The Impacts of Scavenging on the Prey-Predator-Scavenger Fishery Model in the Existence of Harvesting and Toxin

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ABSTRACT

The current paper investigates a prey-predator-scavenger fishery model in which both prey and predator species release harmful toxins during predation interaction. The harvesting of scavengers is introduced due to their economic value and serves as one of the important protein sources for humans. Both predator and scavenger fish species consume prey as their food. Scavengers devour the carrion or carcasses of predators which die of natural causes and are infected by the toxins of their preys. The non-negativity of solutions from the fishery model has been derived to ensure biologically meaningful. The model's equilibria are investigated along with its local stability characteristics. We investigate the threshold conditions triggering the bifurcations to occur in the steady-states concerning the scavenging activities by the scavengers. By formulating a suitable Lyapunov function, the global stability analysis of the non-trivial or coexistence steady-state is implemented. The dynamical behaviours of the fishery model, as well as the persistence and extirpation properties, have been analysed using the bifurcation analysis. From the results, it is found that the scavenging parameter in the fishery model with the presence of harvesting and toxin can drive the fish population towards extinction state which is unstable and unfavourable in nature.

1. Introduction

An ecosystem is the accumulation of all living organisms in a specific location with the existence of natural resources like air, soil and water. Examples of living organisms are plants, animals and microorganisms. A scavenger is an organism that ingests the dead animals that have died naturally, were killed by other animals or have been killed by itself. Obligate and facultative scavengers are the two different categories of scavengers. According to Beasley *et al.,* [1], facultative scavengers will scavenge but do not totally depend on carrion for their survival or procreation, as opposed to obligate scavengers, who fully rely on carrion or carcasses for both their survival and procreation. Jensen [2]

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and Luque *et al.,* [3] discovered that the facultative scavengers exist in the marine ecosystem and include nematodes, crustaceans, echinoderms, molluscs, fish and marine mammals.

Foraging interaction primarily focuses on prey-predator interaction, whereas the use of carrion by scavengers, as the best form of animal carcasses, has received little attention in the ecosystem sense. Scavengers research, according to Satar and Naji [4], is important because they help clean the environment. However, excessive scavenging activities, especially for facultative scavengers which consumed both live and dead animals can make their population dominant in the fishery ecosystem and eventually both prey and predator fish species are driven to extinction. Scavenging ability by scavengers depends on the encounter probability which includes the carrion resource availability, locomotion and sensory detection. A shark is a great example of a facultative scavenger because its big pectoral fins are adapted to cruising swimming [5]. The ability to swim is an important factor in determining an organism's ability to move towards carcasses.

On the other hand, according to work done by Ang *et al.,* [6], as a defensive mechanism to protect themselves from predators and compete with their competitors, several fish species may release poisonous toxins to other aquatic life. Pufferfish, lionfish, boxfish and trunkfish are examples of these kinds of fish species. The research by Ang *et al.,* [7], Ang *et al.,* [8] and Huda *et al.,* [9] assumed that during predation interplay, both prey and predator emit toxins to each other. Kar and Chaudhuri [10] and Haque and Sarwardi [11] analysed two competing fish species which are competing for the same food source. Both competing fish species release toxin substances to each other. Besides of water pollution, the pressure of the rising population, scarcity of environmental consciousness and lack of execution of environmental rules lead to the issue of overexploitation. The rising demand for scavenger fish species like shark, remora, Atlantic cod and mackerel triggers the overharvesting activities that bring the extirpation of these populations. The best and most efficient management of renewable resources such as fisheries is important for the continued availability of protein resources for people without endangering the fish population. Therefore, the existence of unregulated harvesting and toxin in the fishery model are important issues in the fishery ecosystem.

The stability and longevity of every species in an ecosystem are the primary ecological concerns. To solve the problem of ecosystem extinction, mathematical models are used to analyse significant parameters, threshold conditions and stability of a variety of ecological systems. In the existence of harvesting and toxic substances, Satar and Naji [4] examined the stability and bifurcation of a prey-predator-scavenger model. A bifurcation takes place when a parameter in a population system crosses a threshold value, resulting in an abrupt dynamical change [12]. Gupta and Chandra [13] have analysed the prey-predator-scavenger model with quadratic harvesting, where the uncontrolled harvesting activities of predator species can lead to chaos in the system.

According to previous research, there is still a dearth of studies that consider the toxicant and harvesting aspects of the prey-predator-scavenger system. The previous studies that can be reviewed from the papers of several authors in [4,6-11,14-16] have motivated us to analyse the impacts of scavenging by facultative scavengers on the dynamics of the prey-predator-scavenger fishery model in the presence of self-produced toxins by fish populations and harvesting. For the preservation and conservation of the fishery ecosystem, we think it will contribute to richer dynamics in the behaviours of the fishery model.

2. Mathematical Model

To analyse the prey-predator-scavenger fishery model, a model which is a modification of the fishery models introduced by [7,8] is proposed. In contrast with previous models, we investigate the effects of scavenging activities by introducing a facultative scavenger in a fishery system with prey

and predator toxins, as well as facultative scavenger harvesting. Hence, we consider the following dimensional prey-predator-scavenger fishery model

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right) - \alpha_1 XY - \alpha_2 XZ - g_1 X^2 Y ,$$ (1a)

$$\frac{dY}{dT} = b_1 \alpha_1 XY - d_1 Y - g_2 Y^2 X ,$$ (1b)

$$\frac{dZ}{dT} = b_2 \alpha_2 XZ - d_2 Z + \alpha_3 YZ - qEZ ,$$ (1c)

where $X(T), Y(T)$ and $Z(T)$ represent the numbers of prey, predator and scavenger at the time T respectively. The logistical growth rate of the prey fish population X is r, and K represents the environmental carrying capacity. In the non-existence of prey, predator fish dies out exponentially at a rate d_1, whereas scavenger fish dies out exponentially at a rate d_2 in the absence of both prey and predator carcasses. Predator and scavenger maximum attack rates on prey are indicated by the parameters α_1 and α_2, respectively. The parameters b_1 and b_2 are referred to the conversion rates of prey to predator and prey to scavenger, respectively. The parameter α_3 refers to the attack rate of predator carcass by a scavenger. Scavenger has no negative effects on the population that it scavenges. The coefficients of toxicity in prey and predator are denoted by g_1 and g_2, respectively. The scavenger population is not affected by the toxins released by the prey and predator populations due to its physical and biological strength. The scavenger population is harvested with the effort harvesting E and a catchability coefficient q. For biological feasibility, all parameters are considered to have positive values.

2.1 Non-Dimensional Fishery Model

To simplify the model and reduce the number of parameters for easier interpretation, non-dimensionalisation is applied to the system Eq. (1). By introducing the dimensionless variables

$$x = \frac{b_1 \alpha_1}{r} X , y = \frac{\alpha_1}{r} Y , z = \frac{\alpha_2}{r} Z , t = rT,$$

and the dimensionless parameters

$$\alpha = \frac{r}{b_1 \alpha_1 K} , \eta = \frac{g_1 r}{b_1 {\alpha_1}^2} , \sigma = \frac{d_1}{r} , \rho = \frac{g_2 r}{b_1 \alpha_1^2} , \beta = \frac{b_2 \alpha_2}{b_1 \alpha_1} , \theta = \frac{d_2}{r} , \epsilon = \frac{\alpha_3}{\alpha_1} , \delta = \frac{qE}{r} ,$$

the dimensional system Eq. (1) becomes

$$\frac{dx}{dt} = x(1 - \alpha x) - xy - xz - \eta x^2 y ,$$ (2a)

$$\frac{dy}{dt} = xy - \sigma y - \rho y^2 x ,$$ (2b)

$$\frac{dz}{dt} = \beta xz - \theta z + \epsilon yz - \delta z .$$ (2c)

The dimensionless system Eq. (2) is defined on the set

$$\Psi = \{(x, y, z) \in \mathbb{R}^3 | x \geq 0, y \geq 0, z \geq 0\}. \tag{3}$$

3. Non-Negativity of Solutions for System Eq. (2)

In this section, we derive the non-negativity of solutions of the system Eq. (2). This step is to ensure that the system Eq. (2) is biologically meaningful, in which the number of populations predicted are non-negative values.

    i.   **Theorem 1:** The solution $(x(t), y(t), z(t))$ is non-negative for all $t > 0$ if the initial population of the system Eq. (2), $(x(0), y(0), z(0))$ is non-negative.

    ii.   **Proof.** To show the non-negativity of $x(t)$, we first rewrite the first equation of system Eq. (2) which is Eq. (2a) as

$$\frac{dx}{dt} + (y + z + \eta xy - 1 + \alpha x)x = 0. \tag{4}$$

Hence, the integrating factor is denoted as $e^{\int (y+z+\eta xy-1+\alpha x)\, dt}$. Integrating the Eq. (4) with respect to t, we get:

$$x(t) = x(0)e^{-\int_0^t (y+z+\eta xy-1+\alpha x)\, dt}. \tag{5}$$

From Eq. (5), we observe that $x(t) \geq 0$ for all $t > 0$ since $x(0)$ and $e^{-\int_0^t (y+z+\eta xy-1+\alpha x)\, dt} \geq 0$. The same proof can be used to show that $y(t) \geq 0$ and $z(t) \geq 0$.

4. Existence and Stability of Equilibria

In this section, we consider the existence and local asymptotic and global stability analysis of the steady-states found from the system Eq. (2). The threshold condition of a trans critical bifurcation about the non-coexistence steady-state concerning the scavenging parameter ϵ is also examined in this section.

4.1 Existence of Steady-States from System Eq. (2)

    i.   The trivial steady-states $P_1 = (0,0,0)$ and $P_2 = \left(\frac{1}{\alpha}, 0, 0\right)$ of a fishery model Eq. (2) exist regardless of any parametric constraints.

    ii.   The predator-free steady-state is given by $P_3 = \left(\frac{\theta+\delta}{\beta}, 0, \frac{\beta-\alpha(\theta+\delta)}{\beta}\right)$ which exists under the parametric condition $\alpha(\theta + \delta) < \beta$.

    iii.   The scavenger-free steady-state $P_4 = \left(\frac{1-\hat{y}}{\alpha+\eta\hat{y}}, \hat{y}, 0\right)$ exists where $\hat{y}$ is a positive root by solving the quadratic equation of $\rho\hat{y}^2 + \hat{y}(-\rho - 1 - \sigma\eta) + 1 - \sigma\alpha = 0$ is feasible under the conditions $(\rho + 1 + \sigma\eta)^2 > 4\rho(1 - \sigma\alpha)$ and $\hat{y} - 1 < 0$.

    iv.   The coexistence steady-state $P_5 = \left(\frac{\sigma}{1-\rho\tilde{y}}, \tilde{y}, \frac{1-(\tilde{y}(\sigma\eta+1+\rho(1-\tilde{y}))+\alpha\sigma)}{1-\rho\tilde{y}}\right)$ exists where $\tilde{y}$ is a positive root by solving the quadratic equation of $\rho\epsilon\tilde{y}^2 + \tilde{y}(-\rho(\theta + \delta) - \epsilon) + \theta + \delta -$

$\sigma\beta = 0$ is feasible under the conditions $(\rho(\theta + \delta) + \epsilon)^2 > 4\rho\epsilon(\theta + \delta - \sigma\beta)$, $1 - \rho\tilde{y} > 0$ and $\tilde{y}(\sigma\eta + 1 + \rho(1 - \tilde{y})) + \alpha\sigma < 1$.

4.2 Local Stability Analysis of the Non-Coexistence Steady-States

Based on the standard linearisation technique and the Routh-Hurwitz criterion [4,13-15,17-19], we derive the local stability conditions for feasible non-coexistence steady-states of the system Eq. (2). By using the Jacobian matrix

$$J(x,y,z) = \begin{pmatrix} 1 - 2x(\alpha + \eta y) - y - z & -x(1 + \eta x) & -x \\ y(1 - \rho y) & x(1 - 2\rho y) - \sigma & 0 \\ \beta z & \epsilon z & \beta x - \theta + \epsilon y - \delta \end{pmatrix}, \tag{6}$$

the local stability of the steady-states can be further analysed.

4.2.1 Stability of $P_1 = (0,0,0)$

The eigenvalues of the Jacobian matrix in Eq. (6) associated at the trivial steady-state $P_1 = (0,0,0)$ are $\lambda_1 = -(\theta + \delta)$, $\lambda_2 = 1$ and $\lambda_3 = -\sigma$. Hence, the origin is always an unstable saddle point since λ_2 is strictly positive.

4.2.2 Stability of $P_2 = \left(\frac{1}{\alpha}, 0, 0\right)$

The eigenvalues of the Jacobian matrix in Eq. (6) associated at P_2 are $\lambda_1 = \frac{1 - \sigma\alpha}{\alpha}$, $\lambda_2 = -1$ and $\lambda_3 = \frac{\beta - \alpha(\theta + \delta)}{\alpha}$. The steady-state or critical point P_2 becomes a stable node if both conditions $\sigma\alpha > 1$ and $\alpha(\theta + \delta) > \beta$ are satisfied. When at least one of the conditions $\sigma\alpha > 1$ and $\alpha(\theta + \delta) > \beta$ is violated, the steady-state P_2 loses its stability and becomes an unstable saddle point. If critical point P_2 is a stable node, hence the steady-state P_3 is unfeasible according to the feasibility condition of P_3 which is $\alpha(\theta + \delta) < \beta$ not satisfied.

4.2.3 Stability of $P_3 = \left(\frac{\theta+\delta}{\beta}, 0, \frac{\beta - \alpha(\theta+\delta)}{\beta}\right)$

By considering the Jacobian matrix in Eq. (6) about the steady-state or equilibrium point P_3, it gives the eigenvalues of $\lambda_1 = \frac{\theta + \delta - \sigma\beta}{\beta}$ and n_1 where n_1 solves the quadratic equation of

$$n_1^2 + n_1\left(\frac{\alpha(\theta+\delta)}{\beta}\right) + \frac{(\beta - \alpha(\theta+\delta))(\theta+\delta)}{\beta} = 0. \tag{7}$$

Theorem 2: Predator-free equilibrium point P_3 exists and is linearly stable if

$$\sigma\beta > \theta + \delta. \tag{8}$$

Proof. The characteristic equation of steady-state P_3 is

$$A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0 \,, \tag{9}$$

where

$$A_1 = 1, \tag{10a}$$

$$A_2 = \frac{\sigma\beta - (\theta+\delta) + \alpha(\theta+\delta)}{\beta} \,, \tag{10b}$$

$$A_3 = \frac{(\theta+\delta)\big(\alpha(\sigma\beta - (\theta+\delta)) + \beta(\beta - \alpha(\theta+\delta))\big)}{\beta^2} \,, \tag{10c}$$

$$A_4 = \frac{(\sigma\beta - (\theta+\delta))(\beta - \alpha(\theta+\delta))(\theta+\delta)}{\beta^2} \,. \tag{10d}$$

From Eq. (10a), A_1 is always positive. If the condition in Eq. (8) satisfies together with the feasible criteria for the existence of a critical point P_3, then A_2, A_3 and A_4 in Eq. (10b) to Eq. (10d) respectively are positive. Again, if the condition in Eq. (8) holds together with the existence criteria of a critical point P_3, then the condition $A_2 A_3 - A_1 A_4 > 0$ in Eq. (11) such as

$$\frac{(\theta+\delta)\big(\alpha(\sigma\beta - (\theta+\delta))^2 + \alpha^2(\theta+\delta)(\sigma\beta - (\theta+\delta)) + \alpha\beta(\theta+\delta)(\beta - \alpha(\theta+\delta))\big)}{\beta^3} > 0 \tag{11}$$

is satisfied. As a result, it guarantees the local stability of P_3 according to the Routh Hurwitz criterion.

4.2.4 Stability of $P_4 = \left(\frac{1-\hat{y}}{\alpha+\eta\hat{y}}, \hat{y}, 0\right)$

By evaluating the Jacobian matrix in Eq. (6) at the equilibrium point P_4, it gives the eigenvalues of $\lambda_1 = \frac{\beta(1-\hat{y}) - (\theta+\delta-\epsilon\hat{y})(\alpha+\eta\hat{y})}{\alpha+\eta\hat{y}}$ and n_2, where n_2 solves the quadratic equation of

$$n_2^2 + n_2\left(\frac{(\hat{y}-1)\big(1 - (\alpha+\eta\hat{y}+2\rho\hat{y})\big) + \sigma(\alpha+\eta\hat{y})}{\alpha+\eta\hat{y}}\right) + \left(\frac{\hat{y}-1}{(\alpha+\eta\hat{y})^2}\right)$$
$$\Big((\alpha+\eta\hat{y})\big(-\hat{y}(1-\rho\hat{y}) - (\hat{y}-1)(1-2\rho\hat{y}) - \sigma(\alpha+\eta\hat{y})\big) + \eta\hat{y}(1-\rho\hat{y})(\hat{y}-1)\Big) = 0 \,. \tag{12}$$

Theorem 3: If all feasibility conditions of the equilibrium point P_4 are satisfied, then the scavenger-free equilibrium point P_4 exists and is locally stable if

$$(\theta + \delta - \epsilon\hat{y})(\alpha + \eta\hat{y}) > \beta(1 - \hat{y}) \,, \tag{13a}$$

$$\alpha + \eta\hat{y} + 2\rho\hat{y} > 1 \,, \tag{13b}$$

$$\frac{1}{2} < \rho\hat{y} < 1 \,. \tag{13c}$$

Proof. The characteristic equation of steady-state P_4 takes the form of

$$B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0 \tag{14}$$

where

$$B_1 = 1 , \tag{15a}$$

$$B_2 = \frac{(\alpha+\eta\hat{y})[(\theta+\delta-\epsilon\hat{y})+\sigma]+(\hat{y}-1)[\beta+(1-(\alpha+\eta\hat{y}+2\rho\hat{y}))]}{\alpha+\eta\hat{y}}, \tag{15b}$$

$$
\begin{aligned}
B_3 = & \frac{\big((1-\hat{y})(1-(\alpha+\eta\hat{y}+2\rho\hat{y}))-\sigma(\alpha+\eta\hat{y})\big)\big(\beta(1-\hat{y})-(\theta+\delta-\epsilon\hat{y})(\alpha+\eta\hat{y})\big)}{(\alpha+\eta\hat{y})^2} \\
& - \frac{(\hat{y}-1)\big[\hat{y}(1-\rho\hat{y})\big((\alpha+\eta\hat{y})-\eta(\hat{y}-1)\big)+\big((\hat{y}-1)(\alpha+\eta\hat{y})(1-2\rho\hat{y})+\sigma(\alpha+\eta\hat{y})^2\big)\big]}{(\alpha+\eta\hat{y})^2}
\end{aligned} \tag{15c}
$$

$$
\begin{aligned}
B_4 = & \frac{(\hat{y}-1)\big[\hat{y}(1-\rho\hat{y})\big((\alpha+\eta\hat{y})-\eta(\hat{y}-1)\big)+(\hat{y}-1)(\alpha+\eta\hat{y})(1-2\rho\hat{y})+\sigma(\alpha+\eta\hat{y})^2\big]}{(\alpha+\eta\hat{y})^3} \\
& \times [\beta(1-\hat{y})-(\theta+\delta-\epsilon\hat{y})(\alpha+\eta\hat{y})] .
\end{aligned} \tag{15d}
$$

From Eq. (15a), B_1 is always positive. If the conditions in Eq. (13a) until Eq. (13c) hold together with all feasible criteria for the existence of a critical point P_4, then B_2, B_3 and B_4 in Eq. (15b) to Eq. (15d) respectively are positive. Again, if the conditions in Eq. (13a) until Eq. (13c) hold together with all existence criteria of a critical point P_4, then the condition $B_2B_3 - B_1B_4 > 0$ in Eq. (16) such as

$$\frac{[\beta(1-\hat{y})-(\theta+\delta-\epsilon\hat{y})(\alpha+\eta\hat{y})]\big[(\hat{y}-1)\big(1-(\alpha+\eta\hat{y}+2\rho\hat{y})\big)+\sigma(\alpha+\eta\hat{y})\big]}{(\alpha+\eta\hat{y})^3}\times$$

$$\Big[(1-\hat{y})\big[\beta+\big(1-(\alpha+\eta\hat{y}+2\rho\hat{y})\big)\big]-(\alpha+\eta\hat{y})[(\theta+\delta-\epsilon\hat{y})+\sigma]\Big]-$$

$$\frac{\big[(\alpha+\eta\hat{y})\big(\hat{y}(1-\rho\hat{y})+(\hat{y}-1)(1-2\rho\hat{y})+\sigma(\alpha+\eta\hat{y})\big)-\eta\hat{y}(\hat{y}-1)(1-\rho\hat{y})\big]}{(\alpha+\eta\hat{y})^3}\times$$

$$\Big[(\hat{y}-1)\big[\sigma(\alpha+\eta\hat{y})+(\hat{y}-1)\big(1-(\alpha+\eta\hat{y}+2\rho\hat{y})\big)\big]\Big] > 0 \tag{16}$$

is satisfied. As an outcome, it guarantees the local stability of P_4 according to the Routh Hurwitz criterion.

To find the threshold condition for a transcritical bifurcation (TB) point to occur, the conditions in Eq. (13b) and Eq. (13c) need to be satisfied together with all feasible criteria of the equilibrium point P_4 which are $\hat{y} - 1 < 0$ and $(\rho + 1 + \sigma\eta)^2 > 4\rho(1 - \sigma\alpha)$. Then, the threshold condition for TB with ϵ refers to the scavenging parameter as a bifurcation parameter is

$$\epsilon = \frac{-\beta(1-\hat{y})+(\theta+\delta)(\alpha+\eta\hat{y})}{\hat{y}(\alpha+\eta\hat{y})} . \tag{17}$$

If $\epsilon < \frac{(\theta+\delta)(\alpha+\eta\hat{y})-\beta(1-\hat{y})}{\hat{y}(\alpha+\eta\hat{y})}$ same with the condition in Eq. (13a), the equilibrium point P_4 is locally asymptotically stable, whereas if $\epsilon > \frac{(\theta+\delta)(\alpha+\eta\hat{y})-\beta(1-\hat{y})}{\hat{y}(\alpha+\eta\hat{y})}$, equilibrium point P_4 is not stable

4.3 Global Stability Analysis of the Coexistence Steady-State

Since analysing the local stability is time-consuming due to the presence of numerous parameters, we implement the global stability analysis on the coexistence steady-state P_5 by formulating a suitable Lyapunov function. Therefore, we consider Theorem 4.

Theorem 4: Non-trivial steady-state P_5 in the form of $(\tilde{x}, \tilde{y}, \tilde{z})$ is globally asymptotically stable in the sub-domain regions of

$$\frac{(x-\tilde{x})\big((y-\tilde{y})+\eta(xy-\tilde{x}\tilde{y})\big)}{(y-\tilde{y})\big((x-\tilde{x})-\rho(xy-\tilde{x}\tilde{y})\big)} > 0 \tag{18}$$

and

$$\frac{(x-\tilde{x})}{\beta(x-\tilde{x})+\epsilon(y-\tilde{y})} > 0 . \tag{19}$$

Proof. Consider the Lyapunov function

$$V(x,y,z) = \left((x-\tilde{x})-\tilde{x}ln\frac{x}{\tilde{x}}\right) + M_1\left((y-\tilde{y})-\tilde{y}ln\frac{y}{\tilde{y}}\right) + M_2\left((z-\tilde{z})-\tilde{z}ln\frac{z}{\tilde{z}}\right), \tag{20}$$

where the positive functions M_1 and M_2 are to be determined in the subsequent steps. Function $V(\tilde{x}, \tilde{y}, \tilde{z}) = 0$ and $V(x,y,z) > 0$ for all the positive values of x, y and z. The derivative of the Lyapunov function in Eq. (20) is

$$\frac{dV}{dt} = \left(\frac{x-\tilde{x}}{x}\right)\frac{dx}{dt} + M_1\left(\frac{y-\tilde{y}}{y}\right)\frac{dy}{dt} + M_2\left(\frac{z-\tilde{z}}{z}\right)\frac{dz}{dt}$$

$$= -[(x-\tilde{x})\big(\alpha(x-\tilde{x})+(y-\tilde{y})+(z-\tilde{z})+\eta(xy-\tilde{x}\tilde{y})\big) + M_1(y-\tilde{y})$$
$$\big(\rho(xy-\tilde{x}\tilde{y})-(x-\tilde{x})\big) + M_2(z-\tilde{z})\big(-\beta(x-\tilde{x})-\epsilon(y-\tilde{y})\big)]$$

$$= -[\,\alpha(x-\tilde{x})^2 + (x-\tilde{x})(y-\tilde{y})(1-M_1) + (xy-\tilde{x}\tilde{y})\big(\eta(x-\tilde{x})+\rho M_1(y-\tilde{y})\big)$$
$$+(z-\tilde{z})\big((x-\tilde{x})(1-\beta M_2)-\epsilon M_2(y-\tilde{y})\big)]\,.$$

From the latest equation of $\frac{dV}{dt}$, we know that the term $\alpha(x-\tilde{x})^2$ is strictly positive. Therefore, we assume that both terms $(x-\tilde{x})(y-\tilde{y})(1-M_1) + (xy-\tilde{x}\tilde{y})\big(\eta(x-\tilde{x})+\rho M_1(y-\tilde{y})\big) = 0$ and $(z-\tilde{z})\big((x-\tilde{x})(1-\beta M_2)-\epsilon M_2(y-\tilde{y})\big) = 0$. By setting or letting $M_1 = \frac{(x-\tilde{x})\big((y-\tilde{y})+\eta(xy-\tilde{x}\tilde{y})\big)}{(y-\tilde{y})\big((x-\tilde{x})-\rho(xy-\tilde{x}\tilde{y})\big)}$ and $M_2 = \frac{(x-\tilde{x})}{\beta(x-\tilde{x})+\epsilon(y-\tilde{y})}$, we obtain

$$\frac{dV}{dt} = -\alpha(x-\tilde{x})^2. \tag{21}$$

Hence, from Eq. (21), $\frac{dV}{dt} < 0$ in the sub-domain regions in Eq. (18) and Eq. (19) implies a non-trivial steady-state P_5 is globally asymptotically stable. The hypothesis has been proven.

5. Hopf Bifurcation Analysis of the Coexistence Steady-State

The presence of a periodic orbit due to a local switch in the stability of a coexistence critical point is known as a Hopf bifurcation (HB). In this section, we analyse the Hopf bifurcation to investigate the occurrence of periodic oscillations through the coexistence steady-state P_5 as the scavenging parameter ϵ increases. We investigate the Hopf bifurcation around P_5 concerning the bifurcation parameter ϵ with the aid of the Hopf bifurcation theorem.

Theorem 5: The fishery system Eq. (2) undergoes a Hopf bifurcation around the coexistence steady-state P_5 concerning to ϵ if the condition $(C_2 - 3\omega^2)\left(\frac{dC_3}{d\epsilon} - \omega^2 \frac{dC_1}{d\epsilon}\right) + 2C_1\omega^2 \frac{dC_2}{d\epsilon} \neq 0$ is satisfied.

Proof. The coexistence steady-state P_5 depends on the scavenging parameter ϵ. Based on the Hopf bifurcation theorem applied in the research conducted by Satar and Naji [4], Gupta and Chandra [13] and Panja [18], it is known that a system Eq. (2) undergoes Hopf bifurcation if the Jacobian matrix J_{P_5}

$$J_{P_5} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{22}$$

where

$$a_{11} = 1 - 2\tilde{x}(\alpha + \eta\tilde{y}) - \tilde{y} - \tilde{z}, \; a_{12} = -\tilde{x}(1 + \eta\tilde{x}), \; a_{13} = -\tilde{x}, \; a_{21} = \tilde{y}(1 - \rho\tilde{y}),$$
$$a_{22} = \tilde{x}(1 - 2\rho\tilde{y}) - \sigma, \; a_{23} = 0, \; a_{31} = \beta\tilde{z}, \; a_{32} = \epsilon\tilde{z}, \; a_{33} = \beta\tilde{x} - \theta + \epsilon\tilde{y} - \delta$$

has a pair of purely imaginary eigenvalues and the third eigenvalue has a negative real part and the transversality condition $Re\left(\frac{d\lambda}{d\epsilon}\right)_{\epsilon=\epsilon_{HB}} \neq 0$. The notation ϵ_{HB} refer to Hopf bifurcation point. The characteristic equation of P_5 is

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0, \tag{23}$$

where

$$C_1 = -(a_{11} + a_{22} + a_{33}),$$

$$C_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31},$$

$$C_3 = a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31} - a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32}.$$

According to Singh *et al.,* [20], for purely imaginary eigenvalues, coefficients of the characteristic polynomial in Eq. (23) must accomplish the condition $C_1C_2 - C_3 = 0$. It is needed to point out that the other necessary conditions are automatically satisfied. Based on Kuang [21], to derive the transversality condition which ensures the occurrence of periodic oscillations through Hopf

bifurcation, we consider $\lambda = \pm i\omega$ be a pair of purely imaginary eigenvalues concerning to ϵ_{HB}. Now differentiating the characteristic equation in Eq. (23) concerning to ϵ we obtain

$$\frac{d\lambda}{d\epsilon} = -\frac{\lambda^2 \frac{dC_1}{d\epsilon} + \lambda \frac{dC_2}{d\epsilon} + \frac{dC_3}{d\epsilon}}{3\lambda^2 + 2\lambda C_1 + C_2}.$$

From above it is clear that

$$\left(\frac{d(Re(\lambda))}{d\epsilon}\right)_{\lambda=i\omega} = Re\left(-\frac{\lambda^2 \frac{dC_1}{d\epsilon} + \lambda \frac{dC_2}{d\epsilon} + \frac{dC_3}{d\epsilon}}{3\lambda^2 + 2\lambda C_1 + C_2}\right)_{\lambda=i\omega}.$$

Thus, we obtain

$$\left(\frac{d(Re(\lambda))}{d\epsilon}\right)_{\lambda=i\omega} = -\frac{(C_2 - 3\omega^2)\left(\frac{dC_3}{d\epsilon} - \omega^2 \frac{dC_1}{d\epsilon}\right) + 2C_1\omega^2 \frac{dC_2}{d\epsilon}}{(C_2 - 3\omega^2)^2 + (2C_1\omega)^2}. \tag{24}$$

Hence, the fishery system Eq. (2) undergoes a Hopf bifurcation concerning bifurcation parameter ϵ if the condition $(C_2 - 3\omega^2)\left(\frac{dC_3}{d\epsilon} - \omega^2 \frac{dC_1}{d\epsilon}\right) + 2C_1\omega^2 \frac{dC_2}{d\epsilon} \neq 0$ in Eq. (24) is satisfied.

6. Bifurcation Results and Analysis

In this section, we perform the numerical simulations to analyse the impacts of scavenging by scavengers on the dynamical behaviours of the fishery model Eq. (2) by using XPPAUT and MATLAB softwares. We used a set of hypothetical parameter values $\alpha = 0.1, \eta = 2, \sigma = 0.1, \rho = 1.8, \beta = 0.7,$ $\theta = 0.44, \epsilon = 0.3$ and $\delta = 0.40$ to illustrate the bifurcation results. The steady-state P_1 is always an unstable saddle point due to its eigenvalue condition. Steady-state P_2 is also an unstable saddle point since both conditions $\sigma\alpha > 1$ and $\alpha(\theta + \delta) > \beta$ are not satisfied. Equilibrium point P_3 is not stable since the condition in Eq. (8) is unsatisfied. All that conditions make the equilibrium points P_1, P_2 and P_3 are absent in Figure 1 below.

Based on Figure 1 in a region I where $0 \le \epsilon < 1.0144$, the steady-state or critical point P_4 with $(x = 0.5432, y = 0.4533, z = 0)$ is stable whereas coexistence equilibrium point P_5 is unstable as shown in blue solid and red dotted lines respectively. This is because the encounter probability of predator carcasses by scavengers is very low. For example, apart from whale deaths, records of naturally occurring marine carcasses in the sea are scarce, according to Britton and Morton [22] and Smith and Baco [23]. Britton and Morton [22] and Kaiser and Moore [24] revealed that the shortage of naturally occurring carcasses may be due to a few fish dying from natural senescence or to their rapid consumption by another scavenger species. At the level of scavenging rate in the region I, the detection ability of predator carcasses by scavengers is very low also due to predators dying because of the prey's toxin infection during their predation interaction is low. Commonly, the predator has the physical advantage and more dominant compared to the prey. Thus, it will make their population less infected by the prey's toxins. This situation is lowering the death rate of predator and directly reducing the availability of predator carcasses in the fishery system. Scavengers experience less chances to consume preys because predators which are more dominant in terms of their existence in the fishery ecosystem also hunt live preys as their sources of food. Both prey and predator

populations still can survive with their predation interaction, considered that there are no threats from the scavengers.

To obtain the TB point concerning the scavenging parameter ϵ denoted by ϵ_{TB}, we need to consider Eq. (17). The conditions in Eq. (13b) and Eq. (13c) together with all feasible criteria of the equilibrium point P_4 are all satisfied. Therefore, the TB point is at $\epsilon_{TB} = 1.0144$ as shown in Figure 1. After passing through $\epsilon_{TB} = 1.0144$, the critical point P_5 becomes stable. In region II at the intermediate level of scavenging rate $1.0144 < \epsilon < 2.1450$ which is in between TB and HB points, the encounter probability of carrion by scavengers is increasing, which allows the population to increase in the fishery ecosystem as shown in Figure 1(c). However, the number of prey and predator populations that persist in the fishery ecosystem decreases as shown in Figure 1(a) and Figure 1(b) respectively when the scavenging rate by scavengers is increasing. Based on Figure 1(b), the decreasing predator population as shown in region II resulting their carrion existence in the fishery ecosystem, which is likely to promote the scavenging process of predator carcasses by scavengers. Scavengers also hunt live preys as an additional source of food that will increase their growth rate and decrease the growth rate of prey. Predators have less chances to consume preys due to the increasing scavenger population in the fishery system, which will make their death rate to increase.

To get the Hopf bifurcation (HB) point concerning the scavenging parameter ϵ, we need to consider the condition $C_1 C_2 - C_3 = 0$ from the characteristic equation of coexistence steady-state P_5 in Eq. (23). By letting the scavenging parameter ϵ as the unknown, we can solve it and get one HB point which is $\epsilon_{HB} = 2.1450$ as shown in Figure 1 below. Then, we solve the characteristic equation of P_5 and we obtain three eigenvalues which are $\lambda_1 = 0.3900i$ $\lambda_2 = -0.3900i$ and $\lambda_3 = -0.3015$ where $\omega = \pm 0.3900$. By substituting all values of terms in the transversality Eq. (24), we obtain $Re\left(\frac{d\lambda}{d\epsilon}\right)_{\epsilon_{HB}=2.1450} = 0.1347$ that supported Theorem 5.

In region III, at a high level of scavenging rate which passes the HB point ($\epsilon > 2.1450$), both coexistence equilibrium point P_5 and equilibrium point P_4 with scavenger-free are unstable as shown in two red dotted lines. At this level of the scavenging rate, the scavenger population becomes dominant in the fishery system, where they have an abundance source of food from predators' carcasses. Scavengers also receive an additional source of food from prey. Both prey and predator populations in Figure 1(a) and Figure 1(b) respectively in region III decrease as compared with region II when the scavenger population increases as illustrated in Figure 1(c). We believe that at a high rate of scavenging is harmful to the ecosystem as both prey and predator populations trigger to extinction over time. Based on Figure 1 in region III, when the value of the scavenging parameter keeps increasing from $\epsilon_{HB} = 2.1450$, we observe larger periodic oscillations as illustrated in the green open loops around the unstable coexistence steady-states are formed, which ensure that the fishery system is unstable. The upper and lower parts of the loops refer to the maximum and minimum levels of amplitudes of periodic oscillations around the unstable coexistence critical points. The maximum levels of amplitudes of periodic oscillations for each prey, predator and scavenger are different, but their minimum levels of amplitudes of periodic oscillations are the same.

From Figure 1 in region III, we observe that the scavenging activities by the scavengers affect the stability of the fishery system. This situation is quite regular, as according to the research conducted by Mortoja *et al.,* [25], the transition rate from immature to mature prey parameter, the birth rate of immature prey parameter and the attack rate of mature predator to mature prey parameter have a significant role to the prey-predator model with stage-structure on both species and anti-predator behaviour system stability. From their findings, the controlled parameters induce the coexistence steady-state always unstable, where the continuous periodic oscillations appeared from one HB point. On the other hand, the chaos control of the discrete-time prey-predator model has been

studied by Singh and Deolia [26], where a bifurcation diagram concerning the step size parameter is constructed. The result shows that the dense chaos can be reduced to limit cycle or quasiperiodicity to stable periodic oscillations.

(a)

(b)
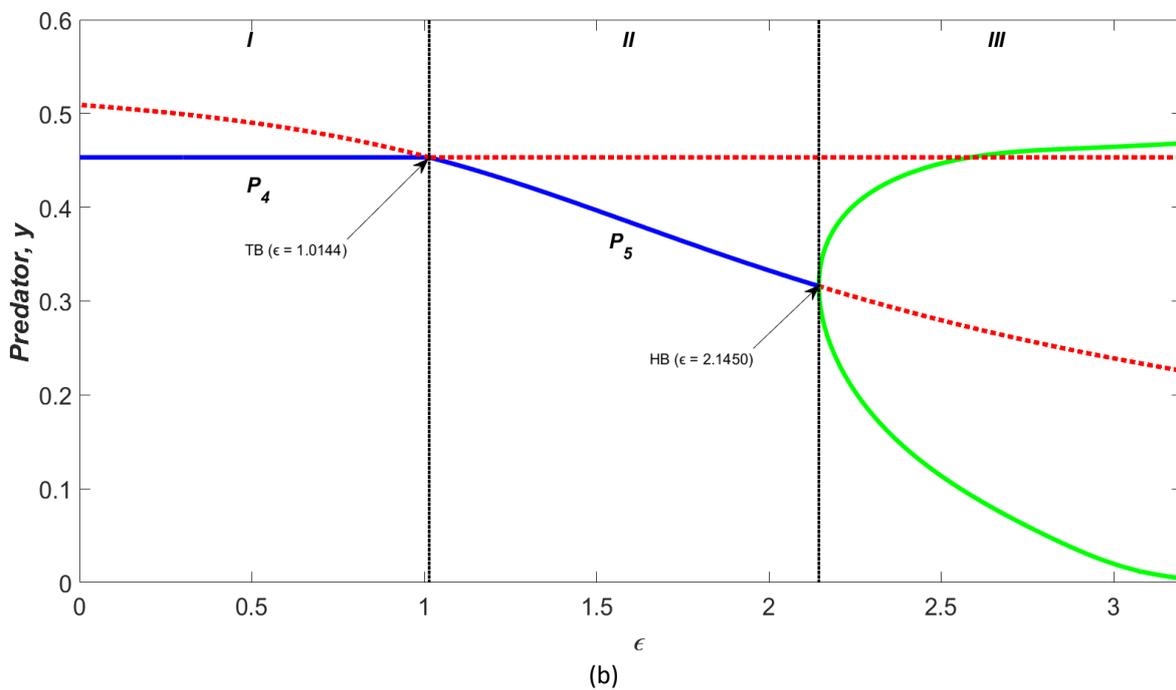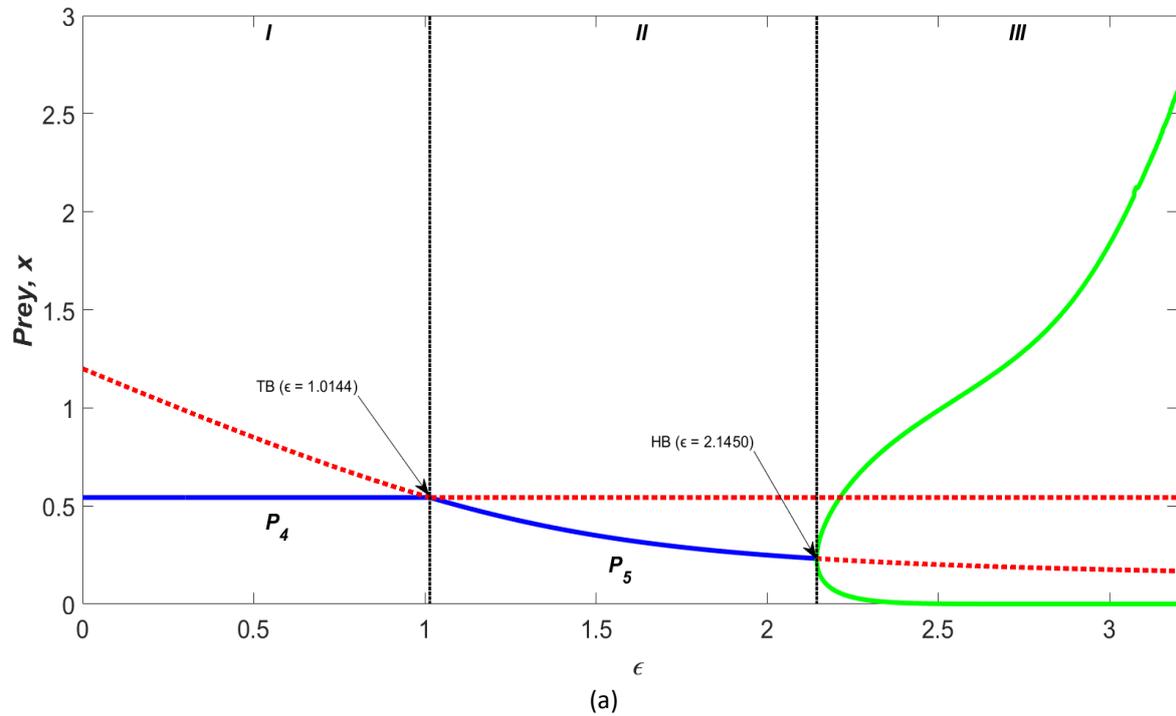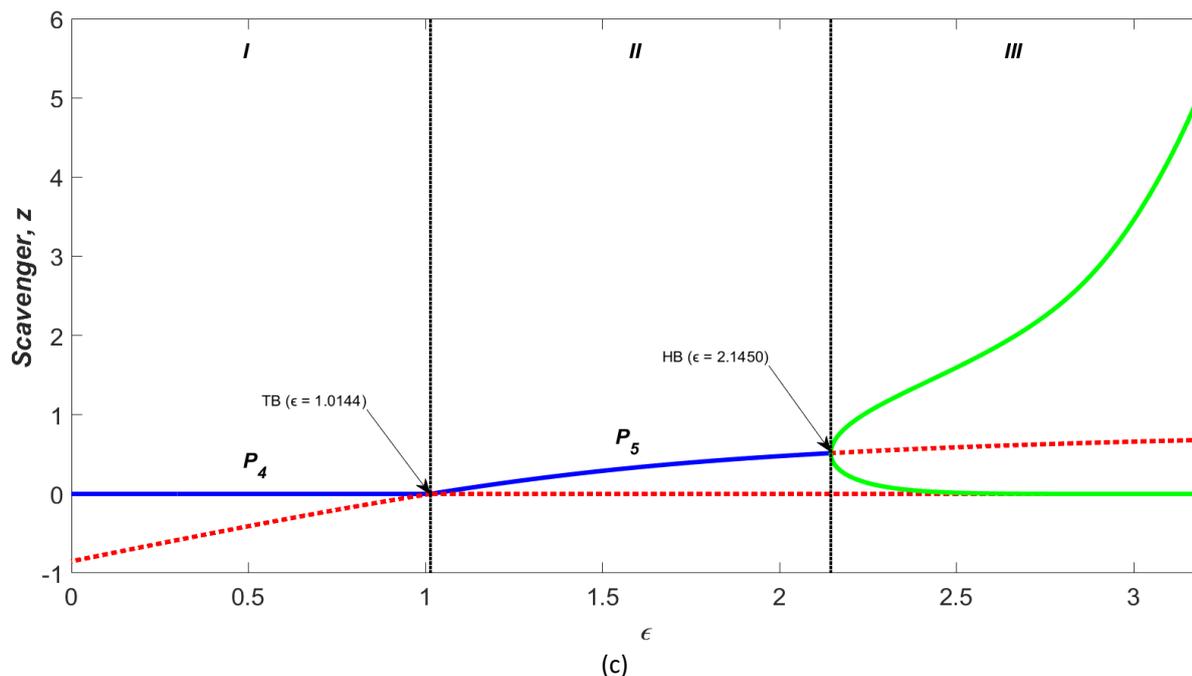
Fig. 1. Bifurcation diagrams of fishery model Eq. (2) concerning scavenging parameter, ϵ for (a) Prey, x (b) Predator, y and (c) Scavenger, z

Based on Figure 2, at $\epsilon = 0.9$ which is lower than the TB point ($\epsilon_{TB} = 1.0144$) and in region I of Figure 1, initially both prey and predator populations are increasing, whereas scavengers are decreasing due to the low scavenging activities of predator carcasses by scavengers. Over time, predators increase to their maximum population, whereas both prey and scavenger populations decrease due to the predation pressure of predators on the prey population. Further, all prey, predator and scavenger converge to their equilibrium number of populations which is $x = 0.5432$, $y = 0.4533$ and $z = 0$. Both prey and predator populations survive in the fishery ecosystem with their stable predation interaction without any threat from the scavenger population. The extinction of the scavenger population is due to several factors which are less opportunity in consuming predator carcasses because of the low death rate of predators and scavengers need to compete with predators which they are more dominant in the fishery system to hunt live preys as sources of their food.
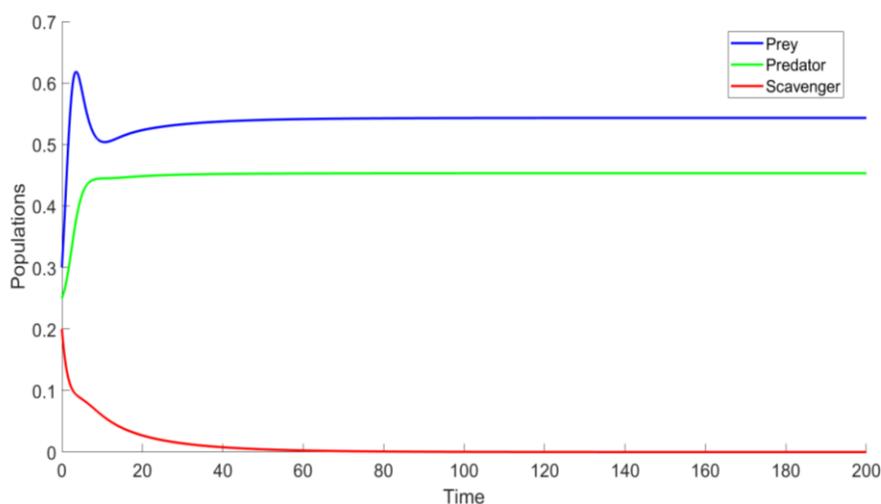
Fig. 2. Time series plot of fishery model Eq. (2) with initial conditions of $(x_0, y_0, z_0) = (0.30, 0.25, 0.20)$ at $\epsilon = 0.9$

From Figure 3, the value of the scavenging parameter in the range of region II in Figure 1, which is $\epsilon = 2.0$, the small periodic oscillations around the coexistence steady-state occurred and finally diminished. This is due to the increased scavenging activities of scavengers on predator carcasses and the predation pressure of both scavengers and predators on the prey population. At this level of the scavenging rate, scavengers get the opportunity to scavenge predator carcasses because of the high death rate of predators as compared to Figure 2. The increase in the number of scavengers in the fishery system can bring down the number of both prey and predator populations that can survive. This is because the predation pressure of scavengers on preys is higher than the predation pressure of predators on preys. Finally, all prey, predator and scavenger converge to their equilibrium number of populations which is $x = 0.2493$, $y = 0.3327$ and $z = 0.4764$ over time.
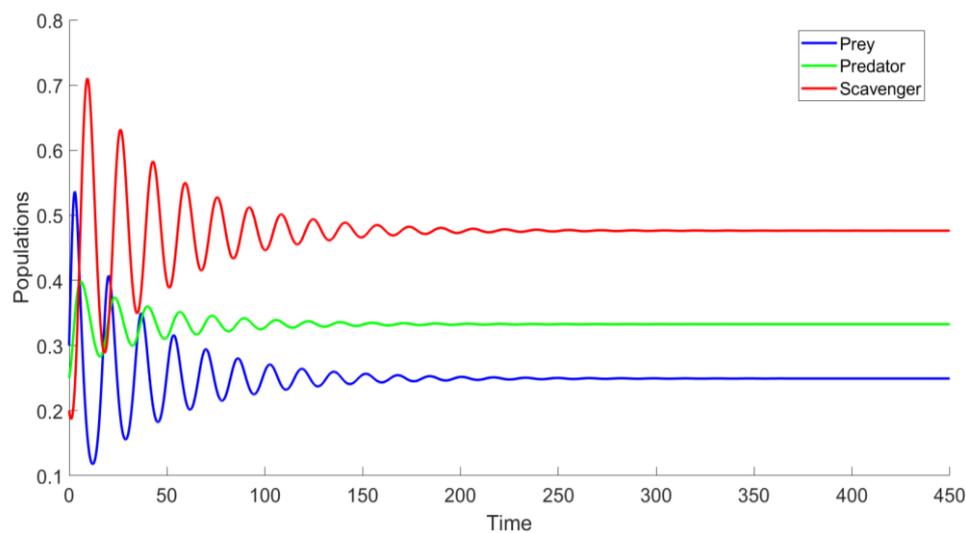
Fig. 3. Time series plot of fishery model Eq. (2) with initial conditions of $(x_0, y_0, z_0) = (0.30, 0.25, 0.20)$ at $\epsilon = 2.0$

Based on Figure 4, the high value of the scavenging rate which passes $\epsilon_{HB} = 2.1450$ and in the range of region III, can trigger a periodic solution around the coexistence steady-state, which causes the coexistence steady-state (0.1684, 0.2257, 0.6815) of the fishery model Eq. (2) to be unstable because prey, predator and scavenger populations do not converge to their critical numbers as time increases. Figure 4 shows the large periodic oscillations of prey, predator and scavenger populations around the unstable coexistence steady-state. At $\epsilon = 3.2$, the growth of both prey and predator populations are affected by the scavengers which becomes dominant in the fishery system and causes the coexistence critical point of the fishery system Eq. (2) always unstable. Based on the research conducted by Wan Hussin *et al.,* [27], large periodic oscillations of both prey and predator populations around their unstable coexistence steady-states occur when the time delays at the predator reaction function which measures the capture rate of prey by a predator exceed their threshold or Hopf bifurcation value. The growth of the predator population is influenced by its population and prey population densities at time delays above the HB point, which makes the coexistence steady-states of the fishery models unstable.
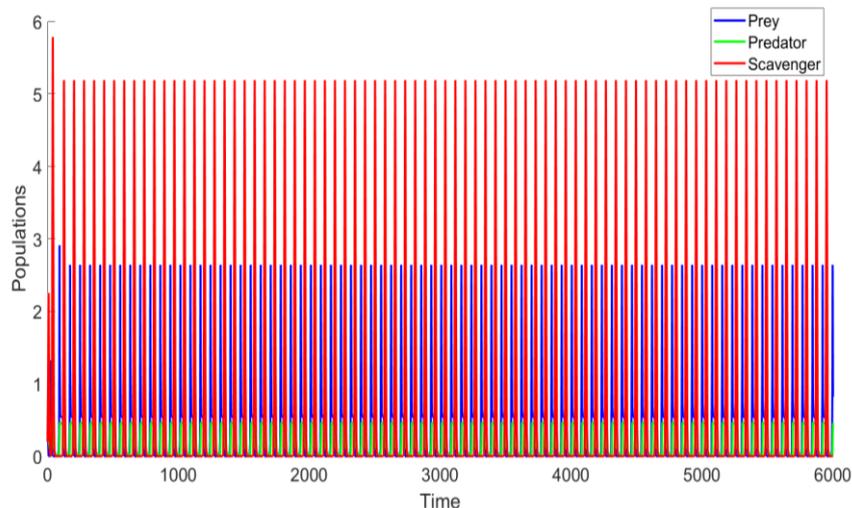
Fig. 4. Time series plot of fishery model Eq. (2) with initial conditions of $(x_0, y_0, z_0) = (0.30, 0.25, 0.20)$ at $\epsilon = 3.2$

7. Conclusions and Discussions

In the current research, the dynamical behaviours of a prey-predator-scavenger fishery model with the existence of scavenger harvesting and self-producing toxins by both prey and predator during their predation interaction are analysed. Based on the previous models cited in [7,8], we have proposed a new model for prey-predator-scavenger, in which the dimensionless model was given in system Eq. (2). We derived the non-negativity solutions of this system so that only biologically meaningful steady-states are considered in the analysis. Then, the local stability criteria of the non-coexistence steady-states are investigated by examining the Jacobian matrices and the associated eigenvalues. The global stability analysis for the coexistence steady-state is investigated by using the Lyapunov function. Based on the bifurcation analysis, the scavenging parameter ϵ of the scavenger has a great impact on the dynamical behaviours of the fishery model Eq. (2). The increasing level of scavenging activities by scavengers allows them to consume both predator carcasses that die naturally and are infected by the prey's toxin giving the appearance of both transcritical and Hopf bifurcations in the numerical simulations.

The existence of TB point at $\epsilon_{TB} = 1.0144$ about the steady-states of the fishery model Eq. (2) shows the interchange of stability behaviours of the critical points, where at $0 \leq \epsilon < 1.0144$, the equilibrium point P_4 with scavenger-free is stable and when $1.0144 < \epsilon < 2.1450$ the coexistence equilibrium point P_5 taking over the stable condition from the equilibrium point P_4. The Hopf bifurcation occurs at $\epsilon_{HB} = 2.1450$, where the coexistence equilibrium point P_5 losses its stability and leads to the appearance of periodic solutions. The periodic oscillations of prey, predator and scavenger populations around their coexistence steady-state become larger as ϵ increases from HB point, which indicates that a coexistence steady-state of a fishery system is always unstable. A favourable state is a stable condition in the fishery system which encourages all populations coexist at the intermediate level of scavenging rate ($1.0144 < \epsilon < 2.1450$). A high scavenging rate may result to the scavenger population to be too dominant. This situation affects the numbers of both prey and predator populations in the fishery system since the predation pressure of scavengers on the preys is higher than the predation pressure of predators on the preys. Consequently, the fishery system is not stable and eventually is driven to extinction.

Therefore, it is very important to maintain the stability of the fishery system as a measure of conserving all species in the ecosystem. The stable condition in the fishery system is when all

populations persist together with their harmonious interaction. One of the conservation measures to maintain the stability of the fishery system is to introduce marine protected areas for threatened fish species to avoid fish overexploitation.

Acknowledgements

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