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Maximizing Economic Benefit: A Case Study of Royalty Payment Optimization using Modified Shooting and Discretization Methods

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ABSTRACT

The shooting method is reviewed as a numerical solution for addressing non-standard optimal control (OC) problems. The non-standard OC problem is taken into account when the final state value's component is unknown and free. As a result, the final shadow value, known as the costate variable, was not equal to zero. The objective function also involves the royalty function, which takes the form of a piecewise function. At a certain time frame, it is, nevertheless, not differentiable. So, to determine the unknown final state value, a new modified shooting method was applied. The model could be differentiated at all times thanks to the simultaneous use of a continuous hyperbolic tangent (tanh) approximation. The Sufahani-Ahmad-Powell-Golden-Royalty Algorithm (SAPGRA) was used to construct the problem in C++ programming language. The findings that meet the optimality criteria were then contrasted with discretization methods such as Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson approximation. This groundbreaking discovery is immensely helpful in resolving practical issues. It can advance the academic discipline so that problem-solving techniques are always current. Meanwhile, this study also discusses the value of fundamental theory in solving real-world economic problems.

1. Introduction

The Optimal Control (OC) problem deals with the finding process of control for a system over a period of time while fulfilling optimality conditions. Young scholars today are drawn to OC issues that relate to economics applications. The optimization technique and non-linear programming (NLP) approach are used to tackle the OC problem.

Xiaobing *et al.,* [1] utilized efficient basis functions to effectively address the complexities of OC problems involving fractional calculus. Concurrently, Xie *et al.,* [2] introduced an innovative hybrid neural network algorithm that integrates L2 and dropout regularization techniques. This approach is

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designed to improve both performance and generalization in applying neural networks to solve engineering-related mathematical problems.

Huang *et al.,* [3] developed an innovative method for addressing nonlinear time-fractional OC problems by applying the space-time Chebyshev spectral collocation technique. Effati *et al.,* [4] introduced an OC-based approach specifically for solving linear Volterra integral equations. This method applies OC theory principles to effectively manage the linear Volterra integral equations commonly found in engineering and scientific contexts, aiming to determine an OC function that meets the integral equation constraints while minimizing an objective function. Additionally, Skandari *et al.,* [5] combined linearization and discretization techniques to efficiently tackle the complexities of certain nonlinear OC problems.

A royalty payment is described as a sum of money given to an owner by users or developers in exchange for the usage of their assets. The payment is paid to the person who has legal ownership of an item, such as computer software, copyrights, trademarks or a patent.

The objective function is not dependent on the value of the final state, $y(T)$, which is the default configuration for the OC problem. The OC problem in this study, however, will only be concerned with the non-standard OC problem, in which the integral of the objective function depends on the state variable's final value, $y(T)$. As a result, the shadow value at the final time, $p(T)$, is not zero. Contrary to the standard OC theory, which states that the costate value, $p(T)$, is equal to zero at the terminal time, T, this is the case. In the context of optimization and decision-making, shadow value refers to the rate at which the value of an objective function changes with respect to a small change in a constraint or parameter. It can also be interpreted as the marginal value of a resource, indicating how much the objective function would improve if the constraint or parameter were relaxed by one unit. The shadow value is a key concept in optimization because it allows decision-makers to determine the most effective allocation of resources in order to achieve their goals. In OC theory, shadow value is also known as costate value.

The royalty function, ρ, a piecewise function in the state, is a component of the objective function, which cannot be differentiated at a specific time frame. To address this issue, a continuous hyperbolic tangent (tanh) approach was adopted. The non-standard OC problem was subsequently tackled using the modified shooting method as an indirect approach, while the discretization method served as a validation process.

2. Royalty Payment Problem and Its Application

The royalty payment is a crucial issue in the field of intellectual property, where authors, artists, musicians and inventors are entitled to receive a share of the revenue generated by their creations. The proper distribution of royalties is essential to ensure that the creators are fairly compensated for their work, which, in turn, encourages innovation and creativity.

The royalty payment problem refers to the challenge of fairly and accurately allocating revenue generated from the sale or use of copyrighted works, such as music, books and films, among the various stakeholders involved in their creation and distribution. In the music industry context, this includes the artists, composers, producers and music labels.

The problem arises because different stakeholders contribute to the creation and success of work and may have different contractual arrangements for revenue sharing. Moreover, the rapid changes in technology and the music industry's business model, such as the rise of streaming services and digital downloads, have made tracking and allocating revenue accurately more challenging.

Various mathematical models and algorithms have been proposed to solve the royalty payment problem, such as multi-criteria decision-making models, linear programming and game theory. This research aims to provide a fair and transparent method for revenue sharing by considering various parameters, such as sales, demand and discount rate.

Yahya *et al.,* [6] propose a music royalty payment scheme using blockchain technology. The authors argue that the decentralized nature of blockchain technology can provide a secure and transparent platform for royalty distribution in the music industry. They describe a prototype system that uses smart contracts and a cryptocurrency-based payment system to facilitate royalty payments to artists and other stakeholders.

Cai *et al.,* [7] examine the impact of a fixed royalty payment on sustainable fashion brand franchising. The authors use game theory to model the interaction between a franchisor and franchisee. The franchisor decides whether to invest in sustainability upfront or later and the franchisee chooses between investing in sustainability or not. The study finds that a fixed royalty payment can lead to a "win-win situation" for both the franchisor and franchisee, as it provides incentives for both parties to invest in sustainability. However, the study also highlights the importance of franchisee selection, as choosing the right franchisee can enhance the effectiveness of the fixed royalty payment mechanism. Overall, the study provides valuable insights into sustainable fashion brand franchising and suggests potential avenues for future research.

Horal *et al.,* [8] discuss the issue of defining and justifying the distribution ratio of oil and gas royalties under decentralization in Ukraine. They argue that the current royalty distribution system in the country is outdated and requires revision. Therefore, a new model was suggested for calculating the royalty payments based on the size and profitability of oil and gas fields, as well as the level of investment in their development. They also emphasize the importance of involving local communities in the decision-making process regarding the distribution of royalties and propose a mechanism for their participation. A comprehensive analysis of the issues was provided related to oil and gas royalties in Ukraine and offered practical recommendations for improving the current system.

Generally, the literature on royalty payment problems provides various approaches to address the issue of fair and efficient distribution of royalties. The proposed models consider different factors, such as sales, airplay, digital downloads and patent licensing, to determine each party's share in the royalty pool. The models are based on different methodologies, such as multi-criteria decision-making, game theory and optimization and provide a fair and transparent method for royalty distribution.

However, most researchers have never seen the approaches through mathematical modelling as interesting. In addition, the fundamental theory of computing a fair royalty payment was not considered. These issues may affect the royalty payment received and it might not be a "win-win situation" between the property owner and developer.

3. Non-Standard Optimal Control Problem with The Royalty Payment

Spence [9] discusses the concept of the learning curve, which refers to the idea that firms can increase their productivity and reduce their costs as they gain experience through repeated production. He also examines how the learning curve affects competition in various industries. He argues that firms with a steeper learning curve (i.e., those that experience more significant productivity gains through experience) have a competitive advantage over firms with a flatter learning curve.

Furthermore, Spence [9] suggests that firms may be incentivized to share information about their production processes with their competitors to reduce their own learning costs. This information sharing can lead to a more efficient allocation of resources in the industry and may ultimately benefit consumers. An economic model was previously proposed by Spence [9].

$$\begin{aligned} J\left[u(t)\right] &= \int_{t_0}^{T} g\left(t, y(t), u(t), y(T)\right) dt \\ &= \int_{t_0}^{T} \left(a(t) u^{1-\alpha} - \left(\rho + m_0 + c_0 e^{-wy(t)}\right) u\right) e^{-rt} dt \end{aligned} \tag{1}$$

where $a(t)$ is denoted as the demand, α is denoted as the price elasticity demand, u is defined as control variable, ρ is defined as royalty payment, m_0 is known as the asymptote of the learning curve, c_0 is denoted as the component of unit cost that is subject to the learning curve, w is the parameter of the speed of learning, y is the state variable, r is the discount component and t is time.

An attempt to solve the economic model using matrix formulation was made by Zinober *et al.,* [10]. In this case, they consider the related parameters, such as the current demand value and discount factor, which may exist for computing optimal royalty payment when maximizing the objective function. However, they face difficulty computing the optimal objective function where the differentiation process cannot continue when the royalty payment level changes.

Cruz *et al.,* [11] introduce a new class of variational problems called non-classical (non-standard). These problems involve an objective functional that is a difference of two integrals. The first integral involves the unknown function and its first derivative and the second integral involves a non-linear function of the anonymous function and its first derivative. They develop the necessary conditions for the optimality of this class of problems using the Pontryagin Maximum Principle. They also proposed a numerical method for solving these problems based on a collocation approach. The unknown function is approximated using a piecewise polynomial function and the objective functional is approximated using quadrature formulas. The proposed method is demonstrated in several examples, including a non-linear boundary value problem and an OC problem for a two-link robotic manipulator. Surprisingly, the findings indicate that the approach is both effective and precise when it comes to addressing these non-standard variational problems. Cruz *et al*., [11] solved an economic problem related to royalty payment in this case. However, the royalty payment is fixed and constant. In our research, these findings were referred to solve the royalty payments problem using the piecewise function.

Later, Zinober *et al.,* [12] resolved a novel OC problem in which a particular function involved the two-stage percentage function. They discuss a non-standard OC problem that arises in an economics application and propose a method to solve the problem using Pontryagin's Maximum Principle, which is a well-known technique in the field of OC. The problem involves finding the optimal strategy for a firm that is competing in a market with other firms. The objective is to maximize the firm's profit over a finite time horizon, subject to constraints on the firm's production and investment decisions. The necessary condition prerequisites are then derived by the authors using Pontryagin's Maximum Principle. A coupled partial differential equations (PDE) system with these conditions can be quantitatively solved using finite difference techniques. A numerical example was provided to illustrate the effectiveness of their method.

Al-Hawasy [13] investigated continuous classical OC for nonlinear hyperbolic PDEs, addressing the complexities introduced by both equality and inequality constraints. Later, Al-Hawasy *et al.,* [14] proposed a numerical method that combines the Galerkin finite element method, an implicit technique and the gradient projection method to solve classical OC problems involving hyperbolic PDEs.

Overall, they provide a practical application of OC theory to an economic problem. Using Pontryagin's Maximum Principle allows for a rigorous and systematic approach to finding the optimal strategy and the numerical examples demonstrate the method's practicality. They observed that the piecewise function in its discrete form should be represented in the continuous form to allow differentiation at each stage. Consequently, they suggested using a continuous hyperbolic tangent (tanh) approach to tackle this issue. This ingenious concept inspired us to address the economic model introduced by Spence [9].

4. Modified Shooting Method

The modified shooting method is a numerical technique for solving boundary value problems. It is an extension of the standard shooting method. The method involves first guessing an initial value for the unknown boundary condition and then using numerical methods to solve the differential equation from both ends, i.e., from the first boundary point and from the guessed value at the other end. The solution is then matched at the intermediate point and the guessed boundary condition is iteratively adjusted until the solution satisfies the desired boundary conditions. The modified shooting method is widely used in various fields of science and engineering, including fluid dynamics, heat transfer and mechanics, to solve boundary value problems that cannot be solved analytically.

In this research, the modified shooting method combined the Powell and Golden Section Search methods to compute the optimal solution. Algorithm 1 and Figure 1, named Sufahani-Ahmad-Powell-Golden-Royalty Algorithm (SAPGRA), show how these combinations work.

Algorithm 1: SAPGRA

Input : Initial time t_0, final time T initial and guessed value of $p(0)$ and y_T, boundary conditions ($y(0)=0$) ordinary Differential Equation (ODE).

Output : Approximation to $t=$time, $y[0]=y(t)$, $y[1]=p(t)$ and $y[2]=\eta_T$

Step 1: Initialization

(a) Define the number of ODEs and the number of guess(ed) values.
(b) Set the initial time, final time, boundary condition, initial values, guess(ed) values.

Step 2: Calculation

(c) Call Golden with three range values of , say: , and . (Golden will calculate and generate value within the range and pass it to Powell) which maximizes the function.
(d) Call Powell solver with two scalar functions and :
(e) Run the ODE solver with initial guess , say .
(f) At (final time), check whether the and become small enough (at the final time , is equal to and is equal to).

if yes, then
Go to (g).

or else
Go to (c) and update f_1 and f_2.
end if
(g) Check whether Golden generates the best y_T value that maximizes the function.
if yes, then
Go to (h).
or else
Go to (c) and update r_1, r_2, r_3. This will update the y_T value.
end if
(h) End and printout solution.

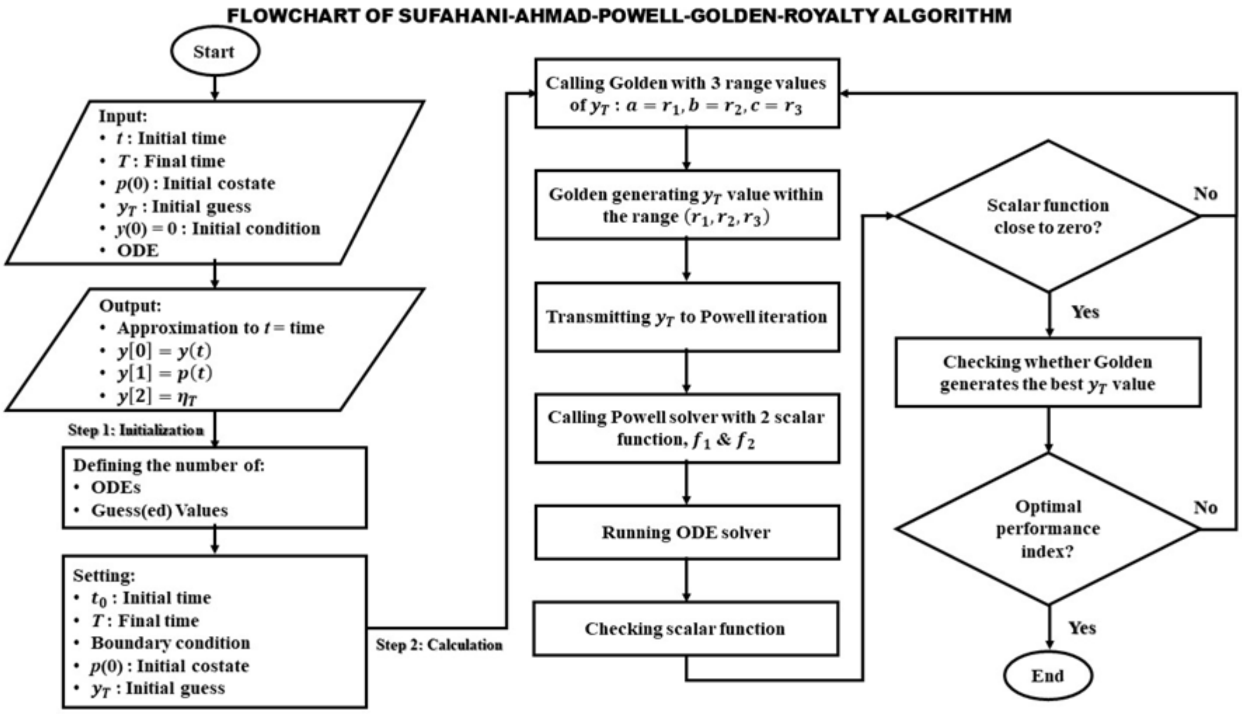


Fig. 1. Flowchart for modified shooting method

The program starts with the initialization process, where the number of ODE and guessed value (s) are defined. In this phase, the setting involved the initial time that is equal to zero, final time and boundary condition. The computation starts by calling Golden Section Search to generate the best final state value. This value will then be transmitted to Powell and the program will run the ODE solver with the guessed value. At the final time, the program will check whether optimality conditions have been fulfilled. If not, the program will be rerun until the optimal solution is yielded while satisfying the optimality condition.

According to Press *et al.,* [15], the Golden Section Search method is classified as a minimization technique. Therefore, to solve a maximization problem, the performance index is multiplied by -1 at the end of the computation. This adjustment enables the minimization algorithm, like the Golden Section Search method, to effectively identify the maximum value of the performance index.

5. Discretization Method

The discretization method was used to approximate continuous functions, equations or systems by dividing them into a finite number of discrete elements. This method is used in various fields, such as numerical analysis, computational physics and engineering. Discretization was used to transform a continuous function or equation into a finite set of equations or variables, making it possible to solve the problem numerically. This involves dividing the continuous domain into a finite set of points or intervals, where the value of the function or equation is computed at each point or interval.

Discretization methods are commonly employed in a wide range of numerical techniques, including finite volume method, finite difference method and finite element analysis. These methods are used to solve differential equations, PDEs and other mathematical problems numerically by approximating the continuous functions or equations using discrete elements or variables.

As a means of validation, this study utilized various discretization methods such as Euler, Runge-Kutta, Trapezoidal and Hermitte-Simpson approximations. The AMPL programming language with MINOS solver was used to conduct the program. The direct method, known for its flexibility, ease of implementation and ability to handle a broad range of problem types, including non-linear and non-convex problems, was applied. It also allows for the inclusion of path constraints and endpoint constraints, which are common in many real-world applications. However, the direct method can suffer from issues such as convergence problems and a lack of global optimality guarantees, particularly for highly non-linear problems.

6. The Illustrative Example

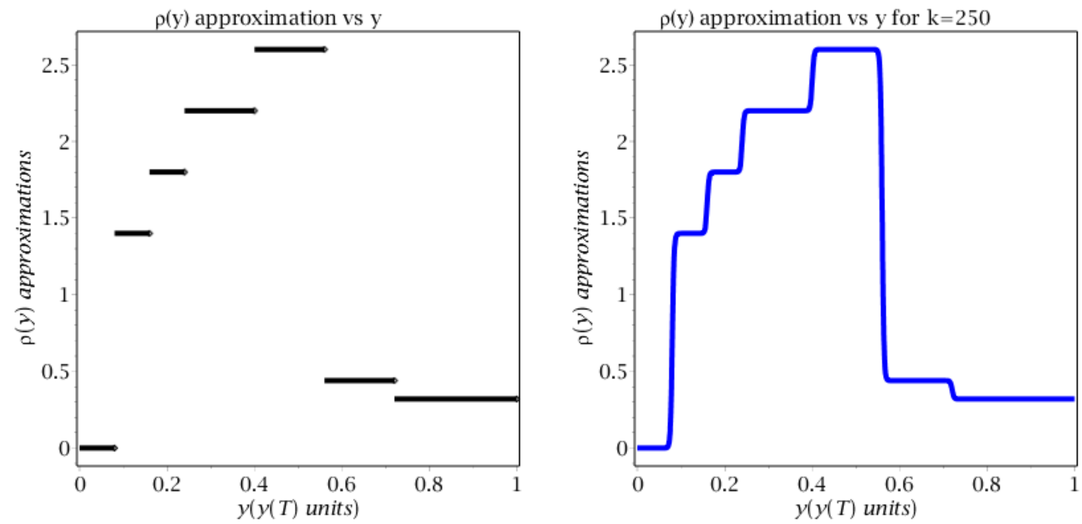
Let us consider the following seven-step constant piecewise function.

$$\rho(y)=\begin{cases} 0 & \text{for} \quad 0 \le y \le 0.08z \\ 1.4 & \text{for} \quad 0.08z < y \le 0.16z \\ 1.8 & \text{for} \quad 0.16z < y \le 0.24z \\ 2.2 & \text{for} \quad 0.24z < y \le 0.4z \\ 2.6 & \text{for} \quad 0.4z < y \le 0.56z \\ 0.44 & \text{for} \quad 0.56z < y \le 0.72z \\ 0.32 & \text{for} \quad 0.72z < y \le z \end{cases} \tag{2}$$

Eq. (2) presented the royalty payment with its constraint that is in terms of the final state variable. The aforementioned system will be transformed into a continuous hyperbolic tangent system (tanh) and the function will be shown below.

$$\begin{aligned} \rho(y) = 0.16 &+ 0.7\tanh\left(k\left(y-0.08z\right)\right) + 0.2\tanh\left(k\left(y-0.16z\right)\right) + 0.2\tanh\left(k\left(y-0.24z\right)\right) \\ &+ 0.2\tanh\left(k\left(y-0.4z\right)\right) - 1.08\tanh\left(k\left(y-0.56z\right)\right) - 0.06\tanh\left(k\left(y-0.72z\right)\right) \end{aligned} \tag{3}$$

Eq. (3) expressed the continuous royalty payment function which is in hyperbolic tangent function with k is denoted as a smoothing value. Both Eq. (2) and Eq. (3) will be illustrated in graphical form, as shown in Figure 2.



(a) (b)

Fig. 2. Piecewise function in (a) discrete piecewise function (b) continuous piecewise function

Eq. (2) was represented as the discrete piecewise function. Now, we can see that the program cannot be differentiable at a certain timeframe when the piecewise function level increases. Therefore, the difficulty was overcome through continuous approximation of hyperbolic tangent (tanh) Eq. (3) and was represented as the continuous piecewise function in Figure 2. Here, the integrand g in Eq. (1) will be solved by inserting Eq. (3) as the royalty function, ρ.

The use of natural boundary conditions in the Calculus of Variations (CoV) was discussed by Malinowska *et al.,* [16]. Natural boundary conditions were applied using a broad framework that demonstrated how they might be utilized to get more precise answers to issues in the CoV. The authors also discuss the relationship between natural boundary conditions and OC problems. The paper includes several examples and numerical simulations to illustrate the use of natural boundary conditions in the CoV. In addition, they highlight the importance of considering natural boundary conditions in analysing variational problems. This brilliant finding was applied in order to resolve the non-zero final costate value, $p(T)$ by proving the fundamental theory to yield the new necessary condition for our non-standard OC problem.

Since $p(T) \neq 0$, mathematically, we need a new equation to solve the non-standard problem. Considering Eq. (1) with the condition of the unknown final state value, $y(T)$ [16], then

$$\frac{d}{dy} g_{\dot{y}}\left(t, y(t), \dot{y}(t), y(T)\right) = g_y\left(t, y(t), \dot{y}(t), y(T)\right) \tag{4}$$

In addition, by Malinowska *et al.,* [16],

$$\begin{aligned} & g_{\dot{y}}\left(\rho(T), y(\rho(T)), \dot{y}(\rho(T)), y(T)\right) + \mu\left(\rho(T) f_y\left(\rho(T)\right), y\left(\rho(T)\right), \dot{y}\left(\rho(T)\right), y(T)\right) \\ & + \int_a^T g_z\left(t, y(t), \dot{y}(t), y(T)\right) dt = 0 \end{aligned} \tag{5}$$

From the properties of delta integral, it follows that.

$$\int_{\rho(T)}^{T} g_y(\dots)\,dt = \mu(\dots) \tag{6}$$

Given that $g_y(\dots)$ is a continuous function, thus,

$$\int_{\rho(T)}^{T} g_y(\dots)\,dt = 0 \tag{7}$$

When the second term of Eq. (5) reduces to zero, as stated in Eq. (7), hence

$$g_{\dot{y}}\left(T, y(T), \dot{y}(T), y(T)\right) = -\int_a^T g_z\left(t, y(t), \dot{y}(t), y(T)\right)dt \tag{8}$$

Eq. (8) was reduced to $g_z\left(t, y(t), \dot{y}(t), y(T)\right) = 0$, since the standard OC problem, the function does not rely upon the final state value, $y(T)$. Due to the fact that the standard OC problem has a final costate value, $p(T) = 0$, the additional necessary condition can be expressed as

$$p(T) = -\int_{t_0}^{T} g_z\,dt \tag{9}$$

By considering the ODE system,

$$\dot{y}(t) = u(t),\ y(0) = 0 \tag{10}$$

and the integrand component in Eq. (1) is subject to

$$a(t) = e^{0.025t}, \alpha = 0.5, m_0 = 1, c_0 = 1, w = 0.12, r = 0.1 \tag{11}$$

Now, the integrand of the objective function can be expressed as in Eq. (11).

$$g = \left(e^{0.025t}u^{0.5} - \begin{pmatrix} 0.7\tanh\left(k\left(y-0.08z\right)\right) + 0.2\tanh\left(k\left(y-0.16z\right)\right) \\ +0.2\tanh\left(k\left(y-0.24z\right)\right) + 0.2\tanh\left(k\left(y-0.4z\right)\right) \\ -1.08\tanh\left(k\left(y-0.56z\right)\right) - 0.06\tanh\left(k\left(y-0.72z\right)\right) \\ +e^{-0.12y} + 1.16 \end{pmatrix} u \right) e^{-0.1t} \tag{12}$$

Hence, the final costate value can be described as

$$p(T)=\int_{t_0}^{T}\begin{bmatrix}\left(-0.056k\left(1-\tanh\left(k\left(y-0.08z\right)\right)^2\right)-0.032k\left(1-\tanh\left(k\left(y-0.16z\right)\right)^2\right)\right.\\ -0.048k\left(1-\tanh\left(k\left(y-0.24z\right)\right)^2\right)-0.08k\left(1-\tanh\left(k\left(y-0.4z\right)\right)^2\right)\\ \left.+0.6048k\left(1-\tanh\left(k\left(y-0.56z\right)\right)^2\right)+0.0432k\left(1-\tanh\left(k\left(y-0.72z\right)\right)^2\right)\right)ue^{-0.1t}\end{bmatrix}dt \tag{13}$$

In accordance, Kirk [17] stressed a useful Hamiltonian function $H=g\left(t,y(t),\dot{y}(t),y(T)\right)+p(t)u(t)$ in solving the non-standard OC problem. Thus,

$$H=\left(e^{0.025t}u^{0.5}-\begin{pmatrix}0.7\tanh\left(k\left(y-0.08z\right)\right)+0.2\tanh\left(k\left(y-0.16z\right)\right)\\ +0.2\tanh\left(k\left(y-0.24z\right)\right)+0.2\tanh\left(k\left(y-0.4z\right)\right)\\ -1.08\tanh\left(k\left(y-0.56z\right)\right)-0.06\tanh\left(k\left(y-0.72z\right)\right)\\ +e^{-0.12y}+1.16\end{pmatrix}u\right)e^{-0.1t}+p(t)u(t) \tag{14}$$

where $p(t)$ is the costate variable and $u(t)$ is the control variable. In mathematics and physics, a Hamiltonian function (or simply Hamiltonian) is a mathematical function that describes the dynamics of a physical system in terms of generalized coordinates and their conjugate momenta. It plays a fundamental role in formulating classical mechanics and is typically denoted by H. The Hamiltonian function provides a powerful mathematical tool for describing and analysing the dynamics of physical systems, offering insights into the fundamental principles of mechanics and related fields. Kirk [17] describes the behaviour of a Hamiltonian function as

$$\begin{aligned}&\dot{y}(t)=H_p\left(t,y(t),u(t),p(t)\right)\\ &\dot{p}(t)=-H_y\left(t,y(t),u(t),p(t)\right)\\ &H_u\left(t,y(t),u(t),p(t)\right)=0\end{aligned} \tag{15}$$

Therefore, we can summarize that

$$\begin{aligned}&\dot{y}(t)=u(t)\\ &\dot{p}(t)=\left(0.7k\left(1-\tanh\left(k\left(y-0.08z\right)\right)^2\right)+0.2k\left(1-\tanh\left(k\left(y-0.16z\right)\right)^2\right)\right.\\ &\qquad +0.2k\left(1-\tanh\left(k\left(y-0.24z\right)\right)^2\right)+0.2k\left(1-\tanh\left(k\left(y-0.4z\right)\right)^2\right)\\ &\qquad -1.08k\left(1-\tanh\left(k\left(y-0.56z\right)\right)^2\right)-0.06k\left(1-\tanh\left(k\left(y-0.72z\right)\right)^2\right)\\ &\qquad \left.-0.12e^{-0.12y}\right)ue^{-0.1t}\\ &u(t)=0.25\left(e^{0.025t}\right)^2\left(e^{-0.1t}\right)^2\left(\rho e^{-0.1t}+e^{-0.12y}e^{-0.1t}+e^{-0.1t}-p\right)^{-2}\end{aligned} \tag{16}$$

Assuming the initial and final times to be zero and ten, respectively, we aim to maximize Eq. (1). The function to be optimized can be expressed as

Maximize $$J\left[u(t)\right]=\int_{0}^{10}\left[\left(e^{0.025t}u^{0.5}-\begin{pmatrix}0.7\tanh\left(k\left(y-0.08z\right)\right)\\+0.2\tanh\left(k\left(y-0.16z\right)\right)\\+0.2\tanh\left(k\left(y-0.24z\right)\right)\\+0.2\tanh\left(k\left(y-0.4z\right)\right)\\-1.08\tanh\left(k\left(y-0.56z\right)\right)\\-0.06\tanh\left(k\left(y-0.72z\right)\right)\\+e^{-0.12y}+1.16\end{pmatrix}u\right)e^{-0.1t}\right]dt \tag{17}$$

Table 1 displays the ideal outcomes for modified shooting and discretization techniques.

Table 1
Optimal results

Methods	Final state value, $y(T)$	Objective function, $J(T)$	Initial costate value, $p(0)$	Final costate value, $p(T)$
Modified shooting method				
SAPGRA	0.317988	0.606969	-1.254380	-0.356766
Discretization method				
Euler	0.319074	0.611833	-1.26011	-
Runge-Kutta	0.323087	0.612860	-1.28002	-
Trapezoidal	0.322194	0.614032	-1.28354	-
Hermite-Simpson	0.328815	0.61295	-1.25833	-

Comparing the modified shooting method to the Euler method, Table 1 shows that it produces an optimal final state value that is within two (2) decimal places. The modified shooting method's computation of the optimal objective function, however, is equal to the discretization findings up to one (1) decimal point. Simultaneously, the Hermite-Simpson approximation and the modified shooting method's initial costate yield are comparable up to two (2) decimal places. Finally, it can be concluded that the objective function can be maximized up to 61% to meet the optimality condition.

In comparison to the modified shooting approach, the discretization method produced less accurate results during the computing process, as seen in Figure 3. This is most likely a result of the process' discretization error [18,19]. Overall, the discretization methods can be applied to solve the OC problem since the delivered results were optimal and similar to the modified shooting method.

Based on the findings, royalty payments can be applied in fields such as the oil and gas industry, where developers pay royalties to property owners. Book publishing is another good example of a place where the publisher pays royalties to the authors for each book sold.

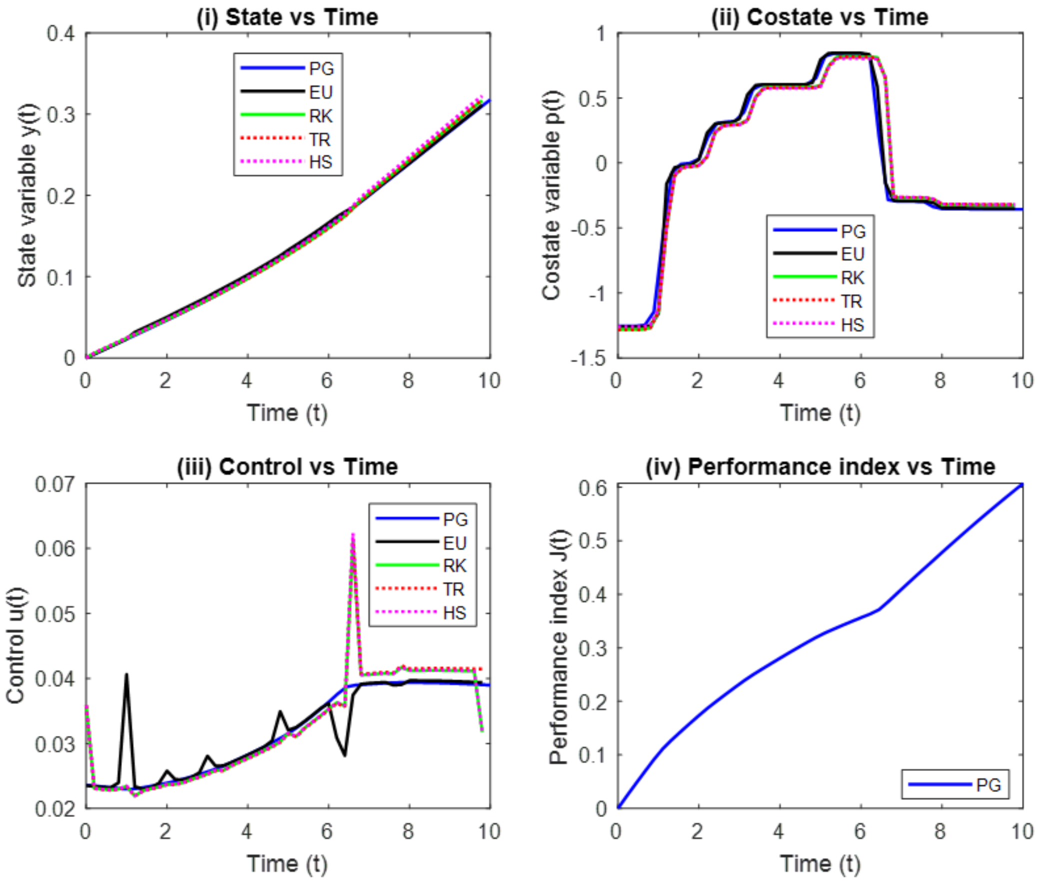


Fig. 3. Plot for optimal state, costate, control and objective function

6.1 Discussion and Limitations

Although the modified shooting method using the combination of Powell and Golden Section Search method yields a more accurate optimal solution, the method tends to be slow in convergence. The computation process takes a longer time compared to discretization methods. Therefore, in the future, other sophisticated methods, such as the Newton method, which was studied by Cheng *et al.,* [20], will be considered to replace the Powell method.

7. Concluding Remark

In conclusion, the modified shooting method delivered a more accurate result that satisfied the optimality condition compared to discretization methods. This study successfully addressed the non-standard OC problem and met the optimality requirement. The unknown final state value was computed, while the new necessary condition was proven in order to resolve the issue related to the final costate value that is not equal to zero. Additionally, the research considers the difficulty of paying the royalty through the piecewise function. Therefore, to overcome the difficulty, the continuous approximation of the hyperbolic tangent (tanh) approach was used. These innovative results could serve as a basis for future researchers to create novel mathematical methods for solving complex economic problems in the real world. Furthermore, the academic field can be more advanced as the problem-solving method is current and up to date. Meanwhile, this research also addresses the importance of fundamental theory in settling real economic problems. These criteria are essential to make a fair royalty payment aside from continuous payment.

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