functions offers numerous advantages in handling non-differentiable piecewise functions, it is essential to consider potential trade-offs and nuances.

i. Accuracy vs. smoothness. The choice of hyperbolic tangent (tanh) smoothing parameters should balance the desire for smoothness with the need to accurately capture the original function's behaviour.
ii. Convergence speed. Smoothing with a hyperbolic tangent (tanh) may slow down convergence compared to using methods tailored explicitly for piecewise functions. However, it often leads to more robust and reliable optimization outcomes.

7. Royalty Payment Example

Mathematically, the royalty payment model can be expressed as follows.

$$\rho\big(y(t)\big) = \begin{cases} \frac{1}{10} & \text{for } 0 \le y(t) \le \frac{1}{4}z \\ \frac{6}{5} & \text{for } \frac{1}{4}z < y(t) \le \frac{1}{2}z \\ \frac{6}{25} & \text{for } \frac{1}{2}z < y(t) \le \frac{3}{4}z \\ \frac{3}{25} & \text{for } \frac{3}{4}z < y(t) \le z \end{cases} \tag{6}$$

The Eq. (6) can be transformed into the hyperbolic tangent (tanh) function. In this study, the smoothing values were determined as $k = 50$ and $k = 250$ to approximate Eq. (7). It is important to note that the larger the smoothing value, the smoother the resulting plot becomes.

$$\rho\big(y(t)\big) = \frac{11}{100} + \frac{11}{20} tanh\left(k\left(y - \frac{1}{4}z\right)\right) - \frac{12}{25} tanh\left(k\left(y - \frac{1}{2}z\right)\right) - \frac{3}{50} tanh\left(k\left(y - \frac{3}{4}z\right)\right) \tag{7}$$

The Hamiltonian function is a fundamental tool in solving optimal control problems. It is used to derive the necessary optimality conditions, including the Hamiltonian's differential equations and boundary conditions. These conditions guide the search for optimal control strategies that maximize or minimize the Hamiltonian function, depending on the nature of the problem (maximization or minimization of a performance index). The Hamiltonian function is a crucial component of optimal control theory, providing a framework for analyzing and solving complex control optimization problems in various fields, including engineering, economics, and physics. In more detail, the Hamiltonian function is defined as follows.

$$H(t, y, u, p) = g(t, y, u) + p(t)u(t) \tag{8}$$

Subject to t is the time, H is the Hamiltonian function, y represents the state vector of the system p is the costate vector, also known as the adjoint vector. While, u stands for the control input or control vector and $g(t, y, u)$ is the system's Lagrangian, which represents the instantaneous cost or performance associated with state y and control u. The Hamiltonian behaviour fulfils the following conditions [20].

$$H_u = 0;\ \dot{y}(t) = H_p;\ \dot{p}(t) = -H_y \tag{9}$$

Therefore, the state dynamic satisfies the Hamiltonian system, where:

$$\dot{y}(t) = (e^{0.025t}u^{0.5} - (\rho + 1 + e^{-0.12y})u)e^{-0.1t} + u \tag{10}$$

The integrand function g is contingent on both (y_t) and ρ, leading to the costate variable adhering to the following relationship.

$$\dot{p} = \left(\frac{109}{100}k - \frac{11}{20}k\,tanh\left(k\left(y - \frac{1}{4}z\right)\right)^2 + \frac{12}{25}k\,tanh\left(k\left(y - \frac{1}{2}z\right)\right)^2 + \frac{3}{50}k\,tanh\left(k\left(y - \frac{3}{4}z\right)\right)^2 - \frac{3}{25}e^{-0.12y}\right)ue^{-0.1t} \quad (11)$$

The stationarity condition can be expressed as $H_u = 0$, where:

$$H_u = \left(\frac{1}{2}e^{0.025t}u^{-0.5} - \rho - e^{-0.12y} - 1\right)e^{-0.1t} + p = 0 \quad (12)$$

where,

$$u(t) = \frac{1}{4}(e^{0.025t})^2(e^{-0.1t})^2(\rho e^{-0.1t} + e^{-0.12y}e^{-0.1t} + e^{-0.1t} - p)^{-2} \quad (13)$$

Additionally, the integral results in the following conditions.

$$p(T) = \int_0^{10}\left(\frac{59}{400}k + \frac{11}{80}k\,tanh\left(k\left(y - \frac{1}{4}z\right)\right)^2 - \frac{6}{25}k\,tanh\left(k\left(y - \frac{1}{2}z\right)\right)^2 - \frac{9}{200}k\,tanh\left(k\left(y - \frac{3}{4}z\right)\right)^2\right)ue^{-0.1t}dt \quad (14)$$

8. Results and Comparative Analysis

To evaluate the performance of each method, several critical metrics were considered. The final state variable, denoted as $y(t_f)$, serves as a pivotal indicator of optimality. The computed values were compared with those obtained via discretization techniques using the proposed modified shooting method. The initial costate value, denoted as $p(t_i)$, plays a crucial role in determining optimal control strategies. The consistency of this value was assessed across methods. This research aims to maximize the performance index, which quantifies the optimality of control strategies. This optimal performance index was computed for each method, and a comparative analysis was conducted.

8.1 Smoothing Value Equals 50

Table 3 presents the results for a smoothing value k equal to 50. As evident from Table 3, the comparative analysis reveals noteworthy insights where the final state values computed through the modified shooting method and the Euler, as well as the Runge-Kutta method, exhibit remarkable similarity up to two decimal places. The Trapezoidal and Hermite-Simpson approximation closely approximates the final state value, consistent up to one decimal place. The costate value at the initial time for the Runge-Kutta and Trapezoidal method aligns with the shooting method, accurate up to three decimal places. The Euler and Hermite-Simpson method yields an initial costate value comparable up to one and two decimal places, respectively. The performance indices, a key measure of optimality, exhibit convergence among the shooting and discretization methods, accurate up to two decimal places.

Table 3
The optimal result generated by the shooting and discretization methods for smoothing value equal to 50

Methods	Final state value	Initial costate value	Final costate value	Performance index
NG	0.585406	-0.278096	-0.108470	0.820501
EU	0.580630	-0.281408	-	0.824692
RK	0.583067	-0.278355	-	0.825111
TR	0.578117	-0.278567	-	0.825874
HS	0.593281	-0.273071	-	0.825276

*NG=Newton and Golden; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson

Figure 1 depicts the optimal solution generated by both the shooting and discretization methods. A noticeable distinction is observed in the smoothness of the curves, with the shooting method yielding a smoother plot compared to the discretized one. The optimal plot curves exhibit a high degree of similarity for all variables except for the control plot, where differences between the discretized values and the shooting values become apparent at a specific time. This discrepancy can be attributed to the discretization error that occurs during the process [21, 22]. It can be inferred that the C++ routine from Numerical Recipe [16] consistently produces highly accurate solutions.

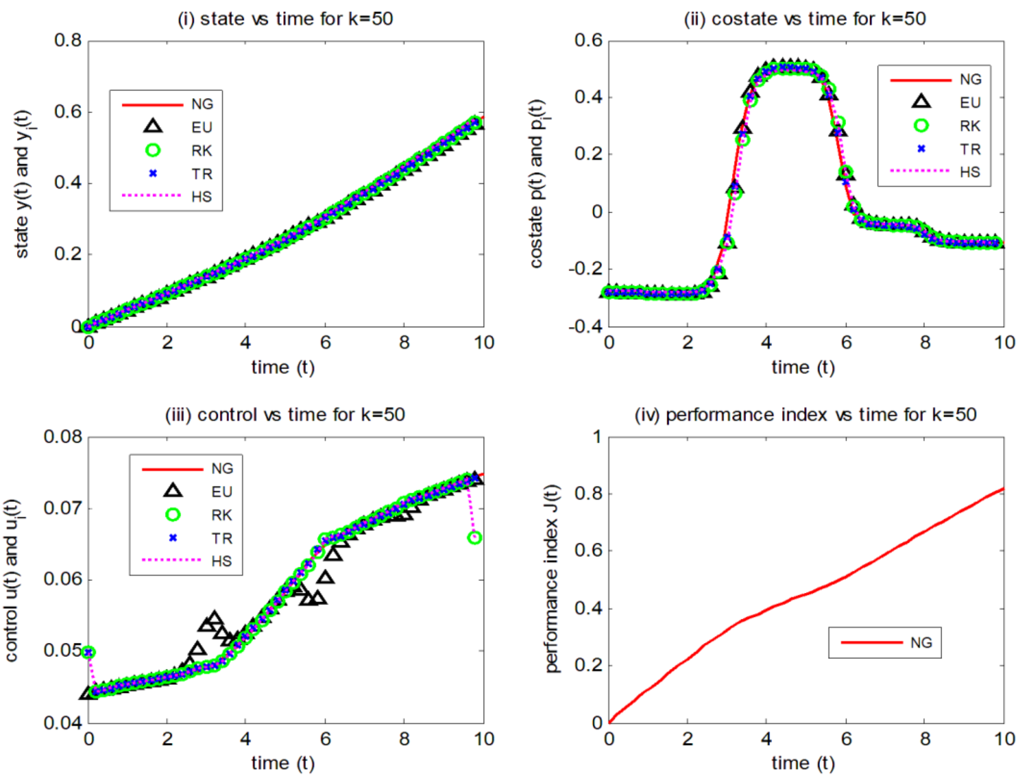

Fig. 1. The optimal plot for smoothing value equal to 50 generated by the shooting and discretization techniques. (NG=Newton and Golden; EU=Euler; RK=Runge-Kutta; TR= Trapezoidal; HS=Hermite-Simpson)

8.2 Smoothing Value Equals 250

Table 4 extends the comparative analysis to a smoothing value k equal to 250. Based on Table 4, implementing comparative analysis highlights the following results. The final state values remain identical up to two decimal places for the Runge-Kutta method. The Euler, Trapezoidal, and Hermite-Simpson approximation converges, accurate up to one decimal place when compared with the shooting result. Initial costate values are consistent up to one decimal place for the shooting method and the Runge-Kutta as well as the Hermite-Simpson method. Performance indices demonstrate convergence across the shooting method, the Runge-Kutta method, the Trapezoidal method and the Hermite-Simpson method, precise up to two decimal places.

Figure 2 presents a graphical representation of the optimal solution for the second case, where the smoothing value is set to 250. The curves in this case bear a resemblance to those in the first case, which k equals 50. However, it is noteworthy that the plot corresponding to $k = 250$ exhibits a higher degree of smoothness compared to the plot presented in the preceding subsection.

Table 4
The result generated by the shooting and discretization methods for smoothing value equal to 250

Methods	Final state value	Initial costate value	Final costate value	Performance index
NG	0.585314	-0.277648	-0.108443	0.820619
EU	0.543349	0.082824	-	0.814081
RK	0.588223	-0.246044	-	0.826710
TR	0.575365	-0.198810	-	0.826766
HS	0.593381	-0.249868	-	0.826571

*NG=Newton and Golden; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson

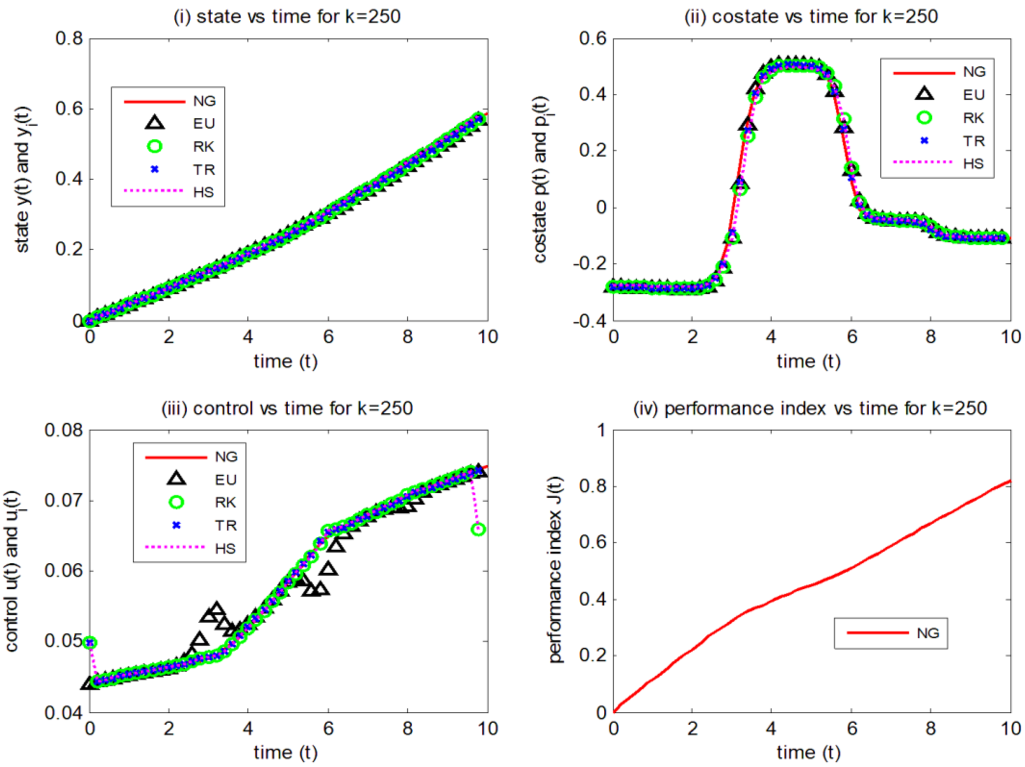

Fig. 2. The optimal plot for smoothing value equal to 250 generated from the shooting method and discretization techniques. (NG=Newton and Golden; EU=Euler; RK=Runge -Kutta; TR=Trapezoidal; HS=Hermite-Simpson)

8.3 Summary

The comparative analysis provides valuable insights into the performance of the proposed modified shooting method in the context of non-standard optimal control problems. Notably, the shooting method consistently yields results comparable to, and often more accurate than, discretization techniques. This underscores the method's effectiveness in addressing complex, non-differentiable functions and scenarios with unknown final state variables.

9. Discussion and Implication for Future Research Directions

A quest was undertaken to address the intricate challenges posed by non-standard optimal control problems, with a specific focus on a case study involving four-stage royalty payment functions. The outcomes of this investigation have illuminated several key aspects. One of the central findings of this research pertains to the efficacy of the modified shooting method. A powerful approach was devised by integrating elements from the Newton and Golden Section Search methods, capable of effectively tackling complex, non-differentiable functions. The comparative analysis revealed that this method consistently produced accurate results, surpassing traditional discretization techniques in several aspects.

The adaptability of the modified shooting method to non-standard scenarios was underscored. The four-stage royalty payment model served as a compelling case study, mirroring real-world complexities often encountered in diverse domains. The model introduced non-differentiability and ambiguity surrounding the final state variable, aligning with the hallmarks of non-standard problems. The successful application of this method in this context highlights its practical utility in addressing challenging optimization scenarios. Incorporating continuous hyperbolic tangent (tanh) functions and establishing a novel natural boundary condition underscore the significance of fundamental theories in addressing real-world challenges. This research demonstrates that a strong theoretical foundation can pave the way for innovative approaches, enabling the optimization of intricate systems.

The findings of this study bear substantial implications for future research trajectories in the realm of non-standard optimal control problems.

i. Methodological advancements. Building upon the success of the modified shooting method, future research endeavours can explore further enhancements and refinements. The integration of advanced optimization algorithms and techniques holds the potential to extend the method's applicability to an even broader range of non-standard scenarios.
ii. Real-world applications. The practical relevance of the research extends beyond theoretical considerations. Future studies can delve into the application of this approach to real-world problems in finance, manufacturing, and resource management, where piecewise payment structures are commonplace. These applications could yield insights and solutions to pressing challenges in these domains.
iii. Educational significance. This research underscores the importance of effective teaching and learning processes, especially in the domains of science and mathematics. Using innovative mathematical approaches to solve complex problems can enhance the educational experience and foster a deeper understanding of optimization techniques.

This research has illuminated the path toward effectively addressing non-standard optimal control problems. The modified shooting method, validated through rigorous comparative analysis, has emerged as a potent tool for optimizing complex, non-differentiable functions with unknown final state variables. These findings serve as a foundation upon which future research endeavours can build, driving advancements in the field of optimization and offering solutions to real-world challenges.

Moving forward, the continuous exploration of non-standard scenarios and the development of innovative methodologies will be essential in pushing the boundaries of what is achievable in optimization and control theory. By doing so, the continued relevance of the academic field and its practical applications in addressing complex real-world problems are ensured.

10. Conclusion

This study introduced a modified shooting method with discretization validation to address the intricate challenges posed by non-standard optimal control problems. The focus of this research centred on a specific case study involving four-stage royalty payment functions, was to optimize these intricate, non-differentiable functions in an efficient manner. Through this approach, the shooting method was adapted and enhanced to facilitate the computation of optimal solutions, with a rigorous validation of its accuracy achieved through the application of discretization techniques. The primary objectives encompassed the maximization of the performance index and the determination of optimal control strategies, particularly in scenarios where the final state variable remains undisclosed. Throughout the investigation, specific scrutiny was applied to royalty payments structured as four-stage piecewise functions, a scenario acknowledged for its propensity to introduce non-differentiability at precise time intervals. To effectively address this challenge, the continuous hyperbolic tangent (tanh) function was introduced, which was found to be highly effective in mitigating issues related to non-differentiability. The implementation of our hybrid shooting method, achieved through the combined application of the Newton and Golden Section Search methods, was executed using the C++ programming language for the purpose of computing the final state value, which remained unknown. Additionally, a novel natural boundary condition, firmly grounded in established theory, was introduced to further streamline and bolster our investigative efforts. Validation of the findings was conducted through the utilization of discretization methods such as Euler, Runge-Kutta, Trapezoidal, and Hermite-Simpson approximations. This validation process was meticulously carried out utilizing the AMPL programming language with the MINOS solver. The comparative analyses conducted throughout this research yielded a noteworthy result: the modified shooting method consistently outperformed discretization methods, conclusively demonstrating its efficacy in effectively addressing non-standard optimal control problems. This outcome underscored the practical utility of our approach, particularly in the realm of complex real-world scenarios. In conclusion, the research conducted herein highlights the critical role played by fundamental theories in addressing substantial real-world challenges, offering valuable insights poised to guide future researchers exploring mathematical approaches within analogous contexts. By contributing to the enduring relevance of the academic field, particularly within the spheres of science and mathematics education, this study provides a foundational cornerstone for the ongoing advancement of solutions addressing intricate real-world optimization problems.

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