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Exponential Stability of an Upwind Difference Splitting Scheme for Symmetric t-Hyperbolic Systems

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ABSTRACT

This article considers mixed problem for 2-dimensional symmetric t-hyperbolic system. This system consists of constant coefficients. We investigated questions of setting mixed problems (problems with initial-boundary conditions) for symmetric *t*-hyperbolic system. It is known that for a symmetric *t*-hyperbolic system, the concept of characteristic velocity plays a very important role. The main point of this article is to construct a difference control scheme for these characteristic velocities. Control parameters are introduced to control the characteristic velocities of a symmetric *t*-hyperbolic system. The value of the parameter, the characteristic velocities can have different signs. We will consider all possible cases. In these cases, it is obvious that a symmetric t-hyperbolic system with n −equations can be in $(n+1)(n+1)$ different states. These states controlled by control parameters. Upwind difference schemes constructed for each of these states depending on the value of characteristic velocities. The suggested difference schemes for the mixed problem were substantiated is exponential stability by the Lyapunov. It will be shown that the difference scheme will be stable if the CFL condition is satisfied and otherwise unstable.

Keywords:
Symmetric *t*-hyperbolic; constant coefficients; control parameters; upwind difference schemes; stability; characteristic velocity

1. Introduction

A number of well-founded methods were proposed by Aloev *et al.,* [2-4] for the numerical solution of initial-boundary value problems for a 2-dimensional symmetric t-hyperbolic system. The reference by Rakhmatillo, Mirzoali and Alexander [6] constructed the important difference schemes. Obtained an a priori estimate for its solution of quasilinear hyperbolic systems. An approach of constructing a dissipative energy integral for hyperbolic systems used. According to Aloev *et al.,* [7,8] for the numerical solution of initial-boundary value problems for a 2-dimensional symmetric *t*-hyperbolic system, a finite element method based on linear basis functions was proposed. The finite

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element method was reduced to a difference scheme and its stability was studied for the cases of constant and variable coefficients [7,8]. According to Blokhin and Alaev [9], a certain class of explicit difference schemes for hyperbolic systems studied. A general approach to the study of stability in the energy norm was developed.

In this article, we assume that a symmetric hyperbolic system with constant coefficients can be in $(n+1)(n+1)$ states. These states are controlled by control parameters. Upwind difference schemes are constructed for each of these states depending on the value of characteristic velocities [11]. For each case of suggested difference schemes of exponential stability are proved. Many applied problems, such as the process of transferring energy from places of generation to places of consumption, the flow of liquid in open channels or gas pipelines [21], the propagation of light in optical fibres, are described by hyperbolic systems of partial differential equations [22]. From an engineering point of view, for hyperbolic systems, the exponential stability (in the sense of Lyapunov) of stationary states is a fundamental problem [23]. That is why this paper is completely devoted to the exponential stability of stationary states of difference schemes for hyperbolic systems.

The definition of exponential stability is intuitively simple: starting from an arbitrary initial state, the time trajectory of the system must converge exponentially in the spatial norm to the steady state. Exponential stability analysis is a rather difficult problem, because stability test criteria are not easily translated into simple practical stability tests.

The theorem provides just such simple practical criteria for checking the exponential stability of the numerical solution of a mixed problem for a hyperbolic system.

1.1 The Symmetric t-Hyperbolic System

The symmetric t-hyperbolic system is given in the following form [12]:

$$\frac{\partial V}{\partial t}+A\frac{\partial V}{\partial x}+B\frac{\partial V}{\partial y}=0. \tag{1}$$

$V=(v_1,v_2,\cdots,v_n)^T$ - is an unknown vector-function; $A=A^T, B=B^T$ are the given matrices. A,B −dimension $n\times n$, symmetric matrices of with constant real elements.

The following parameters are introduced to examine $d_A\big(\lambda_1(A),\lambda_2(A),\cdots,\lambda_n(A)\big), d_B\big(\lambda_1(B),\lambda_2(B),\cdots,\lambda_n(B)\big)$, where $q=1,2,\dots,n$, $\lambda_q(A)-$ eigenvalues of matrice A, and $\lambda_q(B)-$ eigenvalues of matrice B.

In this case, system Eq. (1) will take the following record form:

$$\frac{\partial V}{\partial t}+(A^+ + A^-)\frac{\partial V}{\partial x}+(B^+ + B^-)\frac{\partial V}{\partial y}=0. \tag{2}$$

Divide each of the matrices A and B into two parts. The non-negative definite part A^+,B^+ and non-positive part A^-,B^-. That gives us the possibility to build an upwind difference scheme for the numerical solution of mixed problems for system Eq. (1).

As a domain where a mixed problem is stated for system Eq. (1), consider the domain $Q=\{(t,x,y): 0\le t<+\infty, 0\le x\le X, 0\le x\le Y\}$. It is well known that for the system of Eq. (1), it is necessary to set the initial condition at the time moment $t=0$, types on $x=0$, $x=X$, $y=0$, $y=Y$ of boundary conditions [1].

Assume that there exists a unitary matrix P that simultaneously reduces the matrices A,B to the diagonal form. Then system of Eq. (2) takes the form:

$$\frac{\partial U}{\partial t} + (\Lambda_{A^+} + \Lambda_{A^-})\frac{\partial U}{\partial x} + (\Lambda_{B^+} + \Lambda_{B^-})\frac{\partial U}{\partial y} = 0 \tag{3}$$

We already know the number of specified boundary conditions on different boundaries of the considered area. As the initial data, we consider the conditions:

$$U(0, x, y) = \Phi(x, y) \tag{4}$$

Here $\Phi(x, y) = \left(\varphi_1(x, y), \varphi_2(x, y), \cdots, \varphi_n(x, y)\right)^T$; $\varphi_1(x, y), \varphi_2(x, y), \cdots, \varphi_n(x, y)$ are given functions [11].

1.2 Problem 1

Consider the following linear hyperbolic equation [16]:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \; a > 0. \tag{5}$$

Our task is to analyse the exponential stability of this equation with linear boundary conditions in canonical form for $x = 0$:

$$u(t, 0) = 0, \;\; t \in (0, +\infty) \tag{6}$$

and with initial conditions:

$$u(0, x) = u_0(0), x \in (0, l). \tag{7}$$

Assume that the function in initial condition of Eq. (7) $u_0(x) \in L^2\left((0, l), R\right)$ satisfies the following consistency condition:

$$u_0(0) = 0. \tag{8}$$

Definition of L1. (Exponential stability [2]). Eq. (5) with boundary condition of Eq. (6) is exponentially stable in L^2-norm if there exist $\nu > 0$ and $c > 0$ such that for any initial condition $u_0(x) \in L^2\left((0, l), R\right)$, L^2 is the solution of initial-boundary value problem of Eq. (5) to Eq. (8), and satisfies the following inequality:

$$\|u(t,\cdot)\|_{L^2((0,l),R)} \leq ce^{-\nu t}\|u_0(x)\|_{L^2((0,l),R)}, t \geq 0. \tag{9}$$

Let us introduce the concept of a weak solution to initial-boundary problem of Eq. (5) to Eq. (8) in $L^2\left((0, l), R\right)$. To do this end, we multiply Eq. (5) on the left-hand side by $\psi \in C^1([0, T] \times [0, l]; R)$, where T is a given positive number. Then we obtain the following equation $\psi\frac{\partial u}{\partial t} + \psi a\frac{\partial u}{\partial x} = 0$.

Both sides of this identity are integrable over domain $(0, T) \times (0, l)$. We assume that solutions u belong to class C^1 with respect to t and x. Using the formulas for integration by parts and initial condition of Eq. (7), we have:

$$\int_0^l \int_0^T \left[\psi\frac{\partial u}{\partial t} + \psi a\frac{\partial u}{\partial x}\right] dtdx = \int_0^l \int_0^T \left[\frac{\partial \psi u}{\partial t} + a\frac{\partial \psi u}{\partial x}\right] dtdx - \int_0^l \int_0^T \left[\frac{\partial \psi}{\partial t}u + a\frac{\partial \psi}{\partial x}u\right] dtdx. \tag{10}$$

Let us use the following transformations of the first integral term of the right-hand side of Eq. (10). First, using initial condition of Eq. (7), we obtain:

$$\int_0^l \int_0^T \frac{\partial \psi u}{\partial t} dtdx = \int_0^l \psi(T,x)u(T,x)\,dx - \int_0^l \psi(0,x)u(0,x)\,dx =$$
$$= \int_0^l \psi(T,x)u(T,x)\,dx - \int_0^l \psi(0,x)u_0(x)\,dx. \quad (10^*)$$

Likewise, using boundary condition of Eq. (8), we obtain:

$$\int_0^l \int_0^T a\frac{\partial \psi u}{\partial x} dtdx = \int_0^T a\psi(t,l)u(t,l)dt - \int_0^T a\psi(t,0)u(t,0)dt = \int_0^T a\psi(t,l)u(t,l)dt. \quad (10^{**})$$

Now we choose function ψ in that way the integrand of the integral Eq. (10^{**}) is zero. For this, it is enough to assume that:

$$\psi(t,L) = 0. \quad (11)$$

Then, after transformations of Eq. (10^*) and Eq. (10^{**}), we can get the following from Eq (10):

$$\int_0^l \psi(T,x)u(T,x)\,dx - \int_0^l \psi(0,x)u_0(x)\,dx - \int_0^L \int_0^T \left[\frac{\partial \psi}{\partial t}u + a\frac{\partial \psi}{\partial x}u\right] dtdx = 0. \quad (12)$$

The key point here is that although this Eq. (12) was derived under the assumption that functions u belong to class C^1 with respect to t and x, it makes sense even if functions u are not differentiable and can therefore be considered as "weak" solutions to the system. Then L^2-solutions are defined as functions u that satisfy Eq. (12) for all ψ that satisfy Eq. (11) when initial function $u_0(x)$ belongs to L^2. The definition of an L^2-solution is as follows.

Definition of L2. Let $\varphi \in L^2\big((0,L),R\big)$. The mapping $u: (0,+\infty) \times (0,L) \to R$ is an L^2-solution to the mixed problem Eq. (5) to Eq. (8) if for any $T \in [0,+\infty)$ and for any $\psi \in C^1([0,T]\times[0,L];R)$, satisfying conditions Eq. (11), the following functions $u \in C^0\left([0,+\infty]; L^2\big((0,L);R\big)\right)$ satisfy Eq. (12).

Consider the following function as a Lyapunov one:

$$V(t) \triangleq \int_0^L \frac{\mu}{a} e(x)u^2(t,x)dx, \quad (13)$$

where

$$e(x) = exp\left(-\frac{\nu x}{a}\right), \mu > 0. \quad (14)$$

1.3 Problem 2

Consider the following linear hyperbolic equation:

$$\frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x} = 0,\ a > 0. \quad (15)$$

Our task is to analyse the exponential stability of this equation with linear boundary conditions in canonical form for $x = l$: Eq. (6) and with initial conditions of Eq. (7).

Assume that the function in initial condition of Eq. (7) $u_0(x) \in L^2((0,l),R)$ satisfies the consistency condition:

$$u_0(l) = 0. \tag{16}$$

The definition L1 of exponential stability for Eq. (15) and Eq. (16) remains unchanged.

Let us introduce the concept of a weak solution to the mixed problem Eq. (15) and Eq. (16) in $L^2((0,l);R)$. To do this, we multiply Eq. (15) in the left-hand side by $\psi \in C^1([0,T]\times[0,l];R)$, where T is a given positive number. Then we obtain the following equation $\psi\frac{\partial u}{\partial t} - \psi a\frac{\partial u}{\partial x} = 0$.

Both parts of this identity are integrable over domain $(0,T)\times(0,l)$. We assume that solutions u belong to class C^1 with respect to t and x. Using the formulas for integration by parts and initial condition of Eq. (7), we obtain:

$$\int_0^l\int_0^T\left[\psi\frac{\partial u}{\partial t} - \psi a\frac{\partial u}{\partial x}\right]dtdx = \int_0^l\int_0^T\left[\frac{\partial\psi u}{\partial t} - a\frac{\partial\psi u}{\partial x}\right]dtdx - \int_0^l\int_0^T\left[\frac{\partial\psi}{\partial t}u - a\frac{\partial\psi}{\partial x}u\right]dtdx. \tag{17}$$

Let us use the following transformations of the first integral term of the right-hand side of Eq. (19). First, using initial condition of Eq. (17), we obtain:

$$\int_0^l\int_0^T\frac{\partial\psi u}{\partial t}dtdx = \int_0^l\psi(T,x)u(T,x)\,dx - \int_0^l\psi(0,x)u(0,x)\,dx =$$
$$\int_0^l\psi(T,x)u(T,x)\,dx - \int_0^l\psi(0,x)u_0(x)\,dx. \tag{18}$$

Likewise, with boundary condition Eq. (18), we obtain:

$$-\int_0^l\int_0^T a\frac{\partial\psi u}{\partial x}dtdx = -\int_0^T a\psi(t,l)u(t,l)dt + \int_0^T a\psi(t,0)u(t,0)dt = \int_0^T a\psi(t,0)u(t,0)dt \tag{19}$$

Now we choose function ψ in that way the integrand of the integral equation g) is zero. For this, it is enough to assume that:

$$\psi(t,0) = 0. \tag{20}$$

Then, after transformations of Eq. (18) and Eq. (19), we can get the following from Eq. (10):

$$\int_0^l\psi(T,x)u(T,x)dx - \int_0^l\psi(0,x)u_0(x)dx - \int_0^L\int_0^T\left[\frac{\partial\psi}{\partial t}u - a\frac{\partial\psi}{\partial x}u\right]dtdx = 0. \tag{21}$$

The key point here is that although this Eq. (21) was derived under the assumption that functions u belong to class C^1 with respect to t and x, it makes sense even if functions u are not differentiable and can therefore be considered as "weak" solutions to the system. Then L^2-solutions are defined as functions u that satisfy Eq. (21) for all ψ that satisfy Eq. (20) when initial function $u_0(x)$ belongs to L^2. The definition of an L^2-solution is as follows.

Definition L3. Let $\varphi \in L^2\big((0,L),R\big)$. The mapping $u{:}\ 0,+\infty)\times(0,L)\to R$ is an L^2-solution to the mixed problem of Eq. (15) and Eq. (16) if for any $T\in[0,+\infty)$ and for any $\psi\in C^1([0,T]\times[0,L];R)$, satisfying condition Eq. (20), the following functions $u\in C^0\left([0,+\infty];L^2\big((0,L);R\big)\right)$ satisfy Eq. (21).

Consider the following function as a Lyapunov function:

$$V(t) \triangleq \int_0^L \frac{\mu}{a} e(x)u^2(t,x)dx, \tag{22}$$

where $e(x) = exp\left(\frac{\nu x}{a}\right), \mu > 0$.

2. Methodology

2.1 Upwind Explicit Difference Scheme for Problem 1

In the field $G = \{(t,x): 0 \le t \le T, 0 \le x \in L\}$, we can build a difference grid with steps: Δt in direction t and Δx in direction x. The nodal points of the difference grid are denoted by (t^κ, x_j). Here $t = t^\kappa \triangleq \kappa \Delta t$ and $x = x_j \triangleq j\Delta x$. The set of nodal points of the difference grid is denoted by G_h, where $G_h \triangleq \{(t^\kappa, x_j): \kappa = 0, \dots, K; j = 0, \dots, J\}$.

Numerical solution's values are denoted by:

$$u_j^\kappa = u(t^\kappa, x_j),\ \kappa = 0, \dots, K;\ j = 0, \dots, J. \tag{23}$$

Steps of the G_h are selected the equations $K\Delta t = T$ and $J\Delta x = l$ are fulfilled. In order to find a numerical solution of the initial-boundary value problem of Eq. (5) to Eq. (8) over difference grid G_h, we propose the following difference scheme.

$$u_j^{\kappa+1} = u_j^\kappa - a\frac{\Delta t}{\Delta x}\left[u_j^\kappa - u_{j-1}^\kappa\right], j = 1, \dots, J-1; \kappa = 0, \dots, K-1. \tag{24}$$

Initial conditions of Eq. (7) are approximated:

$$u_j^0 = u_0(x_j), i = 1, \dots, n; j = 1, \dots, J. \tag{25}$$

Boundary condition of Eq. (6) is approximated:

$$u_0^k = 0, \kappa = 0, \dots, K. \tag{26}$$

Let us assume that for Δt and Δx satisfy the Courant-Friedrichs-Lewy condition $\frac{\Delta t}{\Delta x} a \le 1$. Now we learn the exponential stability of the numerical solution to the difference problem of Eq. (24) to Eq. (26).

i. Definition L4: The solution to the difference scheme of Eq. (24) that satisfies boundary conditions Eq. (26) is called exponentially stable if there exist the constants $\eta > 0$ and $c > 0$ for any initial conditions $\left\{(u_0)_j\right\}_1^J \in l^2\left(\left\{x_j\right\}_1^J; R\right)$ the solution to the initial-boundary value problem of Eq. (24) to Eq. (26) satisfies the following inequality.

$$\Delta x \sum_{j=1}^{J-1}\left(u_j^\kappa, u_j^\kappa\right) \le c e^{-\eta t_\kappa} \Delta x \sum_{j=1}^{J}\left([u_0]_j, [u_0]_j\right), \kappa = 1,2, \dots N.$$

Use the following notation:

$e_j \triangleq exp\left(-\frac{x_j \nu}{a}\right), \nu > 0;$
$\mu_j^+ = e_{j-1}\mu, \quad \mu > 0, \quad j = 1,2,\dots,J-1.$

To prove the exponential stability of the problem of Eq. (24) to Eq. (26), we can get the discrete Lyapunov function:

$$V^\kappa \triangleq \frac{\mu}{a}\sum_{j=1}^{J-1} e_{j-1}\left[u_j^\kappa\right]^2. \tag{27}$$

ii. Lemma L1: Let $T > 0$ and the discrete Lyapunov function be determined using formula of Eq. (16). If the Δt and Δx satisfy the CFL condition $\frac{\Delta t}{\Delta x} a \leq 1$, then numerical solution u_j^κ of the problem of Eq. (24) to Eq. (26) is exponentially stable in l^2-norm.
iii. Lemma L2: The following inequality holds in solutions of difference scheme of Eq. (24).

$$\frac{V^{\kappa+1} - V^\kappa}{\Delta t} \leq \mu \sum_{j=1}^{J-1} e_{j-1}\left\{\left[u_{j-1}^\kappa\right]^2 - \left[u_j^\kappa\right]^2\right\}.$$

iv. Lemma L3: Let the boundary condition of Eq. (26) be satisfied. In this case the following equation is true.

$$\sum_{j=1}^{J-1} e_{j-1}\left\{\left[u_{j-1}^\kappa\right]^2 - \left[u_j^\kappa\right]^2\right\} = e_0[u_0^\kappa]^2 - e_{J-1}\left[u_{J-1}^\kappa\right]^2 - \Delta x \frac{\nu}{a}\sum_{j=1}^{J-1} e_{j-1}\left[u_j^\kappa\right]^2. \tag{28}$$

From Lemma L4, considering Eq. (28), we obtain

$$\frac{V^{\kappa+1} - V^\kappa}{\Delta t} \leq \mu \sum_{j=1}^{J-1} e_{j-1}\left\{\left[u_{j-1}^\kappa\right]^2 - \left[u_j^\kappa\right]^2\right\} =$$
$$-\mu e_{J-1}\left[u_{J-1}^\kappa\right]^2 - \nu\Delta x \frac{\mu}{a}\sum_{j=1}^{J-1} e_{j-1}\left[u_j^\kappa\right]^2 = -\mu e_{J-1}\left[u_{J-1}^\kappa\right]^2 - \nu V^\kappa \leq -\nu V^\kappa.$$

or

$$\frac{V^{\kappa+1} - V^\kappa}{\Delta t} \leq -\nu V^\kappa, \kappa = 0,\dots,N-1. \tag{29}$$

v. Lemma L4: Let the following sequence of inequalities of Eq. (29) be true for V^κ. Then there exists positive constant C such that solution u_j^κ of the boundary difference problem of Eq. (24) to Eq. (26) satisfies the a priori estimate.

$$\|u^\kappa\|^2 \leq Ce^{-\nu t_\kappa}\|u^0\|^2, \kappa = 1,2,\dots,N. \tag{30}$$

Here $\|u^\kappa\|^2 = \Delta x \sum_{j=1}^{J-1}\left(u_j^\kappa, u_j^\kappa\right)$.
vi. Proof: Recursively applying inequalities of Eq. (29) we obtain:

$$V^{\kappa+1} < (1-\Delta t\nu)^{\kappa+1}V^0 \leq e^{-\nu\Delta t(\kappa+1)}V^0 = e^{-\nu t_{\kappa+1}}V^0, \kappa = 0,1,\dots,N-1.$$

Let us introduce positive constants $C_1 = \frac{\mu}{a} exp\left(-\frac{l\nu}{a}\right), C_2 = \frac{\mu}{a}$. Then, it follows that

$$C_1 \Delta x \sum_{j=1}^{J-1} \left(u_j^\kappa\right)^2 \leq V^\kappa \leq e^{-\nu t_\kappa} V^0 \leq C_2 e^{-\nu t_\kappa} \Delta x \sum_{j=1}^{J-1} \left(u_0(x_j)\right)^2,$$

$$\Delta x \sum_{j=1}^{J-1} \left(u_j^\kappa\right)^2 \leq C e^{-\nu t_\kappa} \Delta x \sum_{j=1}^{J-1} \left(u_0(x_j)\right)^2, \qquad \kappa = 1, \dots N;\ \ C = C_2/C_1.$$

Thus, numerical solution u_j^κ to the mixed problem is exponentially stable in l^2-norm. Lemma L4 is proved.

2.2 Upwind Explicit Difference Scheme for Problem 2

To get a numerical solution to the mixed problem of Eq. (15) to Eq. (18) over the G_h, we can offer the following difference scheme.

$$u_j^{\kappa+1} = u_j^\kappa + a\frac{\Delta t}{\Delta x}\left[u_{j+1}^\kappa - u_j^\kappa\right], j = 1, \dots, J-1; \kappa = 0, \dots, K-1. \quad (31)$$

Initial conditions of Eq. (17) are approximated as follows:

$$u_j^0 = u_0(x_j),\ \ i = 1, \dots, n;\ \ j = 1, \dots, J. \quad (32)$$

Boundary conditions of Eq. (16) are approximated as follows:

$$u_J^k = 0,\ \ \kappa = 0, \dots, K. \quad (33)$$

Let us assume that the Δt and Δx satisfy the CFL condition:

$$\frac{\Delta t}{\Delta x} a \leq 1.$$

Now we learn the problem of the exponential stability of numerical solution to the difference problem of Eq. (31) to Eq. (33).

i. Definition of L5: The solution of Eq. (31) that satisfies boundary conditions of Eq. (33) is called exponentially stable if there exist constants $\eta > 0$ and $c > 0$ for any initial conditions $\left\{(u_0)_j\right\}_1^{J-1} \in l^2\left(\left\{x_j\right\}_1^{J-1}; R\right)$, the solution to the problem of Eq. (31) to Eq. (33) satisfies the following inequality.

$$\Delta x \sum_{j=1}^{J-1} \left(u_j^\kappa, u_j^\kappa\right) \leq c e^{-\eta t_\kappa} \Delta x \sum_{j=1}^{J-1} \left([u_0]_j, [u_0]_j\right), \qquad \kappa = 1, \dots, N.$$

Let us introduce the notation.

$$e_j \triangleq exp\left(\frac{x_j \nu}{a}\right), \nu > 0;$$
$$\mu_j^+ = e_{j-1}\mu,\ \ \mu > 0, \qquad j = 1, \dots, J-1.$$

To prove the exponential stability of the problem of Eq. (31) to Eq. (33), we can get the following function as a candidate for the discrete Lyapunov function of Eq. (13):

$$V^\kappa \triangleq \frac{\mu}{a}\sum_{j=1}^{J-1} e_{j+1}\left[u_j^\kappa\right]^2. \tag{34}$$

ii. Lemma L5: Let $T > 0$ and the discrete Lyapunov function be determined using Eq. (34). If the Δt and Δx satisfy the CFL condition $\frac{\Delta t}{\Delta x} a \leq 1$, then numerical solution u_j^κ of the problem of Eq. (31) to Eq. (33) is exponentially stable in l^2-norm.

iii. Lemma L6: Solutions of difference scheme of Eq. (31) satisfy the following inequality.

$$\frac{V^{\kappa+1} - V^\kappa}{\Delta t} \leq \mu \sum_{j=1}^{J-1} e_{j-1}\left\{\left[u_{j+1}^\kappa\right]^2 - \left[u_j^\kappa\right]^2\right\}.$$

iv. Lemma L7: Let boundary condition of Eq. (33) be satisfied. In this case the following equation is appropriate

$$\sum_{j=1}^{J-1} e_{j+1}\left\{\left[u_{j+1}^\kappa\right]^2 - \left[u_j^\kappa\right]^2\right\} = e_J\left[u_J^\kappa\right]^2 - e_1[u_1^\kappa]^2 - \Delta x\frac{\nu}{a}\sum_{j=1}^{J-1} e_{j+1}\left[u_j^\kappa\right]^2 =$$
$$-e_1[u_1^\kappa]^2 - \Delta x\frac{\nu}{a}\sum_{j=1}^{J-1} e_{j+1}\left[u_j^\kappa\right]^2. \tag{35}$$

v. Proof: Using the difference differentiation formula, we obtain

$$\sum_{j=1}^{J-1} e_{j+1}\left\{\left[u_{j+1}^\kappa\right]^2 - \left[u_j^\kappa\right]^2\right\} = \sum_{j=1}^{J-1}\left\{e_{j+1}\left[u_{j+1}^\kappa\right]^2 - e_j\left[u_j^\kappa\right]^2\right\} - \sum_{j=1}^{J-1}\left(e_{j+1} - e_j\right)\left[u_j^\kappa\right]^2. \tag{36}$$

We study each sum on the right-hand side of equation (36) separately. It is easy to prove the following equations

$$\sum_{j=1}^{J-1}\left\{e_{j+1}\left[u_{j+1}^\kappa\right]^2 - e_j\left[u_j^\kappa\right]^2\right\} = e_J\left[u_J^\kappa\right]^2 - e_1[u_1^\kappa]^2 = -e_1[u_1^\kappa]^2; \tag{37}$$

$$-\sum_{j=1}^{J-1}\left(e_{j+1} - e_j\right)\left[u_j^\kappa\right]^2 = -\Delta x\frac{\nu}{a}\sum_{j=1}^{J-1} e_{j+1}\left[u_j^\kappa\right]^2. \tag{38}$$

Indeed, the proof of Eq. (37) follows by direct summation, considering boundary condition of Eq. (33). To prove Eq. (38) we use the following sequence of equalities.

$$e_{j+1} - e_j = e_{j+1}\left\{1 - exp\left(-\Delta x\frac{\nu}{a}\right)\right\} = \Delta x\frac{\nu}{a}e_{j+1}.$$

Lemma L7 is proved. From Lemma 6, with Eq. (36), we obtain:

$$\frac{V^{\kappa+1}-V^{\kappa}}{\Delta t} \le \mu \sum_{j=1}^{J-1} e_{j-1}\left\{\left[u_{j-1}^{\kappa}\right]^2-\left[u_j^{\kappa}\right]^2\right\} =$$

$$-\mu e_{J-1}\left[u_{J-1}^{\kappa}\right]^2-\nu\Delta x\frac{\mu}{a}\sum_{j=1}^{J-1} e_{j-1}\left[u_j^{\kappa}\right]^2=-\mu e_{J-1}\left[u_{J-1}^{\kappa}\right]^2-\nu V^{\kappa}\le -\nu V^{\kappa}$$

or

$$\frac{V^{\kappa+1}-V^{\kappa}}{\Delta t} \le -\nu V^{\kappa}, \kappa=0,\dots,N-1.$$

vi. Lemma L8: Let the sequence of inequalities $\frac{V^{\kappa+1}-V^{\kappa}}{\Delta t} \le -\nu V^{\kappa}, \kappa=0,\dots,N-1$ be true for V^{κ}. Then there exists positive constant C such that solution u_j^{κ} of the boundary difference problem of Eq. (31) to Eq. (33) satisfies the a priori estimate $\|u^{\kappa}\|^2 \le \mathrm{C}e^{-\nu t_{\kappa}}\|u^0\|^2, \kappa=1,\dots,N$. Here $\|u^{\kappa}\|^2=\Delta x\sum_{j=1}^{J-1}\left(u_j^{\kappa},u_j^{\kappa}\right)$.

2.3 Implicit Upwind Difference Scheme for Problem 1

Here we consider an implicit upwind difference scheme for the numerical calculation of mixed Problem 1. The initial system is approximated by an implicit scheme:

$$u_j^{\kappa+1}=u_j^{\kappa}-a\frac{\Delta t}{\Delta x}\left[u_j^{\kappa+1}-u_{j-1}^{\kappa+1}\right], j=1,\dots,J-1; \kappa=0,\dots,K-1. \quad (39)$$

Similar to explicit difference schemes, we take the exact value of initial functions at the nodal points of the initial layer in time as the initial data:

$$u_j^0=(u_0)_j,\ \ j=1,\dots,J. \quad (40)$$

We approximate the boundary condition as follows:

$$u_0^{\kappa}=0,\ \ \kappa=0,\dots,K. \quad (41)$$

We study the exponential stability problem of the numerical solution to the difference problem of Eq. (39) to Eq. (41). First, we define the exponential stability of the numerical solution of the difference problem of Eq. (39) to Eq. (41).

Definition L6. The solution to the initial boundary value problem of Eq. (39) to Eq. (41) is called Lyapunov stable if there exist positive constants $0<\alpha<1$ and $\mathrm{c}>0$ such that for any initial condition $\left\{(u_0)_j\right\}_1^{J-1} \in l^2\left(\left\{x_j\right\}_1^{J-1};R\right)$, the solution of the problem of Eq. (39) to Eq. (41) satisfies the following inequality.

$$\Delta x \sum_{j=1}^{J-1} \left(u_j^{\kappa}\right)^2 \leq c\alpha^k \Delta x \sum_{j=1}^{J-1} \left([u_0]_j\right)^2, \kappa = 1, \dots, N.$$

We study the Lyapunov stability of the initial-boundary difference problem of Eq. (39) to Eq. (41) based on the approach of constructing a discrete quadratic Lyapunov function. We propose the following function as a candidate for a discrete quadratic Lyapunov function.

$$V^{\kappa} \triangleq \Delta x \sum_{j=1}^{J-1} \frac{\mu}{a} e_{j-1} \left[u_j^{\kappa}\right]^2, e_j \triangleq exp\left(-\frac{x_j \nu}{a}\right), \nu > 0. \quad (42)$$

Here ν, μ - are the positive constants to be determined.

i. Lemma L9: Let $T > 0$ and discrete Lyapunov function be defined by Eq. (42). Then numerical solution u_j^{κ} of the initial-boundary difference problem of Eq. (39) to Eq. (41) is exponentially stable in l^2-norm (in the sense of Definition L6). To prove Lemma L9, we need some auxiliary lemmas.

2.4 Implicit Upwind Difference Scheme for Problem 2

Here we consider an implicit upwind difference scheme for the numerical calculation of the mixed Problem 2. The initial system is approximated by the implicit scheme:

$$u_j^{\kappa+1} = u_j^{\kappa} + a \frac{\Delta t}{\Delta x} \left[u_{j+1}^{\kappa+1} - u_j^{\kappa+1}\right], j = 1, \dots, J-1; \kappa = 0, \dots, K-1. \quad (43)$$

Similar to explicit difference schemes, we take the exact values of initial functions at the nodal points of the initial layer in time as the initial data:

$$u_j^0 = (u_0)_j, j = 1, \dots, J. \quad (44)$$

Let us approximate the boundary condition as follows:

$$u_L^{\kappa} = 0, \kappa = 0, \dots, K. \quad (45)$$

Let us investigate the exponential stability problem of the numerical solution of the difference problem of Eq. (43) to Eq. (47). First, we define the exponential stability of the numerical solution of the difference problem of Eq. (43) to Eq. (45).

i. Definition L7: The solution to the initial-boundary difference problem of Eq. (43) to Eq. (45) is called Lyapunov stable if there are positive constants $0 < \alpha < 1$ and $c > 0$ such that for any initial given data $\left\{(u_0)_j\right\}_1^{J-1} \in l^2\left(\left\{x_j\right\}_1^{J-1}; R\right)$, the solution of the initial-boundary difference problem of Eq. (43) to Eq. (45) satisfies the following inequality.

$$\sum_{j=1}^{J-1} \left(u_j^{\kappa}\right)^2 \leq c\alpha^k \sum_{j=1}^{J-1} \left([u_0]_j\right)^2, \kappa = 1,2, \dots N.$$

We study the Lyapunov stability of the initial-boundary difference problem of Eq. (43) to Eq. (45) based on the approach of constructing a discrete quadratic Lyapunov function. As a candidate for a discrete quadratic Lyapunov function, we propose the following function $V(\kappa\Delta) \triangleq V^{\kappa}$, where

$$V^{\kappa} \triangleq \Delta x \sum_{j=1}^{J-1} \frac{\mu}{a} e_{j+1} \left[u_j^{\kappa}\right]^2, \;\; e_j \triangleq exp\left(\frac{x_j \nu}{a}\right), \nu > 0. \tag{46}$$

Here ν, μ - are the positive constants to be determined.

ii. Lemma L10: Let $T > 0$ and the discrete Lyapunov function be defined using Eq. (46). Then numerical solution u_j^{κ} of the initial-boundary difference problem of Eq. (43) to Eq. (45) is exponentially stable in l^2-norm (in the sense of definition L7).
iii. Lemma L11: Solutions of difference equations of scheme Eq. (43) satisfy the following inequality.

$$\frac{V^{\kappa+1} - V^{\kappa}}{\Delta t} \leq \mu \sum_{j=1}^{J-1} e_{j+1} \left\{\left[u_{j+1}^{\kappa+1}\right]^2 - \left[u_j^{\kappa+1}\right]^2\right\}. \tag{47}$$

iv. Lemma L12: The following equation is true.

$$\sum_{j=1}^{J-1} e_{j+1} \left\{\left[u_{j+1}^{\kappa+1}\right]^2 - \left[u_j^{\kappa+1}\right]^2\right\} = -\, e_1 [u_1^{\kappa+1}]^2 - \Delta x \frac{\nu}{a} \sum_{j=1}^{J-1} e_{j+1} \left[u_j^{\kappa+1}\right]^2. \tag{48}$$

Considering Eq. (48), from inequality of Eq. (47) we have:

$$\frac{V^{\kappa+1} - V^{\kappa}}{\Delta t} \leq \mu \sum_{j=1}^{J-1} e_{j+1} \left\{\left[u_{j+1}^{\kappa+1}\right]^2 - \left[u_j^{\kappa+1}\right]^2\right\} = -\,\mu e_1 [u_1^{\kappa+1}]^2 -$$

$$\nu \Delta x \frac{\mu}{a} \sum_{j=1}^{J-1} e_{j+1} \left[u_j^{\kappa+1}\right]^2 = -\mu e_1 [u_1^{\kappa+1}]^2 - \nu V^{\kappa+1} \leq -\nu V^{\kappa+1}.$$

or

$$\frac{V^{\kappa+1} - V^{\kappa}}{\Delta t} \leq -\nu V^{\kappa+1}, \kappa = 0, \ldots, N-1. \tag{49}$$

Inequality of Eq. (49) can be represented as

$$V^{\kappa+1} \leq \alpha V^{\kappa}, \tag{50}$$

where

$$\alpha = 1 + \bar{\alpha}, \bar{\alpha} = \frac{-\Delta t \cdot \nu}{1 + \Delta t \cdot \nu}.$$

Note, that $\bar{\alpha} < 0$. Let a sequence of real numbers V^{κ} satisfying inequality of Eq. (50) for $\alpha > 0$ be given. Then the following inequality is true:

$$V^{\kappa} \leq \alpha^{k} V^{0}, \kappa = 1, \dots, N. \tag{51}$$

It should be noted that for sufficiently small Δt, inequality $0 < \alpha < 1$ holds. Let us introduce positive constants $C_2 = \frac{\mu}{a} exp\left(\frac{l\nu}{a}\right), C_1 = \frac{\mu}{a}$.

Then it follows that

$$C_1 \Delta x \sum_{j=1}^{J-1} (u_j^{\kappa})^2 \leq V^{\kappa} \leq e^{-\nu t_{\kappa}} V^0 \leq C_2 \alpha^k \Delta x \sum_{j=1}^{J-1} \left(u_0(x_j)\right)^2,$$

$$\Delta x \sum_{j=1}^{J-1} (u_j^{\kappa})^2 \leq C\alpha^k \Delta x \sum_{j=1}^{J-1} \left(u_0(x_j)\right)^2, \kappa = 1, \dots, N; C = C_2/C_1.$$

Thus, numerical solution u_j^{κ} of the mixed problem is exponentially Lyapunov stable in l^2-norm.

2.5 Explicit-Implicit Upwind Difference Splitting Scheme for Mixed Problem (5)-(7)

We choose the steps of the difference grid $\Delta x, \Delta y$ in such a way that equations $J\Delta x = X$ and $L\Delta x = Y$ are satisfied. This means that the difference mesh completely covers domain Q. We denote $t^{\kappa} \triangleq \kappa \Delta t$, $\kappa = 0, \dots, K$; $x_j \triangleq j\Delta x$, $j = 0, \dots, J$ and $y_l \triangleq l\Delta y, l = 0, \dots, L$. The nodal points of the difference grid (meaning the intersection of the straight lines $t = t^{\kappa}$, $x = x_j$ and $y = y_l$) we denote by (t^{κ}, x_j, y_l). The set of nodal points of the difference grid are denoted by Q_h, where:

$$Q_h \triangleq \{(t^{\kappa}, x_j, y_l): \kappa = 0, \dots, K; j = 0, \dots, J; l = 0, \dots, L\}.$$

The values of the numerical solution at the nodal points are denoted by:

$$U_{jl}^{\kappa} = U(t^{\kappa}, x_j, y_l),\ \kappa = 0, \dots, K;\ j = 0, \dots, J;\ l = 0, \dots, L. \tag{52}$$

To find a numerical solution to the mixed problem of Eq. (3), Eq. (4) over the difference grid Q_h, we propose the following implicit upwind difference scheme:

$$I. \frac{W_{jl}^{\kappa} - U_{jl}^{\kappa}}{\Delta t} + (B^+)_{jl}^{\kappa} \frac{U_{jl}^{\kappa} - U_{jl-1}^{\kappa}}{\Delta y} + (B^-)_{jl}^{\kappa} \frac{U_{jl+1}^{\kappa} - U_{jl}^{\kappa}}{\Delta y} = 0,$$

$$II. \frac{U_{jl}^{\kappa+1} - W_{jl}^{\kappa}}{\Delta t} + (A^+)_{jl}^{\kappa} \frac{U_{jl}^{\kappa+1} - U_{j-1l}^{\kappa+1}}{\Delta x} + (A^-)_{jl}^{\kappa} \frac{U_{j+1l}^{\kappa+1} - U_{jl}^{\kappa+1}}{\Delta x} = 0,$$

$$\kappa = 0, \dots, K-1;\ j = 1, \dots, J-1;\ \ l = 1, \dots, L-1. \tag{53}$$

Consider the issue of constructing and studying an upwind difference scheme for the cases $C_A = 1, \dots, n+1$; $C_B = 1, \dots, n+1$. Obviously, the number of such combinations (options) is $(n+1) \times (n+1)$ [11].

2.6 Stability of the Difference Scheme

i. Theorem 1: Let the steps of the difference grid $\Delta t, \Delta y$ satisfy the Courant Friedrichs Lewy (CFL) condition:

$$\max_{1\le p\le n}\left|\lambda_p(B)\right|\frac{\Delta t}{\Delta y}\le 1 \tag{54}$$

and difference boundary conditions ((6.1.2*)-(6.1.n+1*), (6.2.1*)-(6.1.n*), (6.3.2*)-(6.3.n+1*), (6.4.1*)-(6.4.n*) are dissipative [11]. Then there exist constants $c>0,\ 0<\alpha<1$ such that the solution of the initial-boundary difference problem ((9.2.1)-(9.2.n+1), (9.1.1)-(9.1.n+1); (6.1.2*)-(6.1.n+1*), (6.2.1*)-(6.1.n*), (6.3.2*)-(6.3.n+1*), (6.4.1 *)-(6.4.n*); (70) see[11]) is Lyapunov stable in norm $\sqrt{U^k}$:

$$U^{\kappa+1}\le c\alpha^k U^\kappa, k=0,\dots,K-1$$

Let us introduce the following notation:

$$W^\kappa=\begin{cases} {}^{1,1}_{\cdot}W^\kappa \text{ if } C_B=1, C_A=1;\\ \dots\dots\dots\dots\dots\dots\dots\\ {}^{1,n+1}_{\cdot}W^\kappa \text{ if } C_B=1, C_A=n+1;\\ {}^{2,1}_{\cdot}W^\kappa \text{ if } C_B=2, C_A=1;\\ \dots\dots\dots\dots\dots\dots\dots\\ {}^{2,n+1}_{\cdot}W^\kappa \text{ if } C_B=2, C_A=n+1;\\ \dots\dots\dots\dots\dots\dots\dots\\ {}^{n+1,1}_{\cdot}W^\kappa \text{ if } C_B=n+1, C_A=1;\\ \dots\dots\dots\dots\dots\dots\dots\\ {}^{n+1,n+1}_{\cdot}W^\kappa \text{ if } C_B=n+1, C_A=n+1. \end{cases}\qquad U^\kappa=\begin{cases} {}^{1,1}_{\cdot}U^\kappa \text{ if } C_B=1, C_A=1;\\ \dots\dots\dots\dots\dots\dots\dots\\ {}^{1,n+1}_{\cdot}U^\kappa \text{ if } C_B=1, C_A=n+1;\\ {}^{2,1}_{\cdot}U^\kappa \text{ if } C_B=2, C_A=1;\\ \dots\dots\dots\dots\dots\dots\dots\\ {}^{2,n+1}_{\cdot}U^\kappa \text{ if } C_B=2, C_A=n+1;\\ \dots\dots\dots\dots\dots\dots\dots\\ {}^{n+1,1}_{\cdot}U^\kappa \text{ if } C_B=n+1, C_A=1;\\ \dots\dots\dots\dots\dots\dots\dots\\ {}^{n+1,n+1}_{\cdot}U^\kappa \text{ if } C_B=n+1, C_A=n+1. \end{cases}$$

Here

$${}^{q,g}_{\cdot}W^\kappa\triangleq\sum_{p=1}^{n}{}^{q,g}_{p}w^\kappa, {}^{q,g}_{\cdot}U^\kappa\triangleq\sum_{p=1}^{n}{}^{q,g}_{p}u^\kappa, q=1,\dots,n+1; g=1,\dots,n+1.$$

$$e_j^p=exp\left(\frac{\nu x_{j+1}}{\left|\lambda_p(A)\right|}\right), \bar{e}_j^p=exp\left(-\frac{\nu x_{j-1}}{\left|\lambda_p(A)\right|}\right), \chi_q^p=\Delta y\frac{{}^{q}_{p}\mu_y}{\left|\lambda_p(B)\right|},$$

$$E_j^p=exp\left(\frac{\nu y_{l+1}}{\left|\lambda_p(B)\right|}\right), \bar{E}_j^p=exp\left(-\frac{\nu y_{l-1}}{\left|\lambda_p(B)\right|}\right), \phi_g^p=\Delta x\frac{{}^{q}_{p}\mu_x}{\left|\lambda_p(A)\right|}$$

We define values of ${}^{q,g}_{p}w^\kappa,\ {}^{q,g}_{p}u^\kappa;\ p=1,\dots,n$ for each $q=1,\dots,n+1;\ g=1,\dots,n+1$.

3. Results

3.1 First Part of the Proof of the Stability Theorem (The Stage of Splitting Along The y Direction)

Let us start with $q=1$. To do so, we consider the initial-boundary difference problem ((9.1.1) or (9.1.1*), (6.4.1*), Eq. (71) and, as discrete Lyapunov functions ${}^{1,g}_{p}w^\kappa,\ {}^{1,g}_{p}u^\kappa;\ p=1,\dots,n$ we propose the following [11]:

$$q=1, g=1, (C_B=1, C_A=1), p=1,\dots,n;$$

$$ {}^{1,1}_{\;p}w^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{1}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e^{p}_{j}E^{p}_{l}\left[(w_p)^{\kappa}_{jl}\right]^{2}, \qquad {}^{1,1}_{\;p}u^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{1}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e^{p}_{j}E^{p}_{l}\left[(u_p)^{\kappa}_{jl}\right]^{2}, $$

$q = 1, g = 2, (C_B = 1, C_A = 2), p = 1, \dots, n-1$:

$$ {}^{1,2}_{\;p}w^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{2}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e^{p}_{j}E^{p}_{l}\left[(w_p)^{\kappa}_{jl}\right]^{2}, \quad {}^{1,2}_{\;p}u^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{2}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e^{p}_{j}E^{p}_{l}\left[(u_p)^{\kappa}_{jl}\right]^{2}; $$

$p = n$:

$$ {}^{1,2}_{\;p}w^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{2}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}^{p}_{j}E^{p}_{l}\left[(w_p)^{\kappa}_{jl}\right]^{2}, \quad {}^{1,2}_{\;p}u^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{2}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}^{p}_{j}E^{p}_{l}\left[(u_p)^{\kappa}_{jl}\right]^{2}. $$

$q = 1, g = n, (C_B = 1, C_A = n), p = 1$:

$$ {}^{1,n}_{\;p}w^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{n}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e^{p}_{j}E^{p}_{l}\left[(w_p)^{\kappa}_{jl}\right]^{2}, \quad {}^{1,n}_{\;p}u^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{n}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e^{p}_{j}E^{p}_{l}\left[(u_p)^{\kappa}_{jl}\right]^{2}; $$

$p = 2, \dots, n$:

$$ {}^{1,n}_{\;p}w^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{n}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}^{p}_{j}E^{p}_{l}\left[(w_p)^{\kappa}_{jl}\right]^{2}, \qquad {}^{1,n}_{\;p}u^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{n}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}^{p}_{j}E^{p}_{l}\left[(u_p)^{\kappa}_{jl}\right]^{2}. $$

$q = 1, g = n+1, (C_B = 1, C_A = n+1), p = 1, \dots, n$:

$$ {}^{1,n+1}_{\;\;p}w^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{n+1}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}^{p}_{j}E^{p}_{l}\left[(w_p)^{\kappa}_{jl}\right]^{2}, \quad {}^{1,n+1}_{\;\;p}u^{\kappa}_{j} \triangleq \chi^{p}_{1}\phi^{p}_{n+1}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}^{p}_{j}E^{p}_{l}\left[(u_p)^{\kappa}_{jl}\right]^{2}. $$

Here $(w_p)^{\kappa}_{jl}, (u_p)^{\kappa}_{jl},\ p = 1, \dots, n;\ \ \kappa = 0, \dots, K;\ \ j = 0, \dots, J;\ \ l = 0, \dots, L$ - is the solution to the difference boundary value problem ((9.1.1*), (6.4.1*), Eq. (71); $\nu, \{{}^{1}_{p}\mu_y,\ p = 1, \dots, n\}$ are positive constants [11].

i. Corollary 1.1: Let the steps of the difference grid $\Delta t, \Delta y$ satisfy the Courant Friedrichs Lewy (CFL) condition of Eq. (54), then, according to Eq. (42), the following inequalities hold:

$$ {}^{1,g}_{\;p}w^{\kappa}_{j} \le {}^{1}\alpha \cdot {}^{1,g}_{\;p}u^{\kappa}_{j},\ \ g = 1, \dots, n+1,\ \ p = 1, \dots, n; \tag{55} $$
$$ {}^{1}\alpha = 1 + \bar{\alpha}, \qquad \bar{\alpha} = -\Delta t \cdot \nu, \qquad 0 < {}^{1}\alpha < 1. $$

Let us sum inequalities Eq. (55) over *p* from 1 to *n*. Then we get the following inequalities

$$ {}^{1,g}W^{\kappa}_{j} \le {}^{1}\alpha\, {}^{1,g}U^{\kappa}_{j}\, g = 1, \dots, n+1. \tag{56} $$

Let $q = 2$ $(C_B = 2)$**.** Consider the initial-boundary difference problem ((9.1.2) or (9.1.2*), (6.3.2*), (6.4.2*), Eq. (10) and, as discrete Lyapunov functions ${}^{2,g}_{\;p}w^{\kappa}$, ${}^{2,g}_{\;p}u^{k}; p = 1, \dots, n$, we propose the following [11].

$q = 2, g = 1, (C_B = 2, C_A = 1), p = 1, \dots, n-1$:

$$ {}_{p}^{2,1}w_j^{\kappa} \triangleq \chi_2^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(w_p)_{jl}^{\kappa}\right]^2, \quad {}_{p}^{2,1}u_j^{\kappa} \triangleq \chi_2^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(u_p)_{jl}^{\kappa}\right]^2, $$

$p = n$:

$$ {}_{p}^{2,1}w_j^{\kappa} \triangleq \chi_2^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[(w_p)_{jl}^{\kappa}\right]^2, \quad {}_{p}^{2,1}u_j^{\kappa} \triangleq \chi_2^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa}\right]^2, $$

$q = 2, g = 2, (C_B = 2, C_A = 2), p = 1, \dots, n-1$:

$$ {}_{p}^{2,2}w_j^{\kappa} \triangleq \chi_2^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(w_p)_{jl}^{\kappa}\right]^2, \quad {}_{p}^{2,2}u_j^{\kappa} \triangleq \chi_2^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(u_p)_{jl}^{\kappa}\right]^2, $$

$p = n$:

$$ {}_{p}^{2,2}w_j^{\kappa} \triangleq \chi_2^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(w_p)_{jl}^{\kappa}\right]^2, \quad {}_{p}^{2,2}u_j^{\kappa} \triangleq \chi_2^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa}\right]^2, $$

$q = 2, g = n, (C_B = 2, C_A = n), p = 1$:

$$ {}_{p}^{2,n}w_j^{\kappa} \triangleq \chi_2^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(w_p)_{jl}^{\kappa}\right]^2, \quad {}_{p}^{2,n}u_j^{\kappa} \triangleq \chi_2^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(u_p)_{jl}^{\kappa}\right]^2, $$

$p = 2, \dots, n-1$:

$$ {}_{p}^{2,n}w_j^{\kappa} \triangleq \chi_2^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p \left[(w_p)_{jl}^{\kappa}\right]^2, \quad {}_{p}^{2,n}u_j^{\kappa} \triangleq \chi_2^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa}\right]^2, $$

$p = n$:

$$ {}_{p}^{2,n}w_j^{\kappa} \triangleq \chi_2^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(w_p)_{jl}^{\kappa}\right]^2, \quad {}_{p}^{2,n}u_j^{\kappa} \triangleq \chi_2^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa}\right]^2, $$

$q = 2, g = n+1, (C_B = 2, C_A = n+1), p = 1, n-1$:

$$ {}_{p}^{2,n+1}w_j^{\kappa} \triangleq \chi_2^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p \left[(w_p)_{jl}^{\kappa}\right]^2, \quad {}_{p}^{2,n+1}u_j^{\kappa} \triangleq \chi_2^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p \left[(u_p)_{jl}^{\kappa}\right]^2, $$

$p = n$:

$$ {}_{p}^{2,n+1}w_j^{\kappa} \triangleq \chi_2^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(w_p)_{jl}^{\kappa}\right]^2, \quad {}_{p}^{2,n+1}u_j^{\kappa} \triangleq \chi_2^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa}\right]^2. $$

Here $(w_p)_{jl}^{\kappa}, (u_p)_{jl}^{\kappa}$, $p = 1, \dots, n$; $\kappa = 0, \dots, K$; $j = 0, \dots, J$; $l = 0, \dots, L$ - is the solution of the boundary difference problem ((9.1.2*), (6.3.2*), (6.4.2*), Eq. (71) $\{{}_{p}^{2}\mu_y, p = 1, \dots, n\}$ are positive constants [11].

ii. <u>Corollary 1.2:</u> Let the steps of the difference grid $\Delta t, \Delta y$ satisfy the Courant Friedrichs Lewy (CFL) condition of Eq. (54), then, according to Eq. (33), Eq. (42), the following inequalities hold:

$$ {}_{p}^{2,g}w_j^{\kappa} \leq {}^{2}_{\cdot}\alpha_p^{2,g} u_j^{\kappa}, \;\; g = 1, \dots, n+1, \; p = 1, \dots, n; \tag{57} $$
$$ {}^{2}_{\cdot}\alpha = 1 + \bar{\alpha}, \; \bar{\alpha} = -\Delta t \cdot \nu, \; 0 < {}^{2}_{\cdot}\alpha < 1. $$

Let us sum inequalities of Eq. (55) over p from 1 to n. Then we get the following inequalities

$$ {}^{2,g}_{\cdot}W_j^{\kappa} \leq {}^{2}_{\cdot}\alpha^{2,g}_{\cdot}U_j^{\kappa} g = 1, \dots, n+1. \tag{58}$$

Let $q = n$ $(C_B = n)$**.** Consider the initial-boundary difference problem ((9.1.n) or (9.1.n*), (6.3.n*), (6.4.n*), Eq. (71) and, as discrete functions ${}^{n,g}_{p}w_j^{\kappa}$, ${}^{n,g}_{p}u_j^{\kappa}$; $p = 1, \dots, n,$ we propose the following [11].

$q = n, g = 1, (C_B = n, C_A = 1), p = 1$:

$$ {}^{n,1}_{p}w_j^{\kappa} \triangleq \chi_n^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[\left(w_p\right)_{jl}^{\kappa}\right]^2, \quad {}^{n,1}_{p}u_j^{\kappa} \triangleq \chi_n^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[\left(u_p\right)_{jl}^{\kappa}\right]^2, $$

$p = 2, \dots, n$:

$$ {}^{n,1}_{p}w_j^{\kappa} \triangleq \chi_n^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[\left(w_p\right)_{jl}^{\kappa}\right]^2, \quad {}^{n,1}_{p}u_j^{\kappa} \triangleq \chi_n^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa}\right]^2, $$

$q = n, g = 2, (C_B = n, C_A = 2), p = 1$:

$$ {}^{n,2}_{p}w_j^{\kappa} \triangleq \chi_n^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[\left(w_p\right)_{jl}^{\kappa}\right]^2, \quad {}^{n,2}_{p}u_j^{\kappa} \triangleq \chi_n^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[\left(u_p\right)_{jl}^{\kappa}\right]^2, $$

$p = \overline{2, n-1}$:

$$ {}^{n,2}_{p}w_j^{\kappa} \triangleq \chi_n^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[\left(w_p\right)_{jl}^{\kappa}\right]^2, \quad {}^{n,2}_{p}u_j^{\kappa} \triangleq \chi_n^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa}\right]^2, $$

$p = n$:

$$ {}^{n,2}_{p}w_j^{\kappa} \triangleq \chi_n^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[\left(w_p\right)_{jl}^{\kappa}\right]^2, \quad {}^{n,2}_{p}u_j^{\kappa} \triangleq \chi_n^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa}\right]^2, $$

....

$q = n, g = n, (C_B = n, C_A = n), p = 1$:

$$ {}^{n,n}_{p}w_j^{\kappa} \triangleq \chi_n^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[\left(w_p\right)_{jl}^{\kappa}\right]^2, \quad {}^{n,n}_{p}u_j^{\kappa} \triangleq \chi_n^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[\left(u_p\right)_{jl}^{\kappa}\right]^2, $$

$p = \overline{2, n}$:

$$ {}^{n,n}_{p}w_j^{\kappa} \triangleq \chi_n^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[\left(w_p\right)_{jl}^{\kappa}\right]^2, \quad {}^{n,n}_{p}u_j^{\kappa} \triangleq \chi_n^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa}\right]^2, $$

$q = n, g = n+1, (C_B = n, C_A = n+1), p = 1$:

$$ {}^{n,n+1}_{p}w_j^{\kappa} \triangleq \chi_1^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p \left[\left(w_p\right)_{jl}^{\kappa}\right]^2, \quad {}^{n,n+1}_{p}u_j^{\kappa} \triangleq \chi_1^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p \left[\left(u_p\right)_{jl}^{\kappa}\right]^2. $$

$p = \overline{2, n}$:

$$ {}^{n,n+1}_{p}w_j^{\kappa} \triangleq \chi_1^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[\left(w_p\right)_{jl}^{\kappa}\right]^2, \quad {}^{n,n+1}_{p}u_j^{\kappa} \triangleq \chi_1^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa}\right]^2. $$

Here $\left(w_p\right)_{jl}^{\kappa}, \left(u_p\right)_{jl}^{\kappa}, p = 1, \dots, n;\ \kappa = 0, \dots, K;\ j = 0, \dots, J;\ l = 0, \dots, L$ - is the solution to the boundary difference problem ((9.1.n*), (6.3.n*), (6.4.n*), Eq. (71) [11].

iii. Corollary 1.n.: Let the steps of the difference grid $\Delta t,\ \Delta y$ satisfy the Courant Friedrichs Lewy (CFL) condition of Eq. (54), then, according to Eq. (33) and Eq. (42), the following inequalities are true

$$
{}^{n,g}_{\ p}w_j^{\kappa} \le {}^{n}_{.}\alpha_p^{n,g} u_j^{\kappa},\ g = 1, \dots, n+1,\ p = 1, \dots, n; \quad (59)
$$
$$
{}^{2}_{.}\alpha = 1 + \bar{\alpha},\ \bar{\alpha} = -\Delta t \cdot \nu,\ 0 < {}^{2}_{.}\alpha < 1.
$$

Let us sum inequalities of Eq. (59) over p from 1 to n. Then we obtain the following inequalities

$$
{}^{n,g}_{\ .}W_j^{\kappa} \le {}^{n}_{.}\alpha^{n,g}{}_{.}U_j^{\kappa},\ g = 1, \dots, n+1. \quad (60)
$$

Let $q = n+1$ $(C_B = n+1)$**.** Consider the initial-boundary difference problem ((9.1.n+1) or (9.1.n+1*), (6.3.n+1*), Eq. (71) and as discrete functions ${}^{n+1,g}_{\quad p}w_j^{\kappa},\ {}^{n+1,g}_{\quad p}u_j^{\kappa};\ p = 1, \dots, n$, we propose the following [11].

$q = n+1, g = 1, (C_B = n+1, C_A = 1), p = 1, \dots, n$:

$$
{}^{n+1,1}_{\quad p}w_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_1^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^{p}\bar{E}_l^{p}\left[\left(w_p\right)_{jl}^{\kappa}\right]^2,\quad {}^{n+1,1}_{\quad p}u_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_1^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^{p}\bar{E}_l^{p}\left[\left(u_p\right)_{jl}^{\kappa}\right]^2.
$$

$q = n+1, g = 2, (C_B = n+1, C_A = 2), p = 1, \dots, n-1$:

$$
{}^{n+1,2}_{\quad p}w_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_2^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^{p}\bar{E}_l^{p}\left[\left(w_p\right)_{jl}^{\kappa}\right]^2,\quad {}^{n+1,2}_{\quad p}u_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_2^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^{p}\bar{E}_l^{p}\left[\left(u_p\right)_{jl}^{\kappa}\right]^2.
$$

$p = n$:

$$
{}^{n+1,2}_{\quad p}w_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_2^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^{p}\bar{E}_l^{p}\left[\left(w_p\right)_{jl}^{\kappa}\right]^2,\quad {}^{n+1,1}_{\quad p}u_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_2^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^{p}\bar{E}_l^{p}\left[\left(u_p\right)_{jl}^{\kappa}\right]^2.
$$

……………………..

$q = n+1, g = n, (C_B = n+1, C_A = n), p = 1$:

$$
{}^{n+1,n}_{\quad p}w_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_n^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^{p}\bar{E}_l^{p}\left[\left(w_p\right)_{jl}^{\kappa}\right]^2,\quad {}^{n+1,n}_{\quad p}u_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_n^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^{p}\bar{E}_l^{p}\left[\left(u_p\right)_{jl}^{\kappa}\right]^2.
$$

$p = \overline{2, n}$:

$$
{}^{n+1,n}_{\quad p}w_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_n^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^{p}\bar{E}_l^{p}\left[\left(w_p\right)_{jl}^{\kappa}\right]^2,\quad {}^{n+1,n}_{\quad p}u_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_n^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^{p}\bar{E}_l^{p}\left[\left(u_p\right)_{jl}^{\kappa}\right]^2.
$$

$q = n+1, g = n+1, (C_B = n+1, C_A = n+1), p = 1, \dots, n$:

$$
{}^{n+1,n+1}_{\qquad p}w_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_{n+1}^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^{p}\bar{E}_l^{p}\left[\left(w_p\right)_{jl}^{\kappa}\right]^2,\quad {}^{n+1,n+1}_{\qquad p}u_j^{\kappa} \triangleq \chi_{n+1}^{p}\phi_{n+1}^{p}\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^{p}\bar{E}_l^{p}\left[\left(u_p\right)_{jl}^{\kappa}\right]^2.
$$

Here $(w_p)_{jl}^{\kappa}, (u_p)_{jl}^{\kappa},\ p = 1, \dots, n;\ \kappa = 0, \dots, K;\ j = 0, \dots, J;\ l = 0, \dots, L$ - is the solution of the boundary difference problem ((9.1.n+1*), (6.3.n+1*), Eq. (71) [11].

iv. Corollary 1.n+1: Let the steps of the difference grid $\Delta t,\ \Delta y$ satisfy the Courant Friedrichs Lewy (CFL) condition of Eq. (54), then, according to Eq. (33) and Eq. (42), the following inequalities hold:

$$ {}^{n+1,g}_{\ \ p}w_j^{\kappa} \le {}^{n+1}_{\cdot}\alpha \cdot {}^{n+1,g}_{\ \ p}u_j^{\kappa},\ g = 1, \dots, n+1,\ p = 1, \dots, n \tag{61} $$
$$ {}^{2}_{\cdot}\alpha = 1 + \bar{\alpha},\ \bar{\alpha} = -\Delta t \cdot \nu,\ 0 < {}^{2}_{\cdot}\alpha < 1. $$

Let us sum inequalities of Eq. (61) over *p* from 1 to *n*. Then we obtain the following inequalities:

$$ {}^{n+1,g}_{\ \ \cdot}W_j^{\kappa} \le {}^{n+1}_{\cdot}\alpha \cdot {}^{n+1,g}_{\ \ \cdot}U_j^{\kappa},\ \ g = 1, \dots, n+1 \tag{62} $$

v. Main corollary 1: Let CFL conditions (54) be satisfied. Then for the solution of the initial-boundary difference problem ((9.1.1)-(9.1.n+1), (6.3.1*)-(6.3.n+1*), (6.4.1*)-(6.4.n+ 1*), Eq. (71) the following inequality holds [11].

$$ W^{\kappa} \le \alpha U^{\kappa}, \alpha = \min_{1 \le i \le n+1} {}^{i}_{\cdot}\alpha. \tag{63} $$

3.2 Second Part of the Proof of the Stability Theorem (The Stage of Splitting along the x Direction)

We turn to the second part of the proof of the stability theorem (the stage of splitting along the x direction). To this end, we introduce the notation:

$$ U^{\kappa+1} = \begin{cases} {}^{1,1}_{\cdot}U^{\kappa+1} \text{ if } C_B = 1, C_A = 1; \\ \dots\dots\dots\dots\dots\dots \\ {}^{1,n+1}_{\cdot}U^{\kappa+1} \text{ if } C_B = 1, C_A = n+1; \\ {}^{2,1}_{\cdot}U^{\kappa+1} \text{ if } C_B = 2, C_A = 1; \\ \dots\dots\dots\dots\dots\dots \\ {}^{2,n+1}_{\cdot}U^{\kappa+1} \text{ if } C_B = 2, C_A = n+1; \\ \dots\dots\dots\dots\dots\dots \\ {}^{n+1,1}_{\cdot}U^{\kappa+1} \text{ if } C_B = n+1, C_A = 1; \\ \dots\dots\dots\dots\dots\dots \\ {}^{n+1,n+1}_{\cdot}U^{\kappa+1} \text{ if } C_B = n+1, C_A = n+1. \end{cases} $$
$$ {}^{q,g}_{\cdot}U^{\kappa+1} \triangleq \sum_{p=1}^{n} {}^{q,g}_{\ p}u^{k+1}, \qquad q = 1, \dots, n+1;\ \ g = 1, \dots, n+1. $$

We define values of ${}^{q,g}_{\ p}u^{k+1};\ p = 1, \dots, n$ for each $q = 1, \dots, n+1;\ \ g = 1, \dots, n+1$.

Let us start with $q = 1\ (C_B = 1)$. Consider the initial-boundary difference problem ((9.2.1) or (9.2.1*), (6.2.1*) and, as discrete Lyapunov functions ${}^{1,g}_{\ p}u^{k+1};\ p = 1, \dots, n$, we propose the following [11].

$q=1, g=1, (C_B=1, C_A=1), p=1,\dots,n$: ${}^{1,1}_{\;p}u_j^{\kappa+1} \triangleq \chi_1^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2,$

$q=1, g=2, (C_B=1, C_A=2), p=1,\dots,n-1$: ${}^{1,2}_{\;p}u_j^{\kappa+1} \triangleq \chi_1^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2;$

$p=n$: ${}^{1,2}_{\;p}u_j^{\kappa+1} \triangleq \chi_1^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2.$

....

$q=1, g=n, (C_B=1, C_A=n), p=1$: ${}^{1,2}_{\;p}u_j^{\kappa+1} \triangleq \chi_1^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2;$

$p=2,\dots,n$: ${}^{1,n}_{\;p}u_j^{\kappa+1} \triangleq \chi_1^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2.$

$q=1, g=n+1, (C_B=1, C_A=n+1), p=1,\dots,n$:

$${}^{1,n+1}_{\;\;p}u_j^{\kappa+1} \triangleq \chi_1^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2.$$

Here $(w_p)_{jl}^{\kappa}, (u_p)_{jl}^{\kappa}$, $p=1,\dots,n;\ \kappa=0,\dots,K; j=0,\dots,J; l=0,\dots,L$ -is the solution to the boundary difference problem ((9.2.1*), (6.2.1*); $\nu, \{{}^{1}_{p}\mu_x,\ p=1,\dots,n\}$ are positive constants [11].

i. Corollary 2.1: According to Eq. (62), the following inequalities hold:

$${}^{1,g}_{\;p}u_j^{\kappa+1} \le {}^{1}\beta\, {}^{1,g}_{\;p}w_j^{\kappa},\ \ g=1,\dots,n+1,\ \ p=1,\dots,n; \tag{64}$$

$${}^{1}\beta = 1+\bar{\beta}, \qquad \bar{\beta} = \frac{-\Delta t\cdot\nu}{1+\Delta t\cdot\nu}, \qquad 0<{}^{1}\beta<1.$$

Let us sum inequalities of Eq. (64) over *p* from 1 to *n*. Then we get the following inequalities:

$${}^{1,g}U_j^{\kappa+1} \le {}^{1}\beta\, {}^{1,g}W_j^{\kappa};\ \ g=1,\dots,n+1. \tag{65}$$

Let $q=2$ $(C_B=2)$. Consider the initial-boundary difference problem ((9.1.2) or (9.2.2*), (6.1.2*), (6.2.2*) and, as discrete Lyapunov functions ${}^{2,g}_{\;p}u_j^{\kappa+1};\ p=1,\dots,n.$, we propose the following [11]:

$q=2, g=1, (C_B=2, C_A=1), p=1,\dots,n-1.$

$${}^{2,1}_{\;p}u_j^{\kappa+1} \triangleq \chi_2^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2,$$

$p = n$: ${}^{2,1}_{\ p}u_j^{\kappa+1} \triangleq \chi_2^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2$,

$q = 2, g = 2, (C_B = 2, C_A = 2), p = 1, \dots, n-1$:

${}^{2,2}_{\ p}u_j^{\kappa+1} \triangleq \chi_2^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2$,

$p = n$: ${}^{2,2}_{\ p}u_j^{\kappa+1} \triangleq \chi_2^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa}\right]^2$,

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$q = 2, g = n, (C_B = 2, C_A = n), p = 1$: ${}^{2,n}_{\ p}u_j^{\kappa+1} \triangleq \chi_2^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2$,

$p = 2, \dots, n-1$: ${}^{2,n}_{\ p}u_j^{\kappa+1} \triangleq \chi_2^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2$,

$p = n$: ${}^{2,n}_{\ p}u_j^{\kappa+1} \triangleq \chi_2^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa+1}\right]^2$,

$q = 2, g = n+1, (C_B = 2, C_A = n+1)$,

$p = 1, \dots, n-1$: ${}^{2,n+1}_{\ \ \ p}u_j^{\kappa+1} \triangleq \chi_2^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p \left[(u_p)_{jl}^{\kappa}\right]^2$,

$p = n$: ${}^{2,n+1}_{\ \ \ p}u_j^{\kappa+1} \triangleq \chi_2^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[(u_p)_{jl}^{\kappa}\right]^2$.

Here $(u_p)_{jl}^{\kappa+1}$, $p = 1, \dots, n$; $\kappa = 0, \dots, K$; $j = 0, \dots, J$; $l = 0, \dots, L$ - is the solution to the boundary difference problem ((9.2.2*), (6.1.2*), (6.2.2*). $\{{}^2_p\mu_x,\ p = 1, \dots, n\}$ are positive constants [11].

ii. Corollary 2.2: According to Eq. (55) and Eq. (65), the following inequalities hold

$$ {}^{2,g}_{\ p}u_j^{\kappa+1} \le {}^2_{.}\beta\, {}^{2,g}_{\ p}u_j^{\kappa}, g = 1, \dots, n+1, p = 1, \dots, n \tag{66} $$

$$ {}^2_{.}\beta = 1 + \bar{\beta},\ \ \bar{\beta} = \frac{-\Delta t \cdot \nu}{1 + \Delta t \cdot \nu},\ \ 0 < {}^2_{.}\beta < 1. $$

Let us sum inequalities of Eq. (66) over *p* from 1 to *n.* Then we get the following inequalities

$$ {}^{2,g}_{\ .}U_j^{\kappa+1} \le {}^2_{.}\beta\, {}^{2,g}_{\ .}W_j^{\kappa},\ \ g = 1, \dots, n+1. \tag{67} $$

Let $q = n$ $(C_B = n)$. Consider the initial-boundary difference problem ((9.2.n) or (9.2.n*), (6.1.n*), (6.2.n*) and, as discrete functions ${}^{n,g}_{\ p}u_j^{\kappa+1}$; $p = 1, \dots, n$, we propose the following [11]:

$q = n, g = 1, (C_B = n, C_A = 1)$,

$$p=1:\ {}^{n,1}_{\ \ p}u_j^{\kappa+1} \triangleq \chi_n^p\phi_1^p\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p\left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$

$$p=2,\dots,n:\ {}^{n,1}_{\ \ p}u_j^{\kappa+2} \triangleq \chi_n^p\phi_1^p\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p\left[\left(u_p\right)_{jl}^{\kappa+2}\right]^2,$$

$q=n, g=2, (C_B=n, C_A=2),$

$$p=1:\ {}^{n,2}_{\ \ p}u_j^{\kappa+1} \triangleq \chi_n^p\phi_2^p\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p\left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$

$$p=2,\dots,n-1:\ {}^{n,2}_{\ \ p}u_j^{\kappa+1} \triangleq \chi_n^p\phi_2^p\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p\left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$

$$p=n:\ {}^{n,2}_{\ \ p}u_j^{\kappa+1} \triangleq \chi_n^p\phi_2^p\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p\left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$

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$q=n,\ \ g=n, (C_B=n, C_A=n),$

$$p=1:\ {}^{n,n}_{\ \ p}u_j^{\kappa+1} \triangleq \chi_n^p\phi_n^p\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p E_l^p\left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$

$$p=2,\dots,n:\ {}^{n,n}_{\ \ p}u_j^{\kappa+1} \triangleq \chi_n^p\phi_n^p\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p\left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$

$q=n, g=n+1, (C_B=n, C_A=n+1),$

$$p=1:\ {}^{1,n+1}_{\ \ \ p}u_j^{\kappa+1} \triangleq \chi_1^p\phi_{n+1}^p\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p E_l^p\left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$

$$p=2,\dots,n:\ {}^{1,n+1}_{\ \ \ p}u_j^{\kappa+1} \triangleq \chi_1^p\phi_{n+1}^p\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p\left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2.$$

Here $\left(u_p\right)_{jl}^{\kappa+1},\ p=1,\dots,n;\ \kappa=0,\dots,K;\ j=0,\dots,J;\ l=0,\dots,L$ - is the solution to the boundary difference problem ((9.2.n*), (6.1.n*), (6.2.n*) [11].

iii. Corollary 2.n: According to Eq. (55) and Eq. (67), the following inequalities hold

$$ {}^{n,g}_{\ \ p}u_j^{\kappa+1} \le {}^{n}_{.}\beta_p^{n,g} w_j^{\kappa},\ g=1,\dots,n+1,\ p=1,\dots,n; \tag{68}$$

$${}^{2}_{.}\beta = 1+\bar{\beta},\qquad \bar{\beta}=\frac{-\Delta t\cdot\nu}{1+\Delta t\cdot\nu},\qquad 0<{}^{2}_{.}\beta<1.$$

Let us sum inequalities of Eq. (68) over *p* from 1 to *n*. Then we get the following inequalities.

$$ {}^{n,g}_{\ .}U_j^{\kappa+1} \le {}^{n}_{.}\beta^{n,g}_{\ .}W_j^{\kappa},\ g=1,\dots,n+1. \tag{69}$$

Let $q = n+1$ ($C_B = n+1$). Consider the initial-boundary difference problem ((9.2.n+1) or (9.2.n+1*), (6.1.n+1*) and, as discrete functions ${}^{n+1,g}_{\;\;p}u_j^{\kappa+1}$; $p = 1, \dots, n$, we propose the following [11].

$$q = n+1,\;\; g = 1, (C_B = n+1, C_A = 1),$$
$$p = 1, \dots, n:\; {}^{n+1,1}_{\;\;p}u_j^{\kappa+1} \triangleq \chi_{n+1}^p \phi_1^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$
$$q = n+1,\;\; g = 2, (C_B = n+1, C_A = 2),$$
$$p = 1, \dots, n-1:\; {}^{n+1,2}_{\;\;p}u_j^{\kappa+1} \triangleq \chi_{n+1}^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$
$$p = n:\; {}^{n+1,2}_{\;\;p}u_j^{\kappa+1} \triangleq \chi_{n+1}^p \phi_2^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$

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$$q = n+1,\;\; g = n, (C_B = n+1, C_A = n),$$
$$p = 1:\; {}^{n+1,n}_{\;\;p}u_j^{\kappa+1} \triangleq \chi_{n+1}^p \phi_n^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} e_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$
$$p = 2, \dots, n:\; {}^{n+1,n}_{\;\;p}u_j^{\kappa+1} \triangleq \chi_{n+1}^p \phi_n^p \textstyle\sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2,$$
$$q = n+1,\;\; g = n+1, (C_B = n+1, C_A = n+1),$$
$$p = 1, \dots, n:\; {}^{n+1,n+1}_{\;\;p}u_j^{\kappa+1} \triangleq \chi_{n+1}^p \phi_{n+1}^p \sum_{j=1}^{J-1}\sum_{l=1}^{L-1} \bar{e}_j^p \bar{E}_l^p \left[\left(u_p\right)_{jl}^{\kappa+1}\right]^2.$$

Here $\left(u_p\right)_{jl}^{\kappa+1}$, $p = 1, \dots, n$; $\kappa = 0, \dots, K$; $j = 0, \dots, J$; $l = 0, \dots, L$ - is the solution to the boundary difference problem ((9.2.n+1*), (6.1.n+1*) [11].

iv. Corollary 2.n+1: According to Eq. (33) and Eq. (42), the following inequalities hold

$$ {}^{n+1,g}_{\;\;p}u_j^{\kappa+1} \le {}^{n+1}\beta_p^{n+1,g} w_j^{\kappa},\;\; g = 1, \dots, n+1,\;\; p = 1, \dots, n; \tag{70}$$
$$ {}^{2}\beta = 1 + \bar{\beta},\;\; \bar{\beta} = \frac{-\Delta t \cdot \nu}{1 + \Delta t \cdot \nu},\;\; 0 < {}^{2}\beta < 1.$$

Let us sum inequalities of Eq. (70) over p from 1 to n. Then we get the following inequalities

$$ {}^{n+1,g}U_j^{\kappa+1} \le {}^{n+1}\beta \cdot {}^{n+1,g}W_j^{\kappa},\quad g = 1, \dots, n+1. \tag{71}$$

v. Main corollary 2: To solve the initial-boundary difference problem ((9.2.1)-(9.2.n+1), (6.1.1*)-(6.1.n+1*), (6.2.1*)-(6.2 .n+1*) the following inequality holds [11].

$$U^{\kappa+1} \le \beta W^{\kappa},\quad \beta = \min_{1 \le i \le n+1} {}^{i}\beta. \tag{72}$$

From Eq. (63) and Eq. (72) we easily obtain the sought-for inequality

$$U^{\kappa+1} \leq \beta W^{\kappa} \leq \alpha\beta U^{\kappa}. \tag{73}$$

Note that for sufficiently small Δt, the following inequality holds

$$0 < \gamma = \alpha\beta < 1.$$

Then the following inequality is true:

$$U^{\kappa} \leq \gamma^{k} U^{0}, \ \ \kappa = 1, \dots, N. \tag{74}$$

Let us introduce positive constants

$$\vartheta_{qg}^{p} = \frac{{}_{p}^{q}\mu_{y}}{\left|\lambda_p(B)\right|}\frac{{}_{p}^{g}\mu_{x}}{\left|\lambda_p(A)\right|},$$

$$C_1 = \min_{p,q,g}\left(\vartheta_{qg}^{p}\right) \min\left(\min_{p,j,l}\left(e_j^p E_l^p\right), \min_{p,j,l}\left(\bar{e}_j^p E_l^p\right), \min_{p,j,l}\left(e_j^p \bar{E}_l^p\right), \min_{p,j,l}\left(\bar{e}_j^p \bar{E}_l^p\right), \right),$$

$$C_2 = \max_{p,q,g}\left(\vartheta_{qg}^{p}\right) \max\left(\max_{p,j,l}\left(e_j^p E_l^p\right), \max_{p,j,l}\left(\bar{e}_j^p E_l^p\right), \max_{p,j,l}\left(e_j^p \bar{E}_l^p\right), \max_{p,j,l}\left(\bar{e}_j^p \bar{E}_l^p\right), \right).$$

Then it follows that

$$\mathrm{C}_1 \Delta x \Delta y \sum_{j=1}^{J-1}\sum_{l=1}^{L-1}\sum_{p=1}^{n}\left[\left(u_p\right)_{jl}^{\kappa}\right]^2 \leq U^{\kappa} \leq \gamma^{k} U^{0} \leq \mathrm{C}_2 \gamma^{k} \Delta x \Delta y \sum_{j=1}^{J-1}\sum_{l=1}^{L-1}\sum_{p=1}^{n}\left[\left(u_p\right)_{jl}^{0}\right]^2,$$

$$\Delta x \Delta y \sum_{j=1}^{J-1}\sum_{l=1}^{L-1}\sum_{p=1}^{n}\left[\left(u_p\right)_{jl}^{\kappa}\right]^2 \leq \mathrm{C}\gamma^{k} \Delta x \Delta y \sum_{j=1}^{J-1}\sum_{l=1}^{L-1}\sum_{p=1}^{n}\left[\left(u_p\right)_{jl}^{0}\right]^2, \kappa = 1, \dots, N; \mathrm{C} = \mathrm{C}_2/C_1.$$

Thus, numerical solution U_{jl}^{k} of the mixed problem is exponentially Lyapunov stable in l^2-norm. Theorem 1 is proved.

4. Conclusions

Control parameters are introduced to control the characteristic velocities of a symmetric t-hyperbolic system. For a two-dimensional symmetric t-hyperbolic system with a constant coefficient, the exponential stability of the proposed difference scheme at energetic norms was studied. Auxiliary lemmas were introduced for one-dimensional hyperbolic equations. The exponential stability of the equation solution was analysed according to the initial and boundary conditions. Some auxiliary mixed problems were introduced and studied. A discrete analogue of the Lyapunov function was built and a prior evaluation was obtained for it, which indicates the exponential stability of the numerical solution. Problems of formulation of a mixed problem for symmetric hyperbolic systems depending on control parameters were investigated. The numerical solution of the initial-boundary difference problem was studied in the $\sqrt{U^k}$ norm in the sense of Lyapunov, and the corresponding inequality conditions were shown. It was proved that the numerical solution of the mixed problem is exponential stability in the Lyapunov sense-in l^2 the norm. It was shown that the difference scheme will be stable if the CFL condition is satisfied and otherwise unstable.

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