Journal of Advanced Research in Applied Sciences and Engineering Technology

Journal homepage: https://semarakilmu.com.my/journals/index.php/applied_sciences_eng_tech/index
ISSN: 2462-1943

Approximate Analytical Solutions for Non-linear Telegraph Equations with Source Term

Abdul Rahman Farhan Sabdin[1], Che Haziqah Che Hussin[2,*], Arif Mandangan[1], Graygorry Brayone Ekal[1], Jumat Sulaiman[1]

[1] Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Jalan UMS, 88400 Kota Kinabalu, Sabah, Malaysia
[2] Preparatory Centre of Science and Technology, Universiti Malaysia Sabah, Jalan UMS, 88400, Kota Kinabalu, Sabah, Malaysia

ARTICLE INFO

Article history:
Received 27 April 2023
Received in revised form 25 July 2023
Accepted 1 August 2023
Available online 16 August 2023

Keywords:
Non-linear Telegraph equations; source term; Reduced Differential Transform Method; Adomian polynomials; multistep

ABSTRACT

This study proposed a new version of Multistep Modified Reduced Differential Transform Method (MMRDTM) which is applicable for obtaining semi-analytical approximation on Non-linear Telegraph Equations (NLTEs) with source term. Through the experimental approach, we unveiled various advantages of the proposed method, especially its ability to analytically approximate high-speed converging series. Moreover, a significant reduction in the number of calculated terms can also be observed. Before deploying the multistep technique, the substitution of non-linear term in NLTEs with the corresponding Adomian polynomial takes place. Consequently, a simpler technique for approximating NLTEs with source term was discovered. Besides that, the solutions can be estimated more precisely over a longer time. To justify our findings, solutions of NLTEs with source term are obtained using the MMRDTM. The accuracy of the proposed method for solving these equations was tested by comparing the absolute errors between the solutions obtained by the proposed method and the Modified Reduced Differential Transform Method (MRDTM) with the exact solution. The obtained solutions evidenced that the proposed MMRDTM can deliver highly precise approximations for the NLTEs with source term.

1. Introduction

The following form of one-dimensional Non-linear Telegraph Equation (NLTE) has been taken into consideration [1],

$$v_{tt} - \gamma v_{xx} + \alpha v_t + \delta(v) = h(x,t), \quad (1)$$

where

$$v(x,0) = g_1(x)$$

* *Corresponding author.*
E-mail address: haziqah@ums.edu.my

https://doi.org/10.37934/araset.31.3.238248

$\nu_t(x,0) = g_2(x)$

are the initial conditions while $\alpha > 0$. γ are constants, h, g_1, g_2 are known functions, and δ is a function of ν. This form of NLTE is extensively applied in the studies of electric signal propagation in communication cables and the phenomenon of waves. The interaction of convection and diffusion is the underlying reason for several non-linear phenomena in physical and biological processes [2-4]. This equation is mainly applied in modelling radio frequencies, electromagnetic waves, voltage and current on transmission lines, random walk theory, and oceanic diffusion [5]. Adding a factor to the normal heat or mass transport equation that compensates for the impact of finite velocity can also be used to depict a combination of diffusion and wave propagation [6].

A two-dimensional second-order hyperbolic telegraph equation may be applied to model a number of real-world events in science, engineering, and other domains [7]. In fact, the telegraph equation outperforms the heat equation in portraying certain fluid flow issues involving suspensions [8]. It is also more appropriate than the ordinary diffusion equation to model the reaction-diffusion phenomena involving such suspensions [9]. In order to transmit charged particles in chaotic magnetic fields, such as low-energy cosmic rays in the solar wind, the telegraph equation is considered a better option than the diffusion equation [10]. For instance, biologists employ these equations to investigate pulsing blood flow in arteries and random bug migration along a hedge [11]. Eq. (1) can be used to mathematical explain some of the phenomena, such as acoustic wave propagation in Darcy-type porous media and viscous Maxwell fluids parallel flows [12-14]. The works of Dehghan and Ghesmati [2], Mohebbi and Dehghan [3], and Böhme [15] outlined several discussions about the telegraph equation's derivation.

Researchers used various approaches, whether numerically, analytically, or both, in solving telegraph equations. In order to solve the telegraph equation, Mohebbi and Dehghan [3] examined high-order compact solutions. Gao and Chi [16] presented an unconditionally stable difference technique for a one-space dimensional linear hyperbolic equation. The numerical solution developed by Saadatmandi and Dehghan [17] is based on the Chebyshev Tau method. Furthermore, Yousefi [18] presented the Legendre multi-wavelet Galerkin technique to solve the hyperbolic telegraph equation. Dehghan and Ghesmati [2] established a numerical method according to truly meshless local weak-strong (MLWS) methods for dealing with the second-order two-space dimensional telegraph equation. To obtain the solution of the telegraph equation in researches by Atangana [19] and Biazar and Ebrahimi [20], the researchers used the Adomian decomposition approach. Adomian Decomposition Method with Accelerated Formula of Adomian Polynomial solved non-linear telegraph equations in the study by Sayed *et al.,* [21]. In 2021, a New Homotopy Perturbation Method (NHPM) was proposed to determine the logical methods of linear and non-linear second-order telegraph equation [22]. In addition. Modanli *et al.,* [23] proposed residual power series method (RPSM) for solving pseudo hyperbolic partial differential equations with nonlocal conditions.

Furthermore, several powerful and successful approaches for approximating analytical solutions have been proposed and improved. Examples of these solutions are the Adomian Decomposition Method (ADM), Variation Iteration Method (VIM), Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM), Inverse Scattering Method, Balance Method, Hirota's Bilinear Method, and Differential Transform Method (DTM). However, a modification to the fractional Reduced Differential Transform Method (RDTM) was designed and applied to solve the fractional Korteweg de-Vries (KdV) equation [24]. The non-linear term has been replaced by corresponding Adomian polynomials in this technique. Consequently, it can be solved and achieved more quickly and with fewer computed terms. Later, an adaptive multistep DTM was created for handling solitary perturbation initial-value problems [25]. It proposed a rapid-converging sequence with a longer time-period solution. This study adopted these two approaches for solving second-order non-linear Telegraph equations with source term.

Besides the normal telegraph equations, many numerical and analytical methods have been constructed to obtain solutions of telegraph equations with non-linear source term. A flow (fluid flow, velocity, current, or thermal flow) is created when forcing functions or source terms (pressure, torque, force, voltage, or temperature difference) are applied to an impedance [26]. Cao *et al.,* [27] presented a solution using the generalised trapezoidal formula to solve telegraphic equations with source term numerically. Two-level and three-level compact difference and alternating direction implicit compact difference schemes were proposed to solve one and two-dimensional telegraph equations with non-linear forcing term [28]. Furthermore, the solution for a non-linear telegraph equation was constructed by employing a hyperbolic linear solution of the Klein-Gordon equation [29]. For one and two-dimensional linear telegraph equations and telegraph equations with non-linear forcing term, a numerical solution based on shifted Jacobi–Gauss collocation approach was developed [30]. Then, a novel strategy built on the Haar wavelet collocation technique was developed to solve one and two-dimensional second-order linear and non-linear hyperbolic telegraph equations [31]. A hybrid method was introduced by Arslan [32] to approximate the telegraph equation, incorporating the finite difference and differential transform method. Recently, Partohaghighi *et al.,* [33] presented a method to solve hyperbolic telegraph equations according to the fictitious time integration (FTI) and group preserving (GP) methods. In addition, Kanna *et al.,* [34] proposed the Crank-Nicholson Finite Difference Method to obtain solutions of fractional order telegraph equations.

Apart from that, the proposed Multistep Modified Reduced Differential Transform Method (MMRDTM) has been deployed for approximating various types of well-established equations such as Non-linear Schrodinger Equations (NLSEs), Klein-Gordon equations as well as fractional NLSE [35-37]. Moreover, the proposed MMRDTM was also utilised for approximating the non-linear KdV equation and NLSE with power law non-linearity [38,39]. The MMRDTM has been deployed by Hussin *et al.,* [40] for approximating NKdVEs with compactons. From all these occurrences, the obtained results are not only produced with a minimal number of calculated terms and exceptional precision but the solutions are also obtained with high-speed convergent sequence over a broad time span. Besides that, Hussin *et al.,* [41] also used the method to obtain solitary wave solutions.

In this paper, we introduced a multistep technique and a modification by using Adomian polynomials to approximate NLTEs with source term using the MMRDTM. For that purpose, parametrisation methods were used to produce Adomian polynomials without carrying out time-consuming high-derivative computations [42]. The introduced approach yields a convergence sequence of analytical approximations over a long period. At the same time, the quantity of calculated terms is reduced consequentially.

2. Development of Multistep Modified Reduced Differential Transform Method

Generally, lowercase letters represent the original function. For instance, for the function $\nu(x,t)$, the letter ν is the original function. Conversely, the capital letter V in the function $V_\rho(x)$ denotes the transformed functions. The obtained differential transformation of the function $\nu(x,t) = f(x)g(t)$ can be expanded to form [16]

$$\nu(x,t) = \sum_{\alpha=0}^{\infty} \mathrm{F}(\alpha)x^{\alpha} \sum_{\beta=0}^{\infty} G(\beta)t^{\beta} = \sum_{\rho=0}^{\infty} V_\rho(x)t^{\rho}$$

where $V_\rho(x)$ is known as the function of $v(x,t)$. These are some definitions that describe RDTM's fundamental properties.

Definition 1: Consider an analytical and continuous differential function $v(x,t)$. With respect to time t and space variable x, the differential transformation of $v(x,t)$ is defined by

$$V_\rho(x) = \left[\frac{\partial^\rho}{\partial t^\rho} v(x,t)\right]_{t=0} \tag{2}$$

where the transformed function is $V_\rho(x)$.

Definition 2: The $V_\rho(x)$'s inverse transform is presented as

$$v(x,t) = \sum_{\rho=0}^{\infty} V_\rho(x) t^\rho. \tag{3}$$

Combining Eq. (2) and Eq. (3) yields

$$v(x,t) = \sum_{\rho=0}^{\infty} \frac{1}{\rho!}\left[\frac{\partial^\rho}{\partial t^\rho} v(x,t)\right]_{t=0} t^\rho. \tag{4}$$

By applying MMRDTM basic properties to Eq. (1), we obtain

$$V_{\rho+2,\alpha}(x) = \left(\frac{1}{(\rho+2)(\rho+1)}\right)\left(\frac{\partial^2}{\partial x^2}\left(V_{\rho,\alpha}(x)\right) - \sum_{\rho=0}^{\infty} A_{\rho,\alpha} - \sigma(\rho+1)V_{\rho+1,\alpha} + h(x,t)\right). \tag{5}$$

The initial condition should be written as follows

$$V_0(x) = f(x). \tag{6}$$

The following denotes the non-linear term [23]

$$Nv(x,t) = \sum_{n=0}^{\infty} A_n\left(V_0(x), V_1(x), \dots, V_n(x)\right).$$

A method proposed for computing the Adomian polynomials is shown below [42].

$$A_0 = N\left(V_0(x)\right),$$

$$A_n\left(V_0(x), V_1(x), \dots, V_n(x)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} N\left(\sum_{\rho=0}^{n} V_\rho(x) e^{\alpha\rho x}\right) e^{-\alpha n\lambda} d\lambda \ , n \geq 1.$$

The approach does not require time-consuming computations with high derivatives. Via iterative calculation, Eq. (6) and Eq. (5) are combined, allowing us to obtain the values of $V_\rho(x)$. In addition, the set of inverse transformation values, $\{V_\rho(x)\}_{\rho=0}^{n}$ yields the following approximate solution:

$$v(x,t) = \sum_{\rho=0}^{\infty} V_\rho(x) t^\rho, \qquad t \in [0,T].$$

Equal step size $s = \frac{T}{R}$ is applied, while the interval $[0,T]$ is divided to generate R subintervals $[t_{r-1}, t_r]$ and nodes $t_r = rh$ such that for $r = 1,2,\dots,R$. The upcoming procedures are used to compute MMRDTM. Firstly, the modified RDTM is implemented to the initial value problem (IVP) of the interval $[0, t_1]$. Consider the initial conditions as follows

$$\nu(x,0) = g_0(x), \quad \nu_1(x,0) = g_1(x).$$

The approximation result is obtained as

$$\nu_1(x,t) = \sum_{\rho=0}^{\rho} V_{\rho,1}(x) t^{\rho}, \qquad t \in [0, t_1]$$

At each subinterval $[t_{r-1}, t_r]$, the initial conditions are

$$\nu_r(x, t_{r-1}) = \nu_{r-1}(x, t_{r-1}),$$

$$\left(\frac{\partial}{\partial t}\right) \nu_r(x, t_{r-1}) = \left(\frac{\partial}{\partial t}\right) \nu_{r-1}(x, t_{r-1})$$

The initial conditions are utilised for $r \geq 2$. By integrating multistep RDTM with the IVP on $[t_{r-1}, t_r]$, the term t_0 is replaced by t_{r-1}. For $r = 1,2, \dots, R$, the repetition of the procedures is performed to construct an approximate solutions sequence $\nu_r(x,t)$ as follows

$$\nu_r(x,t) = \sum_{\rho=0}^{\rho} V_{\rho,r}(x)(t - t_{r-1})^{\rho}, \qquad t \in [t_{r-1}, t_r].$$

After all, the following solutions are proposed by the MMRDTM

$$\nu(x,t) = \begin{cases} \nu_1(x,t), \ for\ t \in [0, t_1] \\ \nu_2(x,t), \ for\ t \in, [t_1, t_2] \\ \vdots \\ \nu_R(x,t), \ for\ t \in, [t_{R-1}, t_{R-2}]. \end{cases}$$

Observe that the proposed MMRDTM is straightforward. It offers enhanced computing efficiency regardless of the circumstances of the value of s. It is vital to take note that when the step size $s = T$ in the modified RDTM, MMRDTM is conserved.

3. Numerical Results and Discussion

Two examples have been solved by applying the MMRDTM to assess its superiority and precision in solving NLTEs with source term.

Example 1: Consider the following second-order NLTE [21]

$$u_{xx} = u_{tt} + 2u_t + u^2 - e^{2x-4t} + e^{x-2t}, \tag{7}$$

which is subject to the following initial conditions

$$u(x,0) = e^x,$$

$$u_t(x,0) = -2e^x.$$

The exact solution of this equation is e^{x-2t}.

To obtain Eq. (8), the basic properties of MMRDTM is applied as in Eq. (7)

$$U_{\rho+2,i}(x) = \left(\frac{1}{(\rho+2)(\rho+1)}\right)$$
$$\left(\frac{\partial^2}{\partial x^2}\left(U_{\rho,i}(x)\right) - \sum_{\rho=0}^{\infty} A_{\rho,i} - 2(\rho+1)U_{\rho+1,i} + e^{2x}\left(\frac{(-4)^\rho}{\rho!}\right) - e^x\left(\frac{(-2)^\rho}{\rho!}\right)\right). \quad (8)$$

The following can be written from the initial condition

$U_0(x) = e^x$.

with equal step size $s = 0.1$, where the interval $[0,2]$ is divided into 20 subintervals by using the nodes $t_i = is$ such that $[t_{i-1}, t_i], i = 1, 2, \dots, 20$. The core idea of the MMRDTM is mainly to impose the RDTM on the IVP over the interval $[0, t_1]$. For $i \geq 2$, the initial conditions

$$u_i(x, t_{i-1}) = u_{i-1}(x, t_{i-1}), (\partial/\partial t)u_i(x, t_{i-1}) = (\partial/\partial t)u_{i-1}(x, t_{i-1})$$

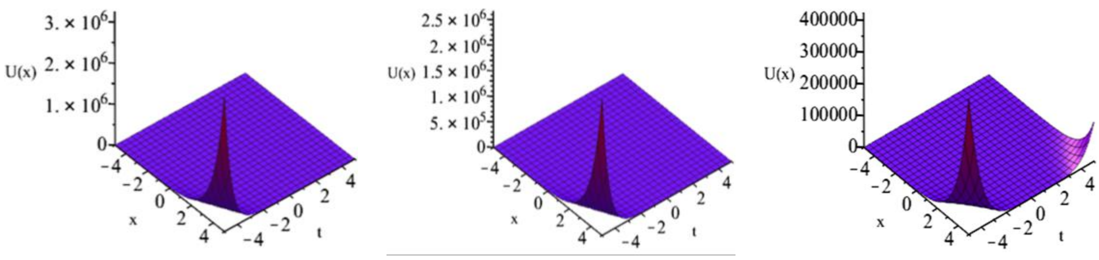
are employed at every subinterval $[t_{i-1}, t_i]$. Then, the MRDTM is deployed to the IVP over the interval $[t_{i-1}, t_i]$, where t_{i-1} substitutes t_0. Next, a multistep procedure for reiterating an operation $u(x, 0) = f_0(x), u_1(x, 0) = a$. The process is continued and repeated to generate an approximate solutions sequence $u_i(x,t), i = 1,2, \dots,20$ to obtain the solution $u(x,t)$ such that:

$$u_i(x,t) = \sum_{\rho=0}^{\mathcal{P}} U_{\rho,i}(x)(t - t_{i-1})^\rho, \qquad t \in [t_{i-1}, t_i].$$

Meanwhile, the MMRDTM yields the following solution

$$u(x,t) = \begin{cases} u_1(x,t) & t \in [0,0.1] \\ u_2(x,t) & t \in [0.1,0.2] \\ \vdots & \vdots \\ u_{20}(x,t) & t \in [1.9,2.0]. \end{cases}$$

Figure 1(a), Figure 1(b), and Figure 1(c) display the graph of the exact solution, approximate solution of MMRDTM for $t \in [-5,5]$ and $x \in [-5,5]$, and approximate solution of RDTM for $t \in [-5,5]$ and $x \in [-5,5]$, respectively. Apparently, the multistep approximate solutions for this sort of NLTEs with source term are significantly closer to the exact solutions.



(a) (b) (c)

Fig. 1. Graphs of semi-analytical methods and exact solution for Example 1, (a) Exact Solution, (b) MMRDTM, (c) MRDTM

Table 1 summarises the error analyses of MMRDTM and MRDTM compared to the exact solution of Example 1.

Table 1
Absolute error comparison of Exact Solution, MMRDTM, MRDTM for Example 1

Time, t	Exact Solution	Absolute Error MMRDTM	Absolute Error MRDTM
0.1	1.349858808	1.000000×10^{-9}	$4.10000000 \times 10^{-9}$
0.2	1.105170918	0	$5.10000000 \times 10^{-7}$
0.3	0.9048374180	8.000000×10^{-10}	$8.51400000 \times 10^{-6}$
0.4	0.7408182207	4.000000×10^{-10}	6.2307700×10^{-5}
0.5	0.6065306597	5.000000×10^{-10}	2.9036380×10^{-4}
0.6	0.4965853038	1.000000×10^{-10}	1.0172416×10^{-3}
0.7	0.4065696597	4.000000×10^{-10}	2.9271258×10^{-3}
0.8	0.3328710837	1.300000×10^{-9}	7.2936741×10^{-3}
0.9	0.2725317930	4.000000×10^{-10}	$1.62832857 \times 10^{-2}$
1.0	0.2231301601	2.000000×10^{-10}	$3.33375932 \times 10^{-2}$
1.1	0.1826835241	1.000000×10^{-10}	$6.36399764 \times 10^{-2}$
1.2	0.1495686192	2.000000×10^{-10}	$1.146699606 \times 10^{-1}$
1.3	0.1224564283	7.000000×10^{-10}	$1.968487101 \times 10^{-1}$
1.4	0.1002588437	7.000000×10^{-10}	$3.242782373 \times 10^{-1}$
1.5	0.08208499862	1.380000×10^{-9}	$5.155764634 \times 10^{-1}$
1.6	0.06720551274	2.74000×10^{-9}	$7.948098663 \times 10^{-1}$
1.7	0.05502322006	6.000000×10^{-11}	1.192525185
1.8	0.04504920239	1.761000×10^{-8}	1.746881320
1.9	0.03688316740	4.260000×10^{-8}	2.504882420
2.0	0.03019738342	3.420000×10^{-9}	3.523712910

Example 2: Consider the following second-order NLTE [29]

$$u_{tt} - u_{xx} + 2u_t + u^2 = e^{-2t}\cosh^2 x - 2e^{-t}\cosh x, \tag{10}$$

which is subjected to the initial condition as follows.

$$u(x,0) = \cosh x,$$

$$u_t(x,0) = -\cosh x,$$

The exact solution of this equation is $e^{-t}\cosh x$. Fundamental principles of MMRDTM are applied to Eq. (10), yielding

$$U_{\rho+2,i}(x) = \left(\frac{1}{(\rho+2)(\rho+1)}\right)$$
$$\left(\frac{\partial^2}{\partial x^2}\left(U_{\rho,i}(x)\right) - \sum_{\rho=0}^{\infty} A_{\rho,i} - 2(\rho+1)U_{\rho+1,i} + \left(\frac{(-2)^\rho}{\rho!}\right)\cosh^2 x - 2\left(\frac{(-1)^\rho}{\rho!}\right)\cosh x\right) \tag{11}$$

We can write the following initial condition for *U* as follows

$$U_0(x) = \cosh x. \tag{12}$$

with equal step size $s = 0.1$. 20 subintervals are produced after dividing the interval $[0,2]$ by using the nodes $t_i = is$ such that $[t_{i-1}, t_i], i = 1, 2, \dots, 20$. The core idea of the MMRDTM is to initially implement the RDTM to the IVP over the interval $[0, t_1]$. For $i \geq 2$, the initial conditions

$$u_i(x, t_{i-1}) = u_{i-1}(x, t_{i-1}), (\partial/\partial t)u_i(x, t_{i-1}) = (\partial/\partial t)u_{i-1}(x, t_{i-1})$$

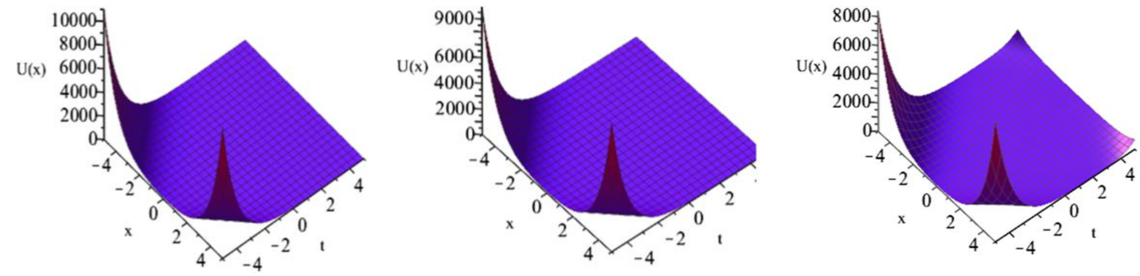
are employed at every subinterval $[t_{i-1}, t_i]$, and the MRDTM is deployed to the IVP over the interval $[t_{i-1}, t_i]$, where t_{i-1} substitutes t_0. Next, a multistep procedure for reiterating an operation $u(x, 0) = f_0(x), u_1(x, 0) = a$. The process is repeatedly executed to generate an approximate solutions sequence $u_i(x, t), i = 1,2, \dots ,20$, obtaining the solution $u(x, t)$ as follows,

$$u_i(x, t) = \sum_{\rho=0}^{\mathcal{P}} U_{\rho,i}(x)(t - t_{i-1})^{\rho}, \qquad t \in [t_{i-1}, t_i].$$

Finally, the MMRDTM yields the following solution

$$u(x, t) = \begin{cases} u_1(x, t), & t \in [0,0.1] \\ u_2(x, t), & t \in [0.1,0.2] \\ \vdots & \vdots \\ u_{20}(x, t), & t \in [1.9,2.0]. \end{cases}$$

Figure 2(a), Figure 2(b), and Figure 2(c) visualise the graph of the exact solution, approximate solution RDTM for $t \in [-5,5]$ and $x \in [-5,5]$, as well as approximate solution MMRDTM for $t \in [-5,5]$ and $x \in [-5,5]$, respectively. Undoubtedly, the solutions of multistep approximation for this type of NLTEs with source term are significantly closer to the exact solutions. Table 2 summarises the error analyses of MMRDTM and MRDTM compared to the exact solution of Example 2.



(a) (b) (c)

Fig. 2. Graphs of semi-analytical methods and exact solution for Example 2, (a) Exact Solution, (b) MMRDTM, (c) MRDTM

Table 2
Absolute error comparison of Exact Solution, MMRDTM, MRDTM for Example 2

Time, t	Exact Solution	Absolute Error MMRDTM	Absolute Error MRDTM
0.1	1.020318167	0	$1.00000000 \times 10^{-9}$
0.2	0.9232220555	0	$2.80000000 \times 10^{-9}$
0.3	0.8353658610	1.000000×10^{-10}	$4.72000000 \times 10^{-8}$
0.4	0.7558702887	0	$3.49000000 \times 10^{-7}$
0.5	0.6839397204	3.000000×10^{-10}	$1.64440000 \times 10^{-6}$
0.6	0.6188542508	3.000000×10^{-10}	$5.82300000 \times 10^{-6}$
0.7	0.5599624824	4.000000×10^{-10}	1.6930400×10^{-5}
0.8	0.5066750067	1.000000×10^{-10}	4.2614700×10^{-5}
0.9	0.4584585049	3.000000×10^{-10}	9.6077200×10^{-5}
1.0	0.4148304099	9.000000×10^{-10}	1.9859120×10^{-4}
1.1	0.3753540770	$4.123105626 \times 10^{-10}$	3.8264200×10^{-4}
1.2	0.3396344138	$2.236067978 \times 10^{-10}$	6.9573200×10^{-4}
1.3	0.3073139261	1.000000×10^{-10}	1.2048972×10^{-3}
1.4	0.2780691394	0	2.0019781×10^{-3}
1.5	0.2516073621	$9.219544457 \times 10^{-10}$	3.2096761×10^{-3}
1.6	0.2276637559	$2.100000002 \times 10^{-9}$	4.9884339×10^{-3}
1.7	0.2059986852	$2.000000250 \times 10^{-10}$	7.5441501×10^{-3}
1.8	0.1863953183	$2.300000020 \times 10^{-9}$	$1.11367864 \times 10^{-2}$
1.9	0.1686574586	5.400000×10^{-9}	$1.60898675 \times 10^{-2}$
2.0	0.1526075793	$2.700000005 \times 10^{-9}$	$2.28009038 \times 10^{-2}$

4. Conclusions

This paper successfully applied a series of solutions to second-order non-linear telegraph equations with source term using MMRDTM. The solutions revealed were evaluated by comparing them to exact and MRDTM solutions. We modified the method by substituting the non-linear term with its Adomian polynomials in the multistep approach. The results showed that the solutions approximated to non-linear telegraph equations with source term can be acquired with high precision. In a nutshell, the proposed MMRDTM performs better than the MRDTM in terms of performance, consistency, and precision for obtaining analytical approximation solutions of NLTEs with source term. All computations in this paper were carried out using the Maple software.

Acknowledgement
This study is funded under the research grant from Universiti Malaysia Sabah (SLB2110).

References
[1] Al-badrani, Hind, Sharefah Saleh, H. O. Bakodah, and M. Al-Mazmumy. "Numerical solution for nonlinear telegraph equation by modified Adomian decomposition method." *Nonlinear Analysis and Differential Equations* 4, no. 5 (2016): 243-257. https://doi.org/10.12988/nade.2016.6418
[2] Dehghan, Mehdi, and Arezou Ghesmati. "Solution of the second-order one-dimensional hyperbolic telegraph equation by using the dual reciprocity boundary integral equation (DRBIE) method." *Engineering Analysis with Boundary Elements* 34, no. 1 (2010): 51-59. https://doi.org/10.1016/j.enganabound.2009.07.002
[3] Mohebbi, Akbar, and Mehdi Dehghan. "High order compact solution of the one-space-dimensional linear hyperbolic equation." *Numerical Methods for Partial Differential Equations: An International Journal* 24, no. 5 (2008): 1222-1235. https://doi.org/10.1002/num.20313
[4] Pozar, David M. *Microwave engineering*. John Wiley & Sons, 2011.
[5] Handibag, S. S., and R. M. Wayal. "Analytical Solutions of Hyperbolic Telegraph Equation by the Laplace Decomposition Method." *Advances and Applications in Mathematical Sciences* 21, no. 3 (2022): 1557-1567.
[6] El-Azab, M. S., and Mohamed El-Gamel. "A numerical algorithm for the solution of telegraph equations." *Applied Mathematics and Computation* 190, no. 1 (2007): 757-764. https://doi.org/10.1016/j.amc.2007.01.091

[7] Ahmad, Imtiaz, Hijaz Ahmad, Ahmed E. Abouelregal, Phatiphat Thounthong, and M. Abdel-Aty. "Numerical study of integer-order hyperbolic telegraph model arising in physical and related sciences." *The European Physical Journal Plus* 135 (2020): 1-14. https://doi.org/10.1140/epjp/s13360-020-00784-z
[8] Eckstein, Eugene C., Jerome A. Goldstein, and Mark Leggas. "The mathematics of suspensions: Kac walks and asymptotic analyticity." In *Proceedings of the Fourth Mississippi State Conference on Difference Equations and Computational Simulations*, vol. 3, pp. 39-50. 1999.
[9] Mittal, R. C., and Rachna Bhatia. "Numerical solution of second order one dimensional hyperbolic telegraph equation by cubic B-spline collocation method." *Applied Mathematics and Computation* 220 (2013): 496-506. https://doi.org/10.1016/j.amc.2013.05.081
[10] Tautz, Robert C., and Ian Lerche. "Application of the three-dimensional telegraph equation to cosmic-ray transport." *Research in Astronomy and Astrophysics* 16, no. 10 (2016): 162. https://doi.org/10.1088/1674-4527/16/10/162
[11] Mohanty, R. K. "New unconditionally stable difference schemes for the solution of multi-dimensional telegraphic equations." *International Journal of Computer Mathematics* 86, no. 12 (2009): 2061-2071. https://doi.org/10.1080/00207160801965271
[12] Evans, David J., and Hasan Bulut. "The numerical solution of the telegraph equation by the alternating group explicit (AGE) method." *International Journal of Computer Mathematics* 80, no. 10 (2003): 1289-1297. https://doi.org/10.1080/0020716031000112312
[13] Jordan, P. M., Martin R. Meyer, and Ashok Puri. "Causal implications of viscous damping in compressible fluid flows." *Physical Review E* 62, no. 6 (2000): 7918. https://doi.org/10.1103/PhysRevE.62.7918
[14] Pascal, Henri. "Pressure wave propagation in a fluid flowing through a porous medium and problems related to interpretation of Stoneley's wave attenuation in acoustical well logging." *International Journal of Engineering Science* 24, no. 9 (1986): 1553-1570. https://doi.org/10.1016/0020-7225(86)90163-1
[15] Böhme, Gert. *Non-Newtonian fluid mechanics*. Elsevier, 2012.
[16] Gao, Feng, and Chunmei Chi. "Unconditionally stable difference schemes for a one-space-dimensional linear hyperbolic equation." *Applied Mathematics and Computation* 187, no. 2 (2007): 1272-1276. https://doi.org/10.1016/j.amc.2006.09.057
[17] Saadatmandi, Abbas, and Mehdi Dehghan. "Numerical solution of hyperbolic telegraph equation using the Chebyshev tau method." *Numerical Methods for Partial Differential Equations: An International Journal* 26, no. 1 (2010): 239-252. https://doi.org/10.1002/num.20442
[18] Yousefi, S. A. "Legendre multiwavelet Galerkin method for solving the hyperbolic telegraph equation." *Numerical Methods for Partial Differential Equations: An International Journal* 26, no. 3 (2010): 535-543. https://doi.org/10.1002/num.20445
[19] Atangana, Abdon. "On the stability and convergence of the time-fractional variable order telegraph equation." *Journal of Computational Physics* 293 (2015): 104-114. https://doi.org/10.1016/j.jcp.2014.12.043
[20] Biazar, J., and H. Ebrahimi. "An approximation to the solution of telegraph equation by Adomian decomposition method." In *International Mathematical Forum*, vol. 2, no. 45, pp. 2231-2236. 2007. https://doi.org/10.12988/imf.2007.07196
[21] Sayed, Ahmed Y., K. M. Abdelgaber, A. R. Elmahdy, and I. L. El-Kalla. "Solution of the telegraph equation using adomian decomposition method with accelerated formula of adomian polynomials." *Information Sciences Letters* 10, no. 1 (2021): 39-46. https://doi.org/10.18576/isl/100106
[22] Kumar, Rajesh, A. K. Singh, and S. S. Yadav. "New Homotopy Perturbation Method For Analytical Solution Of Telegraph Equation." *Turkish Journal of Computer and Mathematics Education (TURCOMAT)* 12, no. 12 (2021): 2144-2155.
[23] Modanli, Mahmut, Sadeq Taha Abdulazeez, and Ahmad Muhamad Husien. "A residual power series method for solving pseudo hyperbolic partial differential equations with nonlocal conditions." *Numerical Methods for Partial Differential Equations* 37, no. 3 (2021): 2235-2243. https://doi.org/10.1002/num.22683
[24] Saha Ray, S. "Numerical solutions and solitary wave solutions of fractional KDV equations using modified fractional reduced differential transform method." *Computational Mathematics and Mathematical Physics* 53, no. 12 (2013): 1870-1881. https://doi.org/10.1134/S0965542513120142
[25] El-Zahar, Essam R. "Applications of adaptive multi step differential transform method to singular perturbation problems arising in science and engineering." *Applied Mathematics & Information Sciences* 9, no. 1 (2015): 223. https://doi.org/10.12785/amis/090128
[26] Ellis, George. *Observers in control systems: a practical guide*. Elsevier, 2002.
[27] Cao, Huai-Huo, Li-Bin Liu, and Yong Zhang. "Numerical Solution of Telegraphic Equations with Source Term Using the Generalized Trapezoidal Formula." In *2011 Fourth International Joint Conference on Computational Sciences and Optimization*, pp. 3-6. IEEE, 2011. https://doi.org/10.1109/CSO.2011.182

[28] Xie, Shu-Sen, Su-Cheol Yi, and Tae In Kwon. "Fourth-order compact difference and alternating direction implicit schemes for telegraph equations." *Computer Physics Communications* 183, no. 3 (2012): 552-569. https://doi.org/10.1016/j.cpc.2011.11.023
[29] Jang, T. S. "A new solution procedure for the nonlinear telegraph equation." *Communications in Nonlinear Science and Numerical Simulation* 29, no. 1-3 (2015): 307-326. https://doi.org/10.1016/j.cnsns.2015.05.004
[30] Hafez, Ramy M. "Numerical solution of linear and nonlinear hyperbolic telegraph type equations with variable coefficients using shifted Jacobi collocation method." *Computational and Applied Mathematics* 37 (2018): 5253-5273. https://doi.org/10.1007/s40314-018-0635-1
[31] Asif, Muhammad, Nadeem Haider, Qasem Al-Mdallal, and Imran Khan. "A Haar wavelet collocation approach for solving one and two-dimensional second-order linear and nonlinear hyperbolic telegraph equations." *Numerical Methods for Partial Differential Equations* 36, no. 6 (2020): 1962-1981. https://doi.org/10.1002/num.22512
[32] Arslan, Derya. "The numerical study of a hybrid method for solving telegraph equation." *Applied Mathematics and Nonlinear Sciences* 5, no. 1 (2020): 293-302. https://doi.org/10.2478/amns.2020.1.00027
[33] Partohaghighi, Mohammad, Mehmet Ali Akinlar, and Gerhard-Wilhelm Weber. "New solutions of hyperbolic telegraph equation." *Journal of Dynamics and Games* 8, no. 2 (2021): 129-138. https://doi.org/10.3934/jdg.2020029
[34] Kanna, M. R. Rajesh, R. Pradeep Kumar, Soner Nandappa, and Ismail Naci Cangul. "On solutions of fractional order telegraph partial differential equation by Crank-Nicholson finite difference method." *Applied Mathematics and Nonlinear Sciences* 5, no. 2 (2020): 85-98. https://doi.org/10.2478/amns.2020.2.00017
[35] Hussin, Che Haziqah Che, Adem Kilicman, and Amirah Azmi. "Analytical Solutions Of Nonlinear Schrodinger Equations Using Multistep Modified Reduced Differential Transform Method." *COMPUSOFT An International Journal of Advanced Computer Technology* 7, no. 11 (2018): 2939-2944.
[36] Che, Hussin Che Haziqah, Ahmad Izani Ismail, Adem Kilicman, and Amirah Azmi. "Analytical solutions of nonlinear Klein-Gordon equations using multistep modified reduced differential transform method." *Thermal Science* 23, no. Suppl. 1 (2019): 317-326. https://doi.org/10.2298/TSCI181015045C
[37] Hussin, Che Haziqah Che, Ahmad Izani Md Ismail, Adem Kilicman, and Amirah Azmi. "Approximate analytical solutions of fractional nonlinear Schrodinger equations using multistep modified reduced differential transform method." In *AIP Conference Proceedings*, vol. 2184, no. 1, p. 060003. AIP Publishing LLC, 2019. https://doi.org/10.1063/1.5136435
[38] Che Hussin, C. H., A. I. M. Ismail, A. Kilicman, and A. Azmi. "Approximate analytical solutions of nonlinear korteweg-de vries equations using multistep modified reduced differential transform method." *Mathematics and Statistics* 8, no. 2A (2020): 9-16. https://doi.org/10.13189/ms.2020.081302
[39] Hussin, Che Haziqah Che, Amirah Azmi, Ahmad Izani Md Ismail, Adem Kilicman, and Ishak Hashim. "Approximate Analytical Solutions of Bright Optical Soliton for Nonlinear Schrödinger Equation of Power Law Nonlinearity." *Baghdad Science Journal* 18, no. 1 (2021): 836-845. https://doi.org/10.21123/bsj.2021.18.1(Suppl.).0836
[40] Hussin, Che Haziqah Che, Amirah Azmi, and Adem Kilicman. "Solitary Wave Solutions with Compact Support for The Nonlinear Dispersive K (m, n) Equations by Using Approximate Analytical Method." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 88, no. 1 (2021): 24-34. https://doi.org/10.37934/arfmts.88.1.2434
[41] Hussin, Che Haziqah Che, Arif Mandangan, Amirah Azmi, and Adem Kilicman. "Solitary Wave Solutions for Forced Nonlinear Korteweg-de Vries Equation by Using Approximate Analytical Method." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 99, no. 2 (2022): 197-206. https://doi.org/10.37934/arfmts.99.2.197206
[42] Kataria, K. K., and P. Vellaisamy. "Simple parametrization methods for generating Adomian polynomials." *Applicable Analysis and Discrete Mathematics* 10, no. 1 (2016): 168-185. https://doi.org/10.2298/AADM160123001K