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Approximate Analytical Solution for Time-Fractional Nonlinear Telegraph Equations with Source Term

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ABSTRACT

In this study, we considered time-fractional nonlinear telegraph equations (TFNLTEs). For solving the TFNLTEs, we deployed a method known as the Multistep Modified Reduced Differential Transform Method (MMRDTM). Prior to the multistep technique, the nonlinear term in TFNLTEs is replaced with corresponding Adomian polynomials. It can be observed that the MMRDTM is much simpler and more straightforward. On top of that, it works exceptionally where the obtained solutions are more accurately approximated over time. To demonstrate the performance of the MMRDTM in terms of its capabilities and accuracies, we provided two numerical examples of solving TFNLTEs by using MMRDTM and Modified Reduced Differential Transform Method (MRDTM). By comparing the absolute errors of the obtained solutions by both methods, we demonstrated that the solutions provided by the MMRDTM much closer to the exact solutions compared to the corresponding solutions yielded by the MRDTM. This justified that the MMRDTM provides highly accurate and precise solutions for the TFNLTEs.

1. Introduction

Various physical phenomena in science and engineering such as diffusion process, rheology, damping laws, electric transmission, viscoelasticity, and fluid mechanics can be effectively enlightened and demonstrated by fractional partial differential equations (FPDEs). Unfortunately, accurate analytical solutions for these kind of equations can only be approximated seldomly. Alternatively, approximation and numerical techniques are employed to address such issues. Various approaches have been proposed such as finite difference method [1], Adomian decomposition method [2], Fourier method [3], variational iteration method [4], wavelet method [5], homotopy analysis method [6], Tau method [7], and the fractional Sumudu decomposition method (FSDM) [8].

In recent years, the telegraph equations have attracted many researchers due to its usage in physical, chemical, and biological sciences. It is a hyperbolic partial differential equation that is used to describe radio frequencies, random walk theory, electromagnetic waves, voltage and current on

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transmission lines, and oceanic diffusion, among other things. It also possesses both diffusion and wave motion characteristics. Besides, the telegraph equation is better at modelling reaction-diffusion than the standard diffusion equation, and it is a preferable model for understanding several fluid flow problems involving suspensions as compared to the heat equation [9]. On the other hand, the non-standard telegraph equations are well defined by a time-fractional derivative of order α (non-integer). Moreover, fractional telegraph equations are the generalisation of the standard telegraph equations, which naturally come from fractal space-time. The time-fractional telegraph equation (TFTE) is important in modeling the Brownian motion. Orsingher and Zhao [10] demonstrated that TFTEs determine the telegraph processes with Brownian time and the law of iterated Brownian motion. Apart from that, Nirmala and Balachandran [11] studied the application of the TFTEs to calculate signal and power losses during transmission media in a communication system. Furthermore, Vyawahare and Nataraj [12] modelled a neutron transport in a nuclear reactor using the TFTE. Recently, Madhukar *et al.,* [13] showed concrete proof for the damped-hyperbolic nature of transient heat conduction in porcine blood and muscle tissue. Through that experiment, they found that the TFTE mimics the wave-like nature of heat conduction and temperature profiles as well as avoids the requirement for additional adjustable parameters.

Several authors suggested different ways to obtain solutions of the TFTEs. For instance, the reproducing kernel theorem was used by Jiang and Lin [14] to find the solution to this equation. Next, Kumar *et al.,* [15] devise a local meshless method to obtain the solution for the TFTEs with linear source term using radial basis functions. This is followed by a numerical experiment on various complicated domains to demonstrate the method's effectiveness, and it shows a pleasant result. Furthermore, Wang and Mei [16] applied the generalized finite difference scheme in time and the Legendre spectral Galerkin technique in space to solve the TFTEs with a forcing term. Furthermore, Kumar *et al.,* [17] presented a finite difference scheme for the generalised TFTEs with forcing term that is defined using generalised fractional derivative terms. Ray [18] formulated and implemented a modification to the Fractional Reduced Differential Transform Method (FRDTM) for solving the fractional Korteweg–De Vries equation (FKdVEs). This strategy used Adomian polynomials to substitute the nonlinear term of the equation. Consequently, the nonlinear problems can have solutions in shorter time with fewer calculated terms. Besides that, Hassani [19] proposed variable-order space–time fractional telegraph equation using transcendental Bernstein series.

In 2018, Hussin *et al.,* [20] proposed the Multistep Modified Reduced Differential Transform Method (MMRDTM) and deployed this method for solving nonlinear Schrodinger equations (NLSE). As a result, the MMRDTM performed outstandingly as the NLSEs are successfully approximated with high accuracy and precision. Furthermore, the MMRDTM was also applied by Hussin *et al.,* [21] to handle Klein-Gordon equations. Once again, the MMRDTM performed outstandingly as approximation of the equations are obtained with great precision and efficiency. Motivated by these results, Hussin *et al.,* [22] implemented the MMRDTM to determine the approximate analytical solutions of the one-dimensional fractional NLSE. As expected, the results are obtained by the MMRDTM with high accuracy to the exact solutions. Hussin *et al.,* also use the method to obtain solitary wave solutions [23-24].

In this paper, we proposed a multistep technique and a variation by using Adomian polynomials to discover the solution to the one-dimensional time-fractional nonlinear telegraph equations (TFNLTEs) with source term using the MMRDTM. To produce Adomian polynomials, we deployed parametrization approaches instead of doing time-consuming high-derivative computations. The following one-dimensional TFNLTEs [25] is considered

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} + \theta\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \gamma\frac{\partial^{2}u(x,t)}{\partial x^{2}} + f(x,t), \tag{1}$$

where $x \in (0, l)$ and $t \in (0, \tau)$. The initial conditions are given by

$u(x, 0) = \varphi(x)$,

$u_t(x, 0) = \psi(x)$,

where α is a parameter defining the fractional derivative's order. Furthermore, the functions $f, \varphi,$ and ψ are sufficiently smooth prescribed functions. Then, the rates θ and γ are arbitrary nonnegative and positive constants, respectively. In the case where $½ < \alpha \leq 1$, the Caputo fractional derivatives are used to describe the time-fractional derivative.

2. Formation of Fractional Multistep Modified Reduced Differential Transform Method

To demonstrate the fundamental concepts behind the use of fractional MMRDTM, let consider a general nonlinear partial differential equation

$$Lu(x, t) + Ru(x, t) + Nu(x, t) = g(x, t)\ , \tag{2}$$

subjected to the initial condition,

$$u(x,\ 0) = f(x), \tag{3}$$

where $L \equiv\ D_t^{\alpha}$, R, $Nu(x, t)$ and $g(x, t)$ are invertible linear operator, remaining part of the linear operator, nonlinear term and inhomogeneous term respectively. Next, the following iteration formula is formed according to the Reduced Differential Transformation Method (RDTM)

$$(r + 1)U_{r+1}(x) = S_r(x) - RU_r(x) - NU_r(x), \tag{4}$$

where $U_r(x)$, $S_r(x)$, $NU_r(x)$ and $RU_r(x)$ represent the transformation functions of $Lu(x, t)$, $Ru(x, t)$, $Nu(x, t)$ and $g(x, t)$ respectively. From the initial condition, we have

$$U_0(x) = f(x). \tag{5}$$

Then, the nonlinear term is written as follows

$$N(u, t) = \sum_{n=0}^{\infty} A_n\big(U_0(x), U_1(x), \dots, U_n(x)\big)t^n, \tag{6}$$

where A_n is the correspond Adomian's polynomials. Recently, a novel technique for calculating the Adomian polynomials was claimed in [26], such as

$$A_0 = N\big(U_0(x)\big),$$

$$A_n(U_0(x), U_1(x), \dots, U_n(x)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} N\Big(\sum_{k=0}^{n} U_k(x)e^{ikx}\Big)\, e^{-in\lambda}\, d\lambda,$$

where $n \geq 1$. As demonstrated, this algorithm does not necessitate the time-consuming calculation of high derivatives. The function $u(x, t)$ can be portrayed on the basis of the differential transformation properties such as

$$u(x,t) = \sum_{r=0}^{\infty} U_r(x) t^{\alpha r}. \tag{7}$$

Definition 1. If function $u(x,t)$ is analytic and differentiated continuously with respect to space x and time t in the desired domain, then let

$$U_r(x) = \frac{1}{\Gamma(r\alpha+1)} \{ D_t^{\alpha r} u(x,t) \}_{t=0}. \tag{8}$$

The generalized fractional RDTM of the function $u(x,t)$ is given by

$$u(x,t) = \sum_{k=0}^{\infty} \left(\frac{1}{\Gamma(k\alpha+1)} \{ D_t^{\alpha k} u(x,t) \}_{t=0} \right) t^{\alpha k}. \tag{9}$$

By applying the Riemann-Liouville integral J^{α} on both sides of Eq. (2), we obtain

$$u(x,t) = J^{\alpha} g(x,t) - J^{\alpha} R u(x,t) - J^{\alpha} N u(x,t) + \Phi, \tag{10}$$

where from the initial condition $\Phi = u(x,0) = f(x)$. Thus, plugging Eq. (8) and Eq. (6), for $u(x,t)$ and $N(u,t)$ respectively, in Eq. (9) yields

$$\sum_{r=0}^{\infty} U_r(x) t^{\alpha r} = f(x) + J^{\alpha}\left(\sum_{r=0}^{\infty} G_r(x) t^{\alpha r}\right) - J^{\alpha}\left(R\left(\sum_{r=0}^{\infty} U_r(x) t^{\alpha r}\right)\right) - J^{\alpha}\left(\sum_{r=0}^{\infty} A_r(x) t^{\alpha r}\right) \tag{11}$$

where $g(x,t) = \sum_{r=0}^{\infty} G_r(x) t^{\alpha r}$, and $G_k(x)$ is the transformed function of $g(x,t)$. After performing Riemann-Liouville integral J^{α}, we obtain [18],

$$\sum_{r=0}^{\infty} U_r(x) t^{\alpha r} = f(x) + \left(\sum_{r=0}^{\infty} G_r(x) \frac{t^{\alpha(r+1)}\Gamma(\alpha r+1)}{\Gamma(\alpha(r+1)+1)}\right) - \left(R\left(\sum_{r=0}^{\infty} U_r(x) \frac{t^{\alpha(r+1)}\Gamma(\alpha r+1)}{\Gamma(\alpha(r+1)+1)}\right)\right) - \left(\sum_{r=0}^{\infty} A_r(x) \frac{t^{\alpha(r+1)}\Gamma(\alpha r+1)}{\Gamma(\alpha(r+1)+1)}\right). \tag{12}$$

Finally, following recursive formula is derived by equating coefficients with similar powers of t

$$U_0(x) = f(x),$$

and

$$U_{r+1}(x) = G_r(x) \frac{\Gamma(\alpha r+1)}{\Gamma(\alpha(r+1)+1)} - R\left(U_r(x) \frac{\Gamma(\alpha r+1)}{\Gamma(\alpha(r+1)+1)}\right) - A_r(x) \frac{\Gamma(\alpha r+1)}{\Gamma(\alpha(r+1)+1)}, \tag{13}$$

for $r \geq 0$. Using the known $U_0(x)$, all $U_1(x), U_2(x), \ldots, U_n(x)$ components can be determined using Eq. (13). Then by replacing these $U_0(x), U_1(x), U_2(x), \ldots, U_n(x)$ in Eq. (8), an approximate solution can be acquired as follows [18]

$$\tilde{u}_p(x,t) = \sum_{r=0}^{p} U_r(x) t^{\alpha r},$$

where p is the order of this approximate solution. Consequently, the following series solution is obtained

$$u(x,t) = \lim_{p\to\infty} \tilde{u}_p(x,t).$$

Finally, follow the multi-step scheme as obtained in *Algorithm 1.*

Algorithm 1 Multi-step scheme for fractional approximate solution as follows

Step 1 : By using the nodes $t_m = kh$, divide the interval $[\,0,T\,]$ into M subintervals $[t_{m-1}, t_m], m = 1,2,\dots,M$, equally sized $h = T/M$.

Step 2 : The RDTM is applied over the interval $[0, t_1]$ to the initial value problem. Then, use the initial conditions $u(x,0) = f_0(x)$, $u_1(x,0) = f_1(x)$ to get the following approximate solution

$$u_1(x,t) = \sum_{r=0}^{R} U_{r,1}(x) t^{\alpha r}$$

where $t \in [0, t_1]$.

Step 3 : Use the following initial conditions

$$u_m(x, t_{m-1}) = u_{m-1}(x, t_{m-1})$$

$$(\partial/\partial t) u_m\,(x, t_{m-1}\,) = (\partial/\partial t) u_{m-1}\,(x, t_{m-1}\,)$$

at each subinterval $[t_{m-1}, t_m]$ for the case where $m \geq 2$. Then, the MMRDTM is used to solve the initial value problem over the interval $[t_{m-1}, t_m]$ where t_{m-1} replaces t_0.

Step 4 : The procedure is reiterated and proceeded in order to yield a sequence of approximate $u_m(x,t), m = 1,2,\dots,M$, for the solutions $u(x,t)$ such as [27-29]

$$u_k(x,t) = \sum_{r=0}^{R} U_{r,m}(x)(t - t_{m-1})^{\alpha r},$$

where $t_{i-1} \leq t \leq t_i$.

Step 5 : In fact, the MsFRDTM is executing the subsequent solution as follows

$$u(x,t) = \begin{cases} u_1(x,t) & , t \in [0, t_1] \\ u_2(x,t) & , t \in [t_1, t_2] \\ & \vdots \\ u_M(x,t) & , t \in [t_{M-1}, t_M]. \end{cases}$$

3. Numerical Results and Discussion

To demonstrate the effectiveness and accuracy of the proposed MMRDTM for approximating solutions of fractional nonlinear Telegraph equation (FNLTE), we provide the following numerical examples.

Example 1 The one-dimensional FNLTE is considered as

$$u_{tt}^{\alpha} + 2u_t^{\alpha} = u_{xx} - u^2 - e^{2x-4t} + e^{x-2t}, \tag{14}$$

with the initial conditions

$u(x,0) = e^x$,

$u_t(x,0) = -2e^x$.

The exact solution of this equation for $\alpha = 1$ is e^{x-2t}.

Applying MMRDTM to Eq. (14) and using the basic properties of MMRDTM, one can obtain

$$U_{R+2,i}(x) = \left(\frac{\Gamma(1+r\alpha)}{\Gamma(1+(2+r)\alpha)}\right)\left(\frac{\partial^2}{\partial x^2}\left(U_{r,i}(x)\right) - \sum_{r=0}^{n} A_{r,i} - 2\left(\frac{\Gamma(1+(1+r)\alpha)}{\Gamma(1+r\alpha)}\right)U_{r+1,i} + e^{2x}\left(\frac{(-4)^r}{r!}\right) - e^{x}\left(\frac{(-2)^r}{r!}\right)\right).$$

From the initial condition, we assign

$U_0(x) = e^x$.

Divide the interval $[0,T]$ into M subintervals by using the nodes $t_i = ih$ of equal step size $h = T/M$ such that $[t_{i-1}, t_i], i = 1,2,\dots,10$. After that, the RDTM is used to solve the initial value problem over the interval $[0, t_1]$. Then, use the initial conditions $u(x,0) = f_0(x)$ and $u_1(x,0) = f_1(x)$ to obtain the following approximate solution

$$u_1(x,t) = \sum_{r=0}^{R} U_{r,1}(x)t^{\alpha r},$$

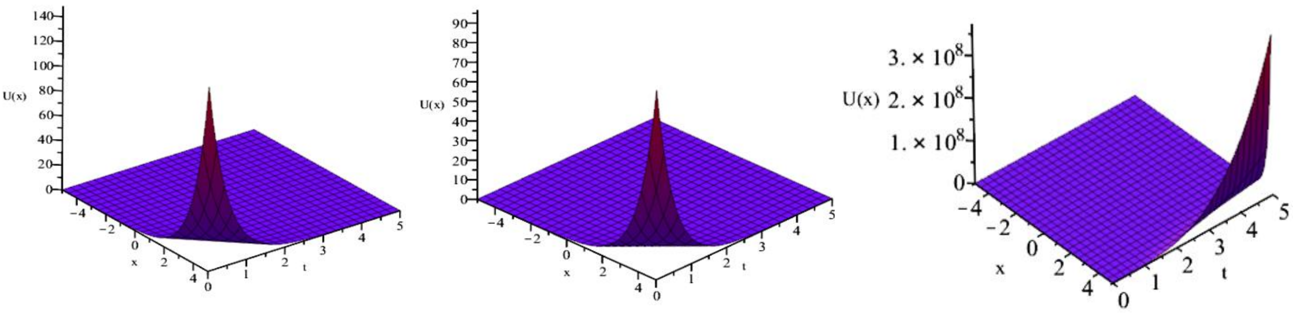
where $0 \le t \le t_1$. For the case where $i \ge 2$, the initial conditions $u_i(x,t_{i-1}) = u_{i-1}(x,t_{i-1})$ and $(\partial/\partial t)u_i(x,t_{i-1})$ is applied at each subinterval $[t_{i-1}, t_i]$. Then, the RDTM is used to solve the initial value problem over the interval $[t_{i-1}, t_i]$, where t_0 is substituted by t_{i-1}. This procedure is repeated continuously to create a series of approximate solutions $u_i(x,t)$, where $i = 1,2,\dots,10$, for the solution $u(x,t)$ such as

$$u_i(x,t) = \sum_{r=0}^{R} U_{r,i}(x)(t - t_{i-1})^{\alpha r},$$

where $t \in [t_{i-1}, t_i]$. In fact, the MMRDTM yields the following solution

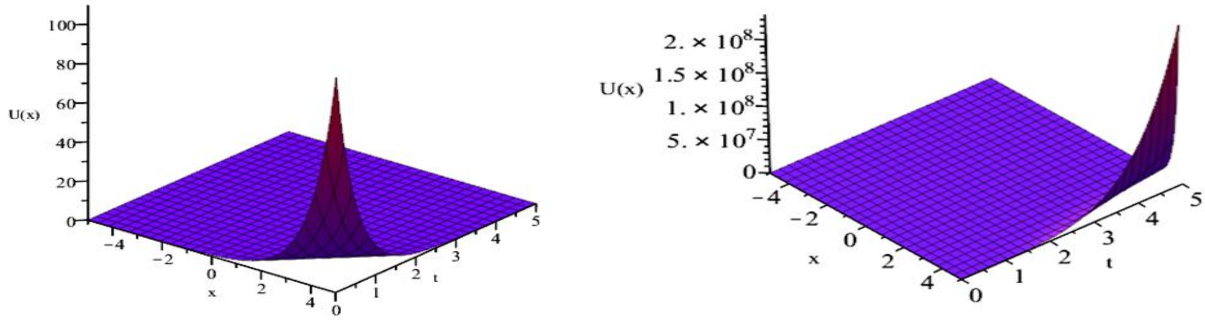
$$u(x,t) = \begin{cases} u_1(x,t) & t \in [0, t_1] \\ u_2(x,t) & t \in [t_1, t_2] \\ \vdots & \vdots \\ u_{10}(x,t) & t \in [t_9, t_{10}] \end{cases}.$$

Figure 1(a) is the exact solution compared to the approximate solutions by the proposed MMRDTM and MRDTM in Figure 1(b) and Figure 1(c) respectively with $\alpha = 0.6$. The outcomes of the approximate solution for $\alpha = 0.8$ are shown in Figure 2(a) for the MMRDTM and Figure 2(b) for the MRDTM respectively. From the obtained results, obviously the approximations produced by the MMRDTM very close to the exact solutions compared to the MRDTM.



(a) Exact solution (b) MMRDTM (c)MRDTM

Fig. 1. Comparison graphs of semi-analytical methods for $\alpha = 0.6$ with exact solution for Example 1



(a) MMRDTM (b) MRDTM

Fig. 2. Comparison graphs of semi-analytical methods with $\alpha = 0.8$ for Example 1

Table 1 and Table 2 show the error analyses of the solutions yielded by the MMRDTM and MRDTM respectively for Example 1 with different values of α.

Table 1
Error comparison between MMRDTM and MRDTM for Example 1 with $\alpha = 0.6$

Time, t	Exact Solution	Absolute Error MMRDTM	Absolute Error MRDTM
0.1	1.349858808	0	0.282102680
0.2	1.105170918	0	0.2377261703
0.3	0.9048374180	$5.000000\text{x}10^{-10}$	0.2476273192
0.4	0.7408182207	$3.000000\text{x}10^{-10}$	0.4234119704
0.5	0.6065306597	$5.000000\text{x}10^{-10}$	0.8803608197
0.6	0.4965853038	$7.000000\text{x}10^{-10}$	1.750204400
0.7	0.4065696597	$3.100000\text{x}10^{-9}$	3.182896167
0.8	0.3328710837	$2.000000\text{x}10^{-9}$	5.346344380
0.9	2.245564710	$2.600000\text{x}10^{-9}$	8.425688724
1.0	0.2231301601	$2.59000\text{x}10^{-8}$	12.62251238

Table 2
Error comparison between MMRDTM and RDTM for Example 1 with $\alpha = 0.8$

Time, t	Exact Solution	Absolute Error MMRDTM	Absolute Error MRDTM
0.1	1.349858808	0	0.130571465
0.2	1.105170918	0	0.1114617051
0.3	0.9048374180	$1.0000000\text{x}10^{-10}$	0.0694857406
0.4	0.7408182207	$2.000000\text{x}10^{-10}$	0.0317849732
0.5	0.6065306597	$1.000000\text{x}10^{-10}$	0.0178225805
0.6	0.4965853038	$8.000000\text{x}10^{-10}$	0.0507412754
0.7	0.4065696597	$1.000000\text{x}10^{-10}$	0.1622436105
0.8	0.3328710837	$7.000000\text{x}10^{-10}$	0.3956705629
0.9	0.2725317930	$2.000000\text{x}10^{-9}$	0.8083715096
1.0	0.2231301601	$1.000000\text{x}10^{-10}$	1.473732968

Example 2 The one-dimensional fractional nonlinear Telegraph equation is taken into consideration

$$u_{tt}^{\alpha} - u_{xx} + 2u_t^{\alpha} + u^2 = e^{-2t}\cosh^2 x - 2e^{-t}\cosh x, \tag{17}$$

subject to the initial condition

$$u(x,0) = \cosh x,$$

$$u_t(x,0) = -\cosh x.$$

The exact solution of this equation for $\alpha = 1$ is $e^{-t}\cosh x$.
Firstly, apply MMRDTM to Eq. (17) and use the basic properties of MMRDTM; we will get

$$U_{r+2,i}(x) = \left(\frac{\Gamma(1+r\alpha)}{\Gamma(1+(2+r)\alpha)}\right)\left(\frac{\partial^2}{\partial x^2}\left(U_{r,i}(x)\right) - \sum_{r=0}^{n} A_{r,i} - 2\left(\frac{\Gamma(1+(1+r)\alpha)}{\Gamma(1+r\alpha)}\right)U_{r+1,i} + \left(\frac{(-2)^r}{r!}\right)\cosh^2 x - \left(\frac{(-2)^r}{r!}\right)\cosh x\right).$$

From the initial condition, we write

$$U_0(x) = e^x.$$

Divide the interval $[0,T]$ into M subintervals by using the nodes $t_i = ih$ of equal step size $h = \frac{T}{M}$ such that $[t_{i-1}, t_i], i = 1,2, \ldots, 10$. After that, the RDTM is used to solve the initial value problem over the interval $[0, t_1]$. Then, use the initial conditions $u(x,0) = f_0(x)$ and $u_1(x,0) = f_1(x)$ to obtain the following approximate solution

$$u_1(x,t) = \sum_{r=0}^{R} U_{r,1}(x) t^{\alpha r},$$

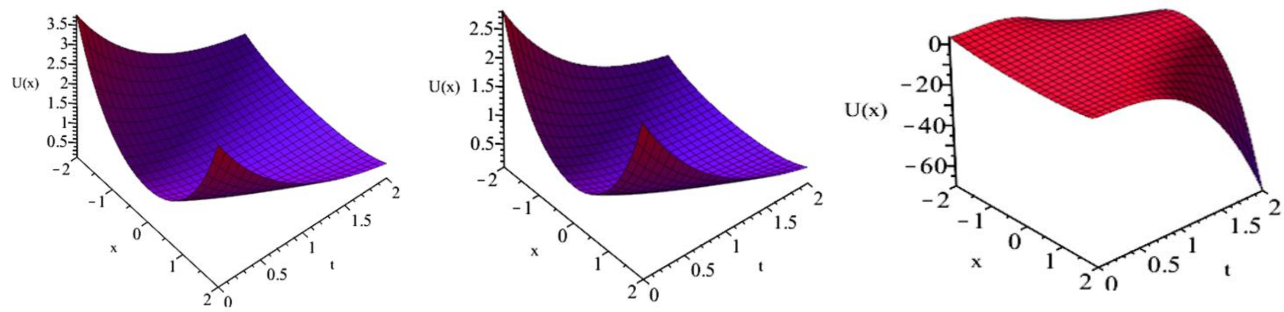
where $0 \le t \le t_1$. For $i \ge 2$, we use the initial conditions $u_i(x, t_{i-1}) = u_{i-1}(x, t_{i-1})$ and $(\partial/\partial t) u_i(x, t_{i-1})$ at each subinterval $[t_{i-1}, t_i]$. Then RDTM is used to solve the initial value problem over the interval $[t_{i-1}, t_i]$, where t_{i-1} replaces t_0. This procedure is repeatedly continued to create a sequence of approximate solutions $u_i(x,t)$, where $i = 1,2,\dots,10$, for the solution $u(x,t)$ such as

$$u_i(x,t) = \sum_{r=0}^{R} U_{r,i}(x) (t - t_{i-1})^{\alpha r}$$

where $t \in [t_{i-1}, t_i]$. In fact, the MMRDTM takes the following solution

$$u(x,t) = \begin{cases} u_1(x,t) & t \in [0, t_1] \\ u_2(x,t) & t \in [t_1, t_2] \\ \vdots & \vdots \\ u_K(x,t) & t \in [t_9, t_{10}]. \end{cases}$$

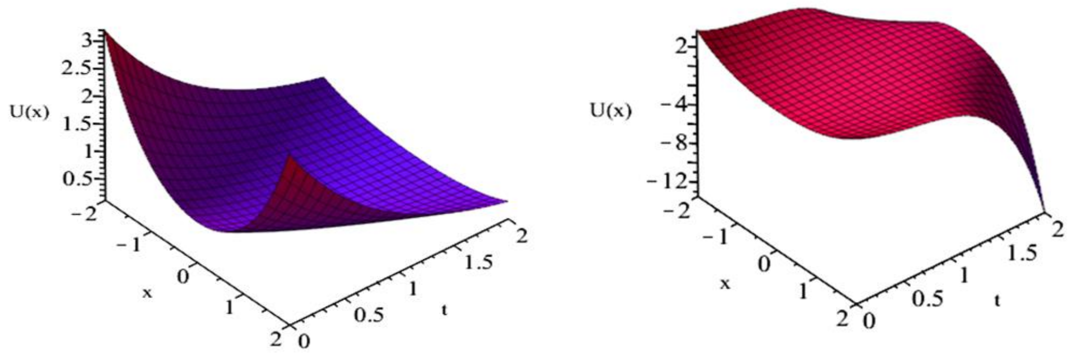
Figure 3(a) is the exact solution compare to the approximate solutions by the proposed MMRDTM and MRDTM in Figure 3(b) and Figure 3(c) respectively with $\alpha = 0.5$ for the Example 2. The outcomes of the approximate solution for $\alpha = 0.8$ are shown in Figure 4(a) for the MMRDTM and Figure 4(b) for the MRDTM respectively. From the obtained results, once again the approximations produced by the MMRDTM look very close to the exact solutions compared to the MRDTM.



(a) Exact solution (b)MMRDTM (c)MRDTM

Fig. 3. Comparison graphs of semi-analytical methods for $\alpha = 0.5$ with exact solution for Example 2

Table 3 and Table 4 respectively tabulate the error analyses of the solutions yeilded by the MMRDTM and MRDTM for Example 2 with different values of α. Just like the results obtained for the Example 1, the applied MMRDTM with $\alpha = 0.5$ and $\alpha = 0.8$ is evidently approximates the exact solution with high accuracy compared to the MRDTM for Example 2.



(a) (b)

Fig. 4. Comparison graphs of semi-analytical methods with $\alpha = 0.8$ for Example 2

Table 3
Error comparison between MMRDTM and MRDTM for Example 2 with $\alpha = 0.5$

Time, t	Exact Solution	Absolute Error MMRDTM	Absolute Error RDTM
0.1	1.020318167	0	0.8428996819
0.2	0.9232220555	$1.000000000\times10^{-10}$	0.7546500574
0.3	0.8353658610	$3.041381265\times10^{-10}$	0.6889462647
0.4	0.7558702887	$1.004987562\times10^{-10}$	0.6262301629
0.5	0.6839397204	$4.000000000\times10^{-10}$	0.5561971833
0.6	0.6188542508	$4.000000000\times10^{-10}$	0.4712445733
0.7	0.5599624824	$8.000000000\times10^{-10}$	0.3648586583
0.8	0.5066750067	$8.000000000\times10^{-10}$	0.2310455469
0.9	0.4584585049	4.200000000×10^{-9}	0.0640865569
1.0	0.4148304099	6.900000000×10^{-9}	-0.141581026

Table 4
Error comparison between MMRDTM and RDTM for Example 2 with $\alpha = 0.8$

Time, t	Exact Solution	Absolute Error MMRDTM	Absolute Error RDTM
0.1	1.020318167	0	0.0554479257
0.2	0.9232220555	0	0.0606845490
0.3	0.8353658610	0	0.0536799417
0.4	0.7558702887	0	0.0413943693
0.5	0.6839397204	0	0.0270231605
0.6	0.6188542508	0	0.0125213638
0.7	0.5599624824	0	0.0005954892
0.8	0.5066750067	1.0×10^{-10}	0.0108908191
0.9	0.4584585049	1.0×10^{-10}	0.0168058678
1.0	0.4148304099	4.0×10^{-10}	0.0165290344

4. Conclusions

We presented a new application of the MMRDTM for solving fractional nonlinear telegraph equations (FNLTEs) with source term. We modified the method by substituting the nonlinear term with its Adomian polynomials in the multi-step approach. Consequently, the approximated solutions are obtained with high accuracy as demonstrated in Example 1 and 2. In a nutshell, MMRDTM outperformed the MRDTM to acquire approximate solution for the FNLTEs in terms of practicality, consistencency and accuracy. All demonstrated calculations in this paper were obtained by using Maple 2022 software.

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