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Analytical Solutions for Fractional Caputo-Fabrizio Casson Nanofluid on Riga Plate with Newtonian Heating

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ABSTRACT

Introduction of fractional derivatives to the mechanics of fluid flow is relatively new. Even though the exact geometrical representations of fractional derivatives on fluid mechanics have not been discovered, recent literatures have proven that it is a paradox that will be useful in the future. Meanwhile, Riga plates are actuators that is convenient for controlling the velocity of fluid flows. Widely used in the field of marine engineering, the properties of fluid flowing over Riga plates are worth investigating. Thus, the aim of this study is to investigate the analytical solutions of an unsteady incompressible Casson nanofluid flowing over a Riga plate with presence of Newtonian heating. Carboxymethyl Cellulose (CMC) water was used as a prime example of Casson fluid with Copper-Oxide (CuO) nanoparticles. Coupled with a non-Newtonian fluid, the Casson fluid, and the Caputo-Fabrizio fractional derivative, the analytical solutions obtained will be beneficial in the engineering world as a tool for validating experimental and numerical studies. Through this study, analytical solutions were obtained and the profiles of both velocity and temperature of fluid with variations in parameters were investigated. It is observed that variations in the fractional derivative parameter produces a spectrum of solutions that abides the initial and boundary conditions set. An amplification of the modified Hartmann number increases both the velocity and temperature profiles, while an amplification of the nanoparticle volume fraction decreases the velocity profile but increase the temperature profile.

Keywords:
Caputo-Fabrizio fractional derivative; Casson nanofluid; Riga plate. Newtonian heating; Laplace transform

1. Introduction

Fluids are generally divided into two categories, Newtonian and non-Newtonian fluids. Newtonian fluids are fluids that behave according to Newton's principle of viscosity, and non-Newtonian fluids are the opposite of a Newtonian fluid. Due to its complexity, there is no comprehensive fluid model that observes every property of a non-Newtonian fluid. Thus, there exists multiple fluid models that focuses on specific behaviours of a non-Newtonian fluid. Maxwell, Orloyd-B, Jefrey and Casson fluid are a few examples of a non-Newtonian fluid model [1]. The most commonly used fluid model is the Casson fluid model. A Casson fluid displays behaviour of a visco-plastic fluid. When shear stress applied on the fluid is less than yield stress, the fluid shows behavior

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of plasticity. On the contrary, the fluid will show Newtonian fluid like behaviour if shear stress applies is greater than yield stress [2]. Extensive analytical studies on Casson fluid have been done over the past few decades. For example, Kumar and Pai [3] conducted a study on Casson fluid flowing through a circular porous bearing. Mohamad et al [4] published a study on a rotating Casson fluid flowing through a disk channel. Hussanan *et al*., [5] introduced the effect of Newtonian heating to Casson fluid flow and treated the problem analytically via Laplace transform. With many other possibilities for application of Casson fluid flow, this area of study has been proven to be very beneficial to the ever-developing scientific community. Another possible attribution to Casson fluid flow, is the introduction of a Riga plate.

A fluid flow can be controlled by introducing an actuator into the system. A prime example of an actuator is a Riga plate, a plate made out of electrodes and magnets arranged alternatingly [6]. With the presence of electrodes and magnets, Riga plates are capable of generating Lorentz force from the resulting electromagnetic field. Depending on its position, Riga plates are able to regulate fluid flow. The most advance usage of Riga plates can be seen on a submarine. Fluid flowing through the submarine can be controlled with the aid of Riga plates reducing turbulence and drag on the submarine [7]. This application is made possible through fundamental studies such as the study of boundary layer flow of a fluid over a Riga plate.

One of the earliest study on Casson fluid flowing over a Riga plate was done by Loganathan and Deepa [8]. Analytical solutions were obtained through Laplace transform for an unsteady Casson fluid flow over a vertical Riga plate. The study observes an increase in fluid velocity with presence of Riga plate. This is due to the direction of the Lorentz force generated by the Riga plate where it is parallel to fluid flow, aiding the velocity of the fluid. The study was repeated with a porous Riga plate with similar findings [9]. Another similar study with similar findings considered an impulsively started horizontal Riga plate [10]. Other studies with similar findings on boundary layer flow over a Riga plate includes Loganathan and Deepa [11], Yusof *et al*., [12], Bilal *et al*., [13], Asogwa *et al*., [14] and Reyaz *et al*., [15]. Almost all of the studies mentioned did not consider the prospect of fractional derivatives into an unsteady Casson fluid flow over a Riga plate.

An unconventional derivative, known as fractional derivative, can provide solutions that are applicable to current engineering or mathematical problems. A fractional, arbitrary or complex number order of a typical derivative is the definition of a fractional derivative. The concept was first introduced by Leibniz and L'Hospital [16]. Since the discovery, many mathematicians have been trying to mathematically define a fractional derivative, producing multiple definitions that can used for various problems. These definitions include the Abel, Riemann, Riemann-Liouville, Caputo and Caputo-Fabrizio fractional definition [17]. Although the fractional derivative is purely theoretical, past studies have proven that by considering fractional derivative into existing engineering problems, a different paradoxical solution is obtained, commencing boundless possibilities for future studies on fractional derivative [19].

Specifically in the study of boundary layer flow of fluid mechanics, the geometrical representation of fractional derivative is still unknown. That being said, referencing past literature, it is not an area that is not worth studying. Over the past few years, studies on fractional derivative on various types of fluid and geometry has been published. The pioneer study on boundary layer flow with fractional derivative was conducted by Khan *et al*., [20]. A study on Casson fluid flow over an infinite vertical plate was conducted and the Caputo fractional derivative was considered. Findings of the study suggested that obtained analytical solutions, with special Wright functions integrated within, can be particularized for any typical Casson fluid. The study was later extended to obtain analytical solutions for Caputo fractional derivative on Casson fluid within a microchannel [21]. The same approach was adapted by Ali *et al*., [22] when studying a free convection flow of Casson fluid. The authors later continued the study by applying the model to a blood flow problem within an axisymmetric cylindrical channel [23]. A similar study was conducted by Maiti *et al*., [24], however instead of Caputo, the non-singular kernel of Caputo-Fabrizio were considered. Other similar studies on fractional Casson fluid includes Abdullah *et al*., [25], Jamil *et al*., [26], Sene [27] and Reyaz *et al*., [28].

Studies above did not take into consideration the effects of introducing nanofluids into their problem. A nanofluid, a fluid containing nano sized particles, are considered to be a very effective solution for an inexpensive but highly conductive fluid [29,30]. Nanoparticles acts as a conductor that aids the process of heat transfer. One of the most stable and common nanofluid used is the Copper-Oxide (CuO) nanofluid [31-35]. Nanofluid was introduced to a fractional Casson fluid flow model by Aman et al [36]. In this study, a sodium alginate base solution was considered as the Casson fluid. Copper and Alumina were the nanoparticles introduced into the sodium alginate fluid. Analytical solutions were obtained and findings suggested that with the introduction of nanoparticles, heat transfer rate were elevated while fractional derivative parameter provided a dissimilar solution. Similar findings were obtained by Raza *et al*., [37] when kerosene oil and water based fractional Casson fluid were investigated. Other studies on fraction nanofluid model includes Anwar and Rasheed [38], Aman *et al*., [39], Abro and Khan [40] and Fetecau *et al*., [41]. Studies where Carboxymethyl Cellulose (CMC) water was used as a prime example of Casson fluid have suggested that this particular fluid ideally resembles properties of non-Newtonian fluid exhibited by a Casson fluid [42-44]. However, CMC water based nanofluids with CuO nanoparticle and fractional derivatives are highly scarce.

Thus, the objective of this study is to conduct an analytical investigation on an unsteady CMC water based nanofluid with CuO nanoparticles flowing over a vertical Riga plate treated with the Caputo-Fabrizio fractional derivative via the Casson fluid model with effects of Newtonian heating. Newtonian heating is considered due to its ideal properties of heat transfer between plate surface and fluid [45,46]. The Zakian's method of inverse Laplace transform is employed to generate final graphical solutions [47-49]. This is to overcome the complexity of the final analytical solution in Laplace domain as well as reduce run time on graphical software. Contribution in this study includes analytical solution in Laplace domain for fractional Casson nanofluid over Riga plate with Newtonian heating effect as well the introduction of the Caputo-Fabrizio fractional derivative to the analytical study of an unsteady Casson nanofluid flowing over a vertical Riga plate with Newtonian heating effect.

2. Problem Formulation

An unsteady free convection flow of a Casson nanofluid past a semi-infinite vertical Riga plate is considered with Carboxymethyl cellulose-water (CMC-water) as the base solution and Copper Oxide (CuO) nanoparticles as the solid particles. The x-axis is taken along the Riga plate in the vertical direction and the y-axis is taken normal to the plate. The fluid is considered to be flowing along the x-direction and to only occupy the space of $y > 0$. Initially, at $t = 0$, both the fluid and the plate are at rest and their temperature is $T(y,t) = 0$. When $t > 0$, the plate begins to move with a uniform velocity in the x-direction. Meanwhile, the heat transfer between the plate and the fluid is proportional to the local surface temperature, T. This effect is known as Newtonian heating. The electromagnetic field induced from the Riga plate, generates an upthrust Lorentz force, F. The Reynold number is assumed to be very minute. Therefore, the magnetic field induced by the movement of the fluid is negligible. A permeated uniform thermal radiation, q_r parallel to the x-axis is applied to the fluid. Velocity, U, and temperature, T, are dependent on space variable, y, and time, t. Figure 1 shows a geometrical representation of the fluid flow as well as an example of a Riga plate.

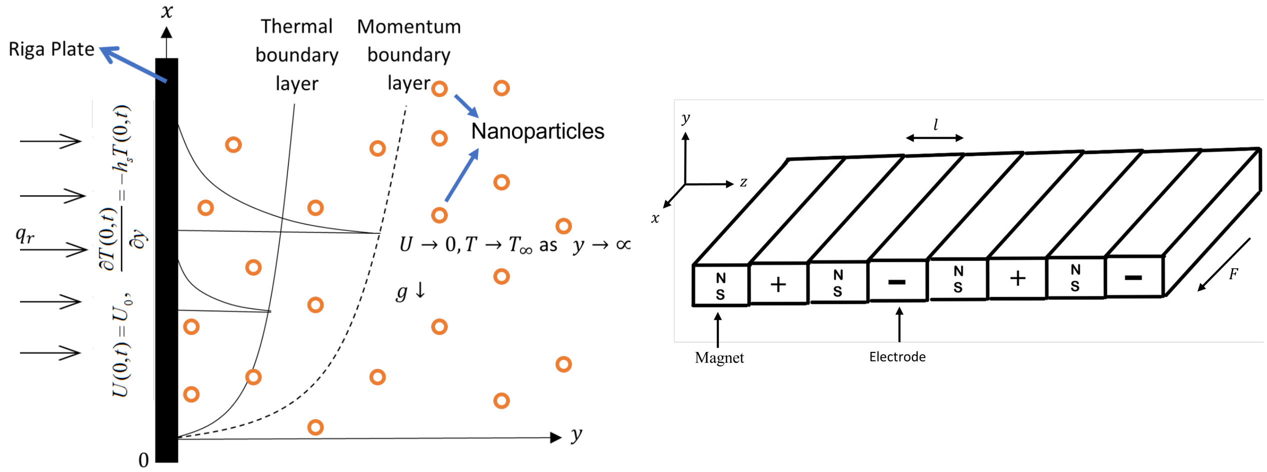

Fig. 1. Representation of unidirectional Casson nanofluid flow over a vertical Riga plate and an example of a Riga plate

Based on these assumptions and taking the Boussinesq's approximation into consideration, the governing momentum and energy equations are written as:

$$\rho_{nf} \frac{\partial U(y,t)}{\partial t} = \mu_{nf}\left(1+\frac{1}{\beta}\right)\frac{\partial^2 U(y,t)}{\partial y^2} + g(\rho\beta_T)_{nf}(T-T_\infty) + \frac{\pi J_0 M_0}{8}\exp\left(-\frac{\pi}{l}y\right), \tag{1}$$

$$\frac{\partial T(y,t)}{\partial t} = k_{nf}\frac{1}{\left(\rho C_p\right)_{nf}}\frac{\partial^2 T(y,t)}{\partial y^2} - \frac{\partial q_r}{\partial y}. \tag{2}$$

Both Eq. (1) and Eq. (2) are subjected to conditions:

$$\begin{aligned} &u(y,0)=0, && T(y,0)=0, && \forall y>0,\\ &u(0,t)=U_0, && \frac{\partial T(0,t)}{\partial y} = -h_s T(0,t), && \forall t>0,\\ &u(\infty,t)\to 0, && T(\infty,t)\to T_\infty, && \forall t>0. \end{aligned} \tag{3}$$

Where ρ is the density of fluid, μ is the dynamic viscosity, β is the Casson fluid parameter, g is the gravitational acceleration, β_T is the thermal expansion coefficient, J_0 is the density of electrical current, M_0 is the magnitude of magnetization of magnets, l is the width of magnets and electrodes of the Riga plate, C_p is the specific heat capacity of the fluid at a constant density, k is the thermal conductivity parameter and q_r is the thermal radiation parameter. The subscript of nf signifies the parametric values for the nanofluid and are defined as follows:

$$\mu_{nf} = \frac{\mu_f}{(1-\varphi)^{2.5}}, \tag{4}$$

$$\rho_{nf} = (1-\varphi)\rho_f + \varphi\rho_s, \tag{5}$$

$$(\rho\beta_T)_{nf} = (1-\varphi)(\rho\beta_T)_f + \varphi(\rho\beta_T)_s, \tag{6}$$

$$(\rho C_P)_{nf} = (1-\varphi)(\rho C_P)_f + \varphi(\rho C_p)_s, \tag{7}$$

$$k_{nf} = k_f\left[\frac{(k_s+2k_f)-2\varphi(k_f-k_s)}{(k_s+2k_f)+\varphi(k_f-k_s)}\right], \tag{8}$$

The symbol φ, signifies nanoparticle volume fraction. Meanwhile, the subscript f, and s, symbolizes thermophysical properties for base fluid and nanoparticle, respectively. The thermophysical properties of base fluid, f, and nanoparticles, s, are defined in Table 1 as follows:

Table 1
Thermophysical properties of base fluid and nanoparticle

Thermophysical properties	Base fluid, f Carboxymethyl Cellulose Water (CMC-Water)	Nanoparticle, s Copper Oxide (CuO)
ρ (kg/m^3)	997.1	6320
C_P (J/Kg · K)	4179	531.8
k (W/m · K)	0.613	76.5
β_T (K^{-1})	21	1.8

All of Eq. (4) to Eq. (8) is employed to Eq. (1) and (2) to reduce them to a non-dimensional equation, so that it may be solved analytically. Also, Rosseland's approximation for thermal radiation is then utilised. The resulting radiative heat flux from Rosseland's approximation is defined as:

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}. \tag{9}$$

Where both σ^* and k^* are the Stefan-Boltzmann constant and mean absorption coefficient. Eq. (9) is used in the governing energy equation from Eq. (2). Next, a set of non-dimensional parameters is applied to both Eq. (1) and Eq. (2) to convert them into a pair dimensionless governing equations. The non-dimensional parameters are defined as:

$$U^* = \frac{U}{U_0}, \quad t^* = \frac{U_0}{\upsilon}t, \quad y^* = \frac{U_0}{\upsilon}y, \quad T^* = \frac{T-T_\infty}{T_\infty}. \tag{10}$$

After applying Eq. (10) into both Eq. (1) and the modified Eq. (2), and removing the asterisk, the Caputo-Fabrizio fractional derivative is applied and is define as:

$$D_t^{\alpha} f(x,t) = \frac{1}{1-\alpha} \int_0^t \frac{\partial f(x,s)}{\partial x} exp\left(-\alpha \frac{t-s}{1-\alpha}\right) ds \tag{11}$$

is applied. The final governing momentum and and energy equations in dimensionless forms are written as:

$$D_t^{\alpha} U(y,t) = \frac{\phi_3}{\phi_4} \beta_0 \frac{\partial^2 u(y,t)}{\partial y^2} + \frac{\phi_5}{\phi_4} Gr\, T(y,t) + \frac{1}{\phi_4} E \exp\left(-L\, y\right), \tag{12}$$

$$D_t^{\alpha} T(y,t) = \mathrm{Pr}_0 \frac{\partial^2 T(y,t)}{\partial y^2}. \tag{13}$$

Both Eq. (12) and Eq. (13) are bounded by dimensionless conditions:

$$\begin{aligned} &U(y,0) = 0, \quad T(y,0) = 0, && \forall y > 0, \\ &U(0,t) = 1, \quad \frac{\partial T(0,t)}{\partial y} = -\gamma\left[1 + T(0,t)\right], && \forall t > 0, \\ &U(\infty,t) \to 0, \quad T(\infty,t) \to 0, && \forall t > 0, \end{aligned} \tag{14}$$

Where $D_t^{\alpha}(\cdot)$ is the fractional derivative notation where α is the fractional parameter, ϕ_i for $i = 1,2,3,4,5$ is the dimensionless nanoparticle volume fraction, β_0 is the dimensionless Casson parameter, Gr is the Grashof number, E is the modified Hartmann number, L is a dimensionless constants for width of electrodes and magnets, Pr_0 is a dimensionless constant and γ is the Newtonian heating parameter. These dimensionless constants are better defined as:

$$\begin{aligned} &\beta_0 = 1 + \frac{1}{\beta}, \quad Gr = \frac{(\rho\beta_T)_f}{\rho_f} \frac{\upsilon_f g}{U_0^3} T_\infty, \quad E = \frac{\upsilon \pi J_0 M_0}{8 U_0 \rho_f}, \quad L = -\frac{\upsilon \pi}{U_0 l}, \quad \mathrm{Pr}_0 = \frac{1}{\mathrm{Pr}}\left(\frac{\phi_2}{\phi_1} + \frac{4}{3}\frac{N}{\phi_1}\right), \\ &\phi_1 = \frac{(\rho C_P)_{nf}}{(\rho C_P)_f}, \quad \phi_2 = \frac{k_{nf}}{k_f}, \quad \phi_3 = \frac{\mu_{nf}}{\mu_f}, \quad \phi_4 = \frac{\rho_{nf}}{\rho_f}, \quad \phi_5 = \frac{(\rho\beta_T)_{nf}}{(\rho\beta_T)_f}. \end{aligned} \tag{15}$$

Here, Pr is the Prandtl number and N is the dimensionless thermal radiation parameter.

3. Analytical Solutions

Solving Eq. (12) and Eq. (13) requires a few conventional Laplace transform formulas as well as the Laplace transform of the Caputo-Fabrizio fractional derivative from Eq. (11). The Laplace transform of Eq. (11) is written as:

$$\mathrm{L}\{D_t^{\alpha} f(y,t)\} = \frac{q\bar{f}(y,q) - f(y,0)}{q + \alpha(1-q)}. \tag{16}$$

where the $\mathrm{L}\{\cdot\}$ notation signifies the Laplace transform. Via Laplace transform, a set of governing ordinary differential equations are obtained. Applying the method of undetermined coefficients, the resulting solutions of velocity and temperature in the frequency domain is written as:

$$\bar{U}(y,q)=\left[\frac{1}{q}-\frac{m_1}{\sqrt{\frac{1}{\Pr_0}\frac{a_0 q}{q+a_1}}-\gamma}\frac{1}{q}\left(\frac{q+a_1}{a_0 q}\right)-\frac{m_6}{q}\left(\frac{qm_4+m_5}{qm_2+m_3}\right)\right]\exp\left(-y\sqrt{\frac{1}{\beta_1}\frac{a_0 q}{q+a_1}}\right)$$
$$+\frac{m_1}{\sqrt{\frac{1}{\Pr_0}\frac{a_0 q}{q+a_1}}-\gamma}\frac{1}{q}\left(\frac{q+a_1}{a_0 q}\right)\exp\left(-y\sqrt{\frac{1}{\Pr_0}\frac{a_0 q}{q+a_1}}\right)+\frac{m_6}{q}\left(\frac{qm_4+m_5}{qm_2+m_3}\right)\exp\left(-Ly\right), \quad (17)$$

$$\bar{T}(y,q)=\frac{\gamma}{\sqrt{\frac{1}{\Pr_0}\frac{a_0 q}{q+a_1}}-\gamma}\frac{1}{q}\exp\left(-y\sqrt{\frac{1}{\Pr_0}\frac{a_0 q}{q+a_1}}\right). \quad (18)$$

Here, the variables of a_0, a_1 and m_i for $i=1,2,...,6$ is define as:

$$a_0=\frac{1}{1-\alpha},\quad a_1=\alpha a_0,\quad m_1=-\frac{\phi_5}{\phi_3}\frac{Gr\gamma}{\beta_0}\left(\frac{1}{\Pr_0}-\frac{\phi_4}{\phi_3}\frac{1}{\beta_0}\right)^{-1},\quad m_2=L^2\phi_3\beta_0-\phi_4 a_0,$$
$$m_3=L^2\phi_3\beta_0 a_1,\quad m_4=\phi_3\beta_0,\quad m_5=\phi_3\beta_0 a_1,\quad m_6=-\frac{E}{\phi_3\beta_0}. \quad (19)$$

As they are now, the solutions of velocity and temperature equation from Eq. (17) and Eq. (18) are too complex to perform analytical inverse Laplace transform. Thus, Zakian's method of inverse Laplace transform is employed. Zakian's inverse Laplace transform uses the following equation:

$$f(t)=\frac{2}{t}\sum_{j=1}^{n}\mathrm{Re}\left\{K_j F\left(\frac{\alpha_j}{t}\right)\right\}. \quad (20)$$

In this study, the value of n is taken as 5. Meanwhile, the values of K_j and α_j for $j=1,2,3...5$ is displayed in Table 2 below:

Table 2
Zakian's method of inverse Laplace transform parametric values

j	K_j	α_j
1	-36902.08210+196990.4257i	12.83767675+1.666063445i
2	61277.02524-95408.62551i	12.22613209+5.012718792i
3	-28916.56288+18169.18531i	10.93430308+8.409673116i
4	4655.361138-1.901528642i	8.776434715+11.92185389i
5	-118.7414011-141.3036911i	5.225453361+15.72952905i

Through Zakian's method of inverse Laplace transform, final graphical solutions are obtained and analysed.

4. Results and Discussions

Employing Zakian's method of inverse Laplace transform from Eq. (20) and using Mathcad 15 as a tool to graph out final solutions from Eq. (17) and Eq. (18), velocity and temperature profiles with variations of parameters are obtained and displayed in Figures 1 to Figure 12.

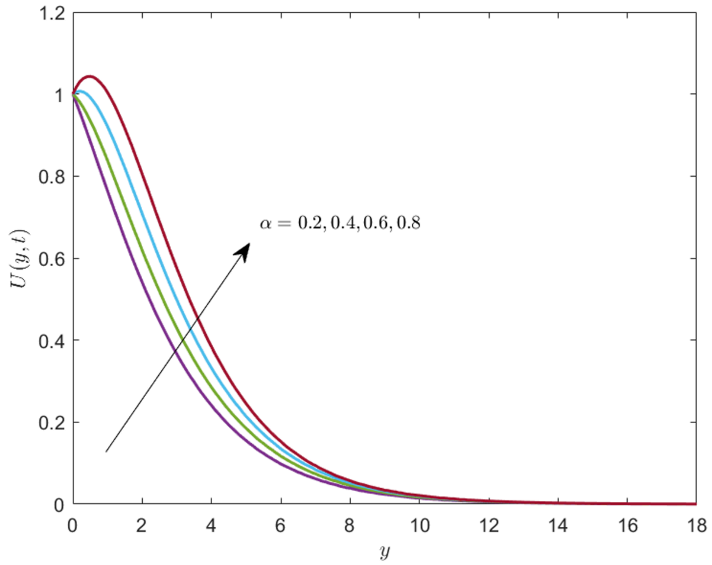

Fig. 2. Effects on the velocity profile with variations in fractional parameter, α

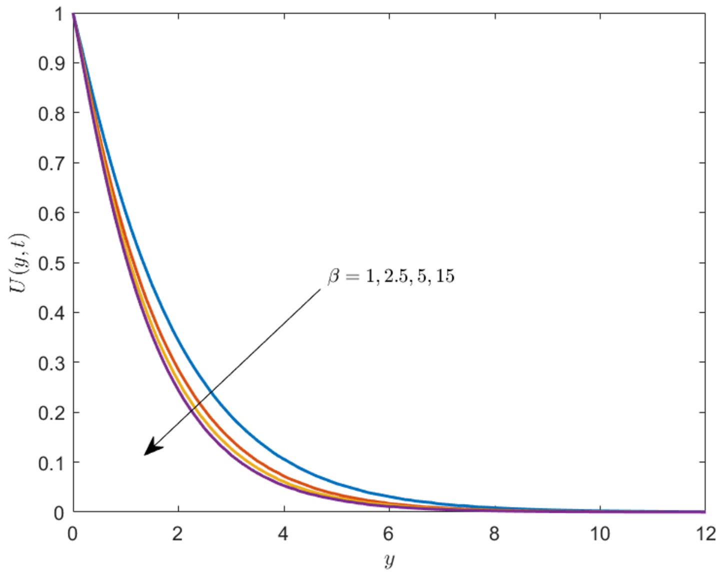

Fig. 3. Effects on the velocity profile with variations in Cassor parameter, β

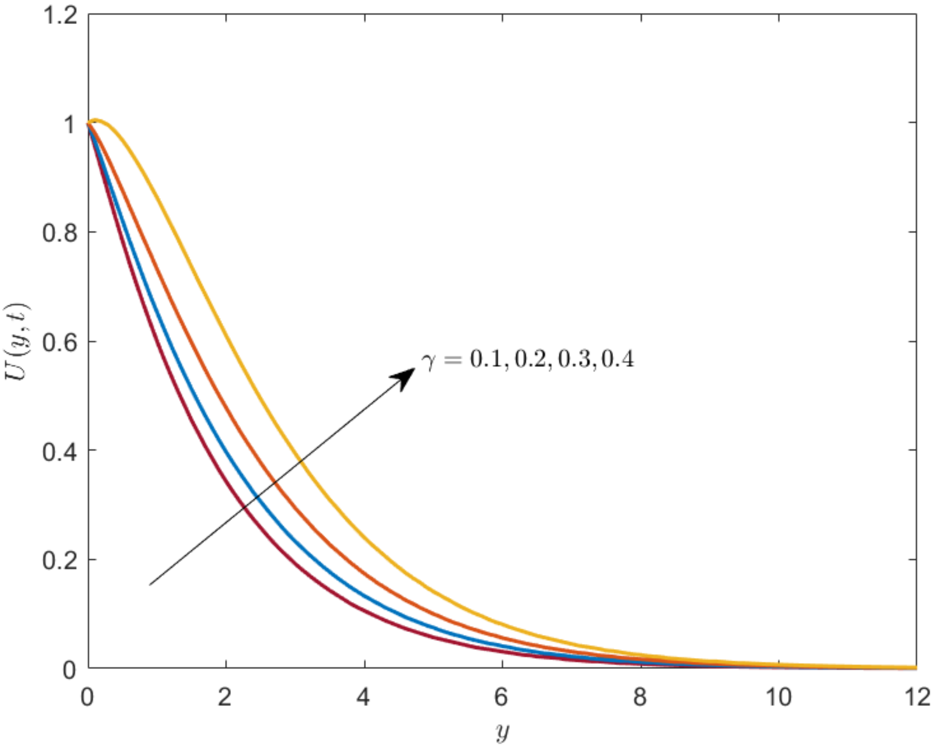

Fig. 4. Effects on the velocity profile with variations in Newtonian heating parameter, γ

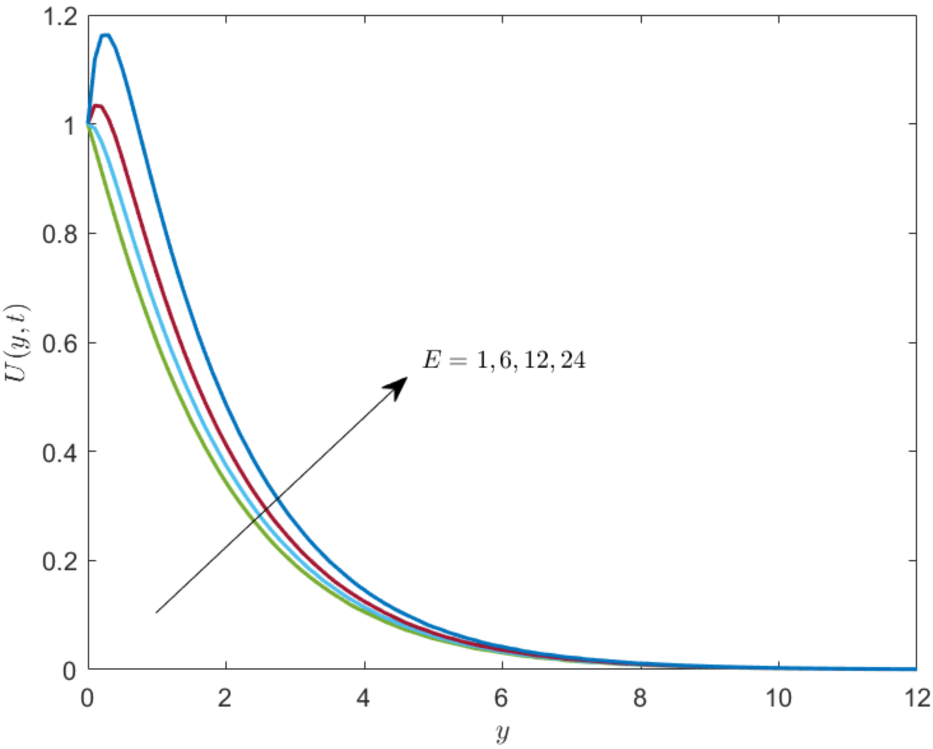

Fig. 5. Effects on the velocity profile with variations in the modified Hartman number, E

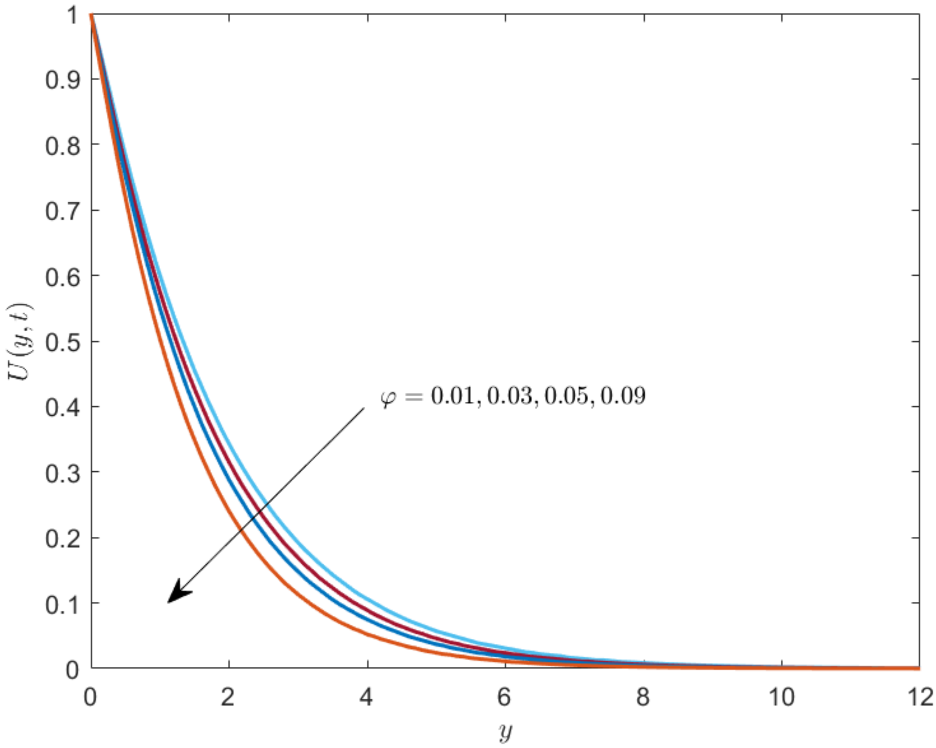

Fig. 6. Effects on the velocity profile with variations in nanoparticle volume fraction parameter, φ

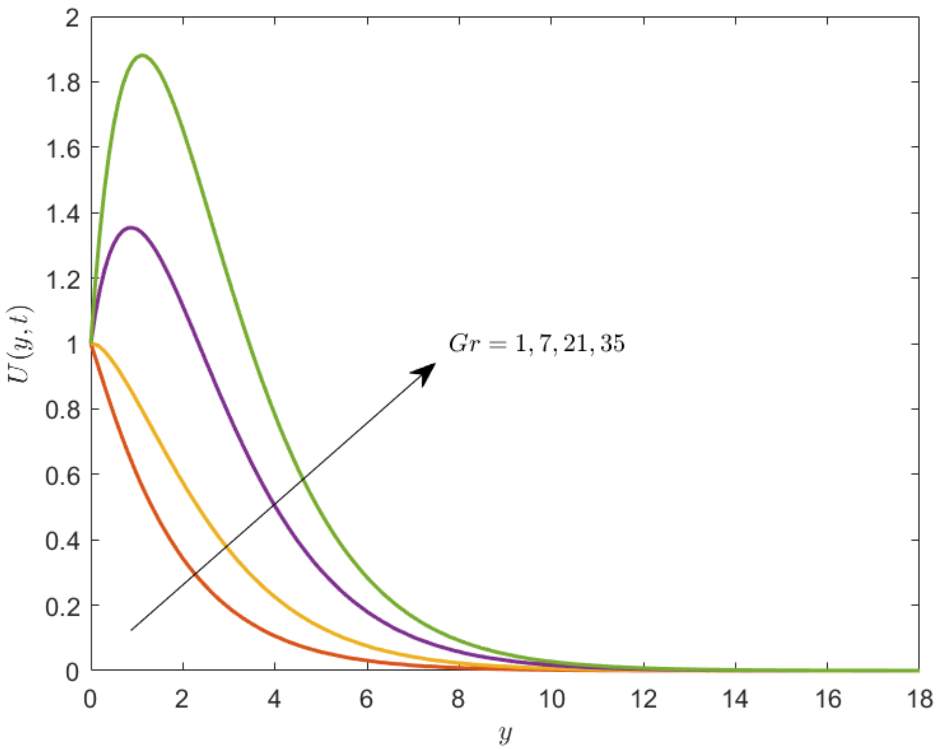

Fig. 7. Effects on the velocity profile with variations in the Grashof number, Gr

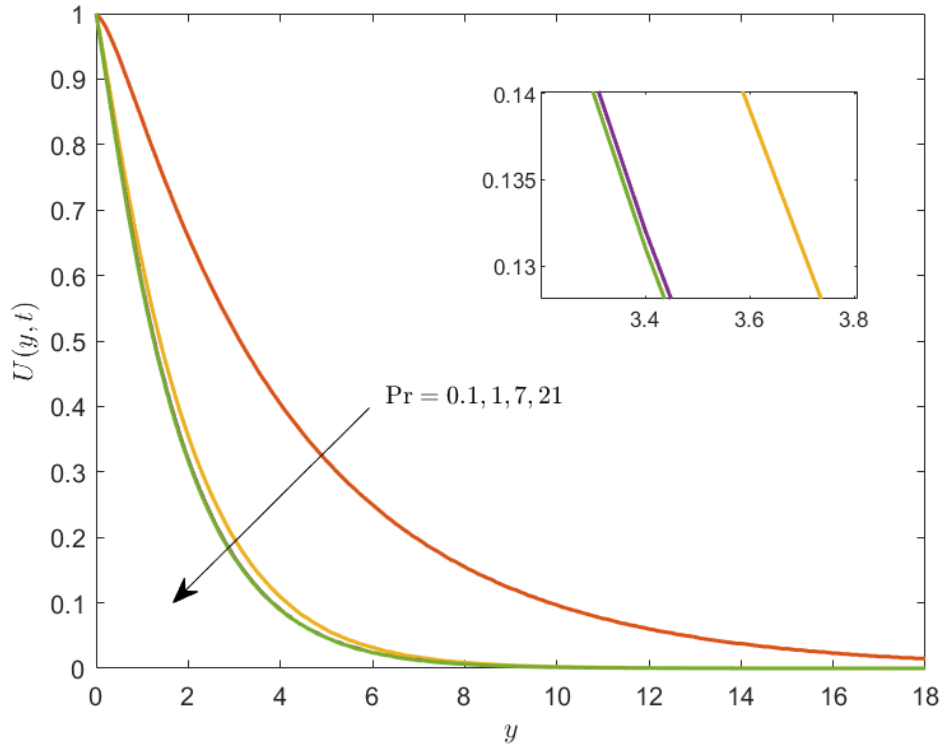

Fig. 8. Effects on the velocity profile with variations in the Prandtl number, $\Pr$

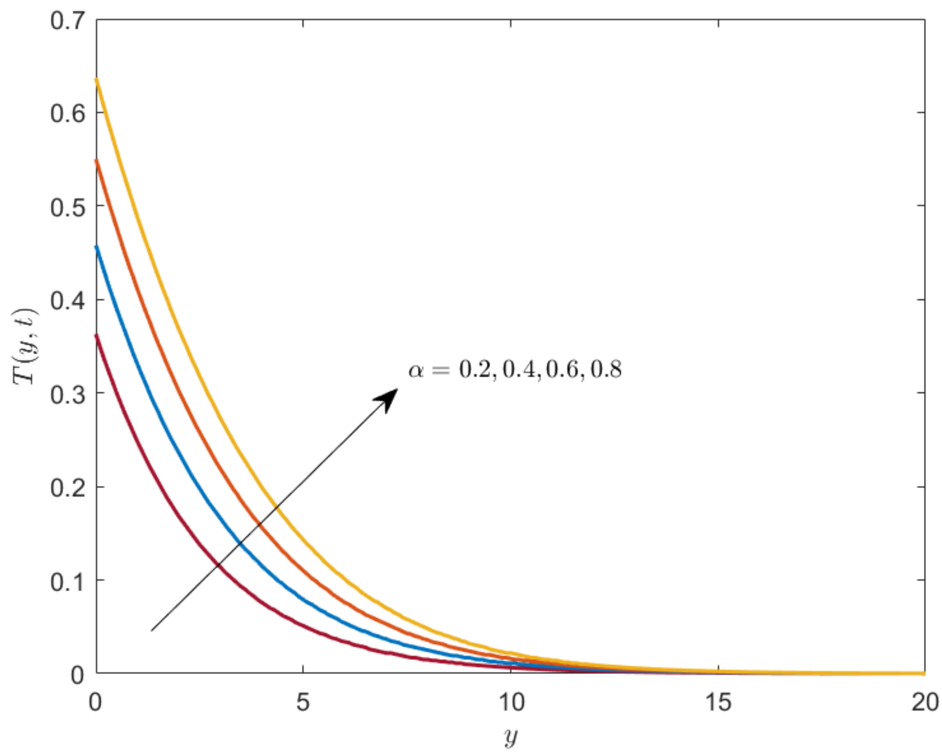

Fig. 9. Effects on the temperature profile with variations in fractional parameter, α

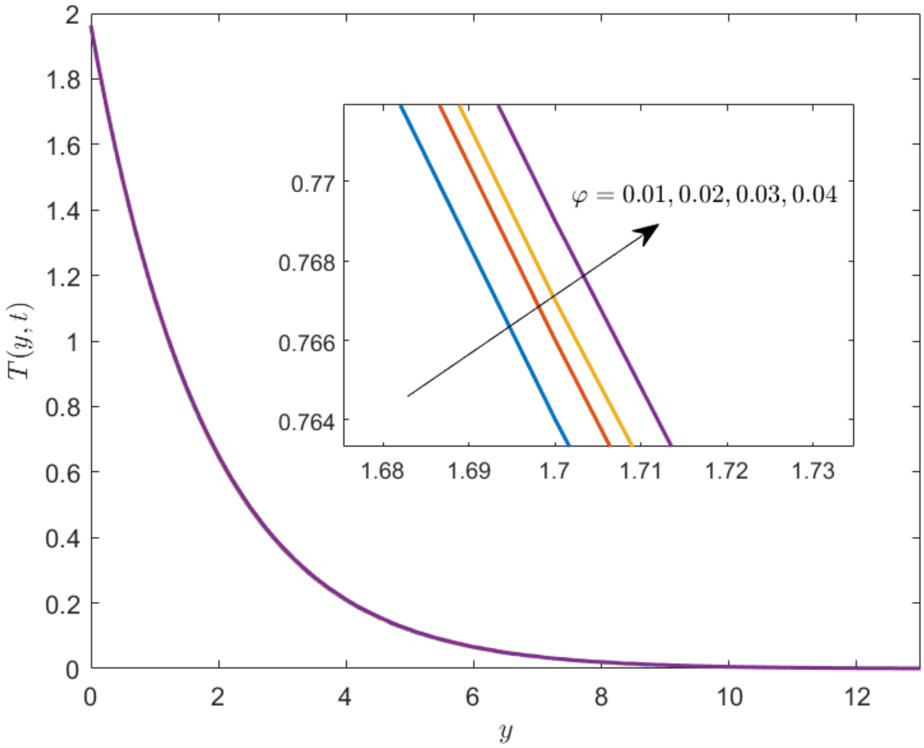

Fig. 10. Effects on the temperature profile with variations in nanoparticle volume fraction parameter, φ

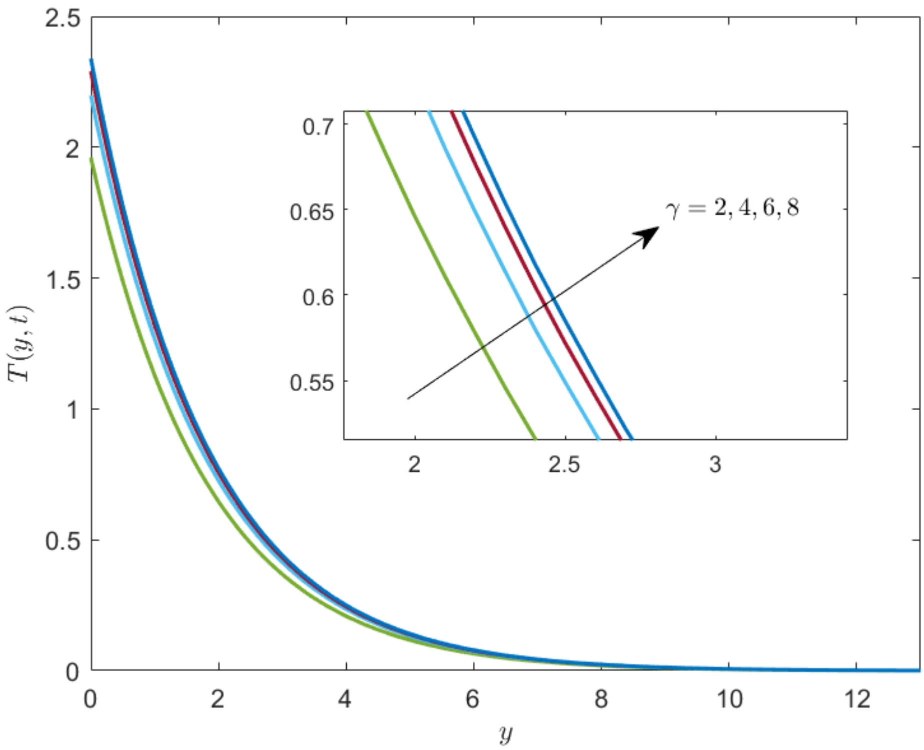

Fig. 11. Effects on the temperature profile with variations in Newtonian heating parameter, γ

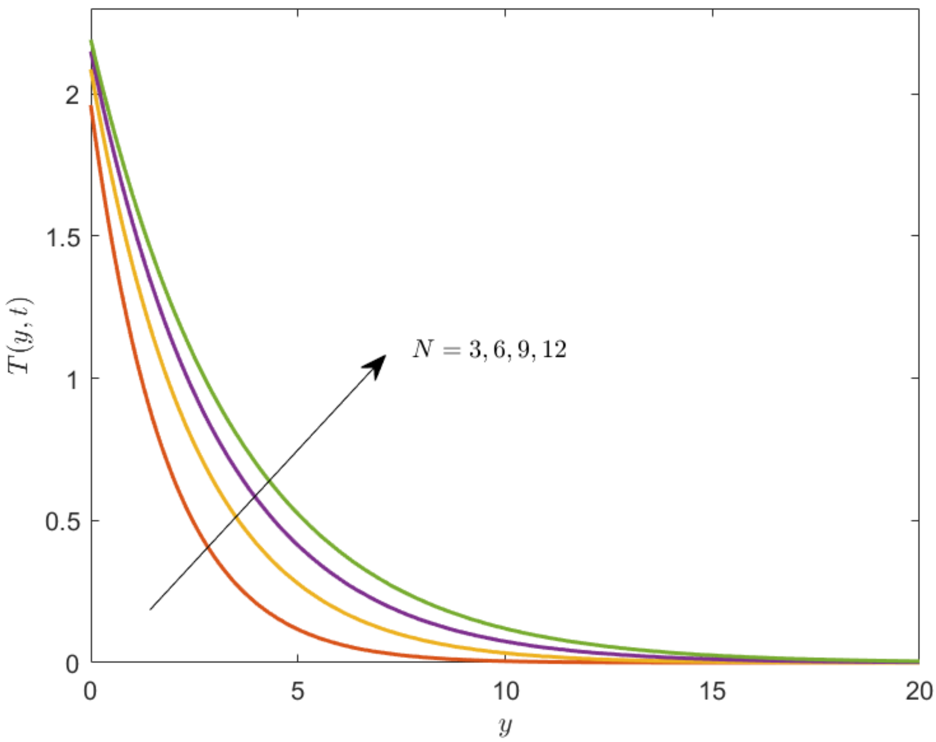

Fig. 12. Effects on the temperature profile with variations in thermal radiation parameter, N

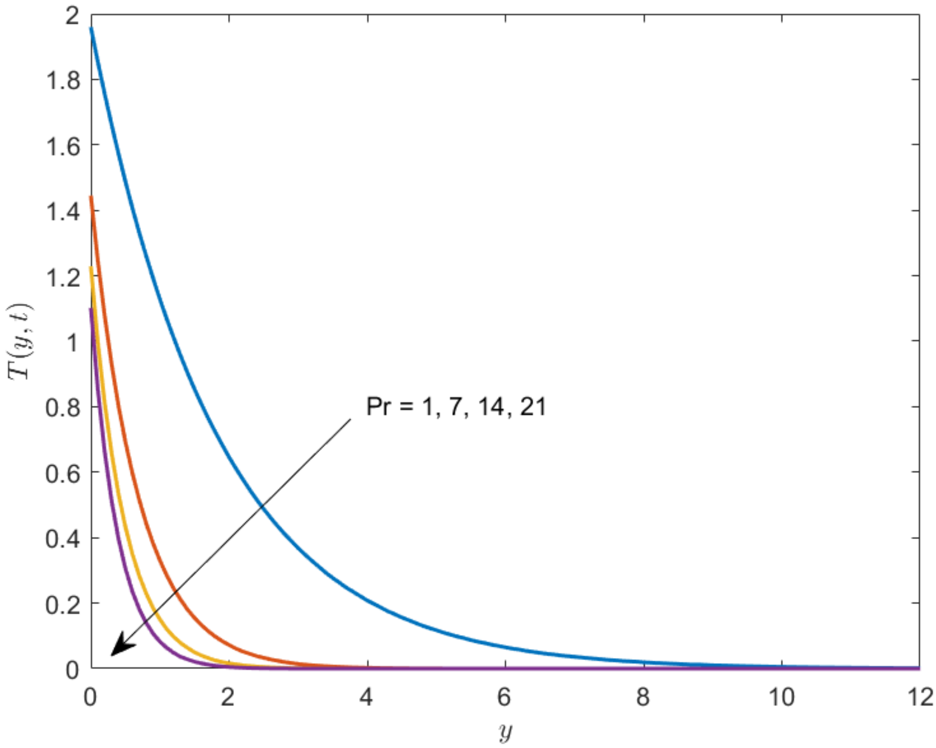

Fig. 13. Effects on the temperature profile with variations in the Prandtl number, Pr

The impact of fractional parameter, α is observed in Figure 2. As the value of α is increased, velocity of fluid is also increased. The geometrical representation of a fractional derivative in fluid is still unknown. However, according to Figure 2, there is a variation when the values of α is changed. This goes to show that the possibility of a physical representation in the future might exist. Till then, solutions of fractional derivative in fluid mechanics will be useful as a reference for experimental or numerical studies in the near future.

Meanwhile, Figure 3 showcases the fluid velocity with different values of Casson parameter, β. In this study, Carboxy-Methyl (CMC) water as base fluid is considered as a prime example of a Casson fluid. The parameter, β, determines the plasticity of the CMC fluid. As the value of β increases, so does the plasticity of the CMC fluid, becoming more and more rigid. As a result, with elevations of β, fluid velocity will decrease, as observed in Figure 3. It is also worth to note that a Casson fluid behaves such that it is viscous if shear stress applied is less than shear stress. The result in Figure 3 observes that as β increases, velocity profile of fluid decreases at a constant behaviour. Showcasing that shear stress applied did not cross the yield stress value.

On the other hand, Figure 4 shows that elevations in the Newtonian heating parameter, γ, fluid velocity will increase. Newtonian heating describes an ideal condition where the heat transfer from the surface of the plate is equal to that of the local surface temperature. When γ is elevated, the local surface temperature is increased, directly increasing heat transfer rate to fluid from surface. In turn, the kinetic energy within the fluid is increased and fluid velocity is amplified.

The influence of the modified Hartmann number, E, is observed in Figure 5. The presence of a Riga plate produces Lorentz force, an electromagnetically induced filed that is able to push and pull objects with magnetic properties. Increasing the modified Hartmann number signifies the amplification of Lorentz force output due to physical changes made to the Riga plate. A fluid is flowing along the direction of Lorentz force output increase fluid velocity as the Lorentz force aids fluid flow. In this study, the Riga late is considered to be producing a Lorentz force that is parallel to the fluid flow. Thus, an increase in the modified Hartmann number resulted in the elevation of fluid velocity.

In Figure 6, it is observed that as the nanoparticle volume fraction, φ, is increased, the fluid velocity is dampened. Higher values of φ is due to higher volumes of nanoparticle within the fluid. As nanoparticles naturally add weight to the fluid, nanofluid tends to be heavier than a classical fluid and a heavy fluid restricts fluid velocity. Thus, the higher values of φ decreases fluid velocity.

In contrast, Figure 7 observes an increase fluid velocity with an increase in the Grashof number, Gr. An increase in the Gr physically entails the buoyancy force acting on fluid is increased and at the same time, viscosity of fluid in decreased. Increasing buoyancy force acting on fluid and decreasing the viscosity of fluid aids in fluid flow. Thus, by increasing Gr, fluid velocity increases.

By comparison, fluid velocity decrease with an amplification of the Prandtl number, $\Pr$, as observed in Figure 8. An increase in $\Pr$ signifies an increase in momentum diffusivity and a decrease in thermal diffusivity. The increase in momentum diffusivity might be due to shear stress exerted between surface and fluid. This in turn slows down the fluid. Thus, an increase in $\Pr$ resulted in the decrease of fluid velocity.

Figure 9 showcases the effect on temperature profile of fluid with variations in fractional parameter, α. As discussed previously, currently there are no physical representation for fractional derivatives and fractional parameters in fluid mechanics. However, the result obtained in this study will be beneficial for future numerical and experimental studies.

Meanwhile, in Figure 10 it is observed that as the nanoparticle volume fraction parameter, φ, increases, fluid temperature is also increased. Nanoparticles have great thermal conductivity properties. As a result, by introducing nanoparticles within fluids, the rate of heat transfer between surface of plate and fluid is increases. Thus, increasing the temperature of fluid.

The same behaviour can be seen in Figure 11, where the fluid temperature increases as the Newtonian heating parameter, γ, is increased. As discussed earlier, Newtonian heating is a behaviour when the rate of heat transfer is directly proportional to local surface temperature. Increasing γ signifies increasing the local surface temperature, directly increasing the heat transfer rate. Thus, as γ increases, the fluid temperature is also increase.

In Figure 12, the effect of thermal radiation parameter, N, is analysed. An increase in fluid temperature is observed when the value of N is increased. An increase in N is due to an increase in the amount of thermal radiation supplied to surface. Since Newtonian heating is considered in this study, as the amount of thermal radiation is increased, the surface temperature also increases, increasing rate of heat transfer between plate surface and fluid. Thus, amplifying fluid temperature.

By comparison, the fluid temperature decreases with an increase in $\Pr$, as observed in Figure 13. As discussed earlier, increasing $\Pr$ increases the momentum diffusivity and decreases thermal diffusivity. A decrease in thermal diffusivity disrupts heat transfer rate and limits the amount of heat supplied to the fluid. Thus, an increase in $\Pr$ decreases the fluid temperature.

5. Conclusion

An analytical study on fractional Caputo-Fabrizio Casson CMC nanofluid over a Riga plate with Newtonian heating has been conducted. Findings of this study includes:

1. Analytical solution in Laplace domain for an unsteady Casson nanofluid over a Riga plate with presence of Newtonian heating and the Caputo-Fabrizio fractional derivative.
2. Fractional parameter, α, effected the fluid such that a higher value of α increase the fluid velocity and temperature.
3. Nanoparticle volume fraction parameter, φ, effected the fluid such that an increase in φ decreases fluid velocity but increase fluid temperature.
4. Newtonian heating parameter, γ, effected the fluid such that an increase in γ increases both the fluid velocity and temperature.
5. The modified Hartmann number, E, and Grashof number, Gr effected the fluid such that an increase in either E or Gr increases the fluid velocity.
6. Prandtl number, $\Pr$, effected the fluid such that an increase in $\Pr$ dampens both the fluid velocity and temperature.
7. Thermal radiation parameter, N, effected the fluid such that an increase in N increases the fluid temperature.

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