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A New Conjugate Gradient Parameter via Modification of Liu-Storey Formula for Optimization Problem and Image Restoration

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ABSTRACT

The conjugate gradient (CG) algorithms are one of the efficient numerical algorithms that are characterized by simplicity and nice convergence properties. However, recent modification of the CG method has complicated algorithms and might fail to converge under certain line search procedures. Also, the performance of some of the classical methods are yet to be tested on real-life application problems. This study presents a new modification of conjugate gradients (CG) algorithm for optimization problems and image restoration. The new formula is a modification of the Liu-Storey (LS) CG formula that generates the descent direction for objective functions. We established the convergence of our formula under suitable line search condition. Results from computational experiments were obtained and they showed that our new approach outperforms other existing algorithms such as Hestenes-Steifel (HS), LS, Dai-Yuan (DY) and Rivaie-Mustafa-Ismail-Leong (RMIL) in terms of both iteration numbers and based on CPU time. To further illustrate the efficiency of our proposed method, the formula was extended to restore images corrupted by impulse noise and the results obtained showed that the method was able to restore images with better accuracy which further confirmed the efficiency and robustness of our method.

1. Introduction

The CG algorithms are among the widely considered iterative procedures for solving application problems in medicine, engineering, sciences, and many more [1-5]. For instance, in [6], the CG was applied to solve regression analysis problem and [7-9] employ new CG strategy for motion control problem. Also, studies investigating the performance of CG methods on portfolio selection problems follows from [10-12]. The algorithms have also been considered for solution of the following unconstrained optimization problems (UOP):

$$\min\{f(x) \mid x \in R^n\} \tag{1}$$

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where $f: R^n \to R$ is a smooth function with gradient $\nabla(f(x)) = g(x)$. The efficacy of all CG algorithms depends on the memory requirements and their abilities of obtaining minimum values of Eq. (1) [10,13,14]. The iterative points of the CG formulas are generated via:

$$x_{k+1} = x_k + \alpha_k d_k, \qquad k = 1,2,.. \tag{2}$$

where $\alpha_k > 0$ is the step size obtained using any condition of the line search (exact or inexact) along the direction of search d_k. A review of different literatures show that most studies considered the inexact search method because of rapid convergence. But, the major drawback of this line search is that it only generates approximate solutions and not the real solution. Therefore, this study will consider the exact minimization method that requires α_k to satisfy:

$$f(x_k + \alpha_k d_k) = \min_{a \geq 0} f(x_k + \alpha d_k). \tag{3}$$

Another important component of the CG algorithm is the direction of search d_k computed as:

$$d_k = \begin{cases} -g_k, & \text{for} \quad k = 0 \\ -g_k + \beta_k d_{k-1}, & \text{for} \quad k \geq 1 \end{cases} \tag{4}$$

where the scalar β_k is the parameter differentiating the CG formulas. The following are among the classical formulas of the CG methods:

$$\beta_k^{HS} = \frac{g_k^T (g_k - g_{k-1})}{d_{k-1}^T (g_k - g_{k-1})} \tag{5}$$

$$\beta_k^{FR} = \frac{g_k^T g_k}{\|g_{k-1}\|^2} \tag{6}$$

$$\beta_k^{RMIL} = \frac{g_k^T (g_k - g_{k-1})}{\|d_{k-1}\|^2} \tag{7}$$

$$\beta_k^{CD} = -\frac{g_k^T g_k}{d_{k-1}^T g_{k-1}} \tag{8}$$

$$\beta_k^{PR} = \frac{g_k^T (g_k - g_{k-1})}{\|g_{k-1}\|^2} \tag{9}$$

$$\beta_k^{DY} = \frac{g_k^T g_k}{d_{k-1}^T (g_k - g_{k-1})}. \tag{10}$$

The above formulas are given by Hestenes-Steifel (HS) [15], Fletcher-Reeves (FR) [16], Rivaie-Mustafa-Ismail-Leong (RMIL) [17], Conjugate Descent (CD) [18], Polak-Ribiere (PR) [19], and the latest by Dai-Yuan (DY) [13]. The HS, PRP, and RMIL formulas are characterized by efficient computational performance, but their convergence fails under uncertain conditions. On the other hands, the FR, CD, and DY formulas possess excellent convergence results but their numerical performance is affected by jamming phenomena. For more references on the convergence of different CG formulas (see; [9,11,20-29]).

In this work, we are the interested in applying a new approach to define an efficient variant of Liu-Storey scheme for the unconstrained form of Eq. (1) and demonstrate the application in restoring corrupted images. The work is motivated by the following facts. Few numbers of modification of LS method for solving Eq. (1). The modifications of LS that exist are only applied to solve unconstrained optimization but their performance has not been evaluated on real-life problems. Some of the formulas available are very complicated.

In this study, we aim to construct an LS-type method that would address the drawbacks discussed above. A new type of LS CG method will be developed for unconstrained optimization. We shall apply the new formula to define a new search direction which will be different from those in the literature. The new method will globally converge globally under some suitable assumptions. The performance of the new algorithm will be compared with other existing methods with similar characteristics. Furthermore, the proposed method will be used to restore corrupted images.

This study will develop a new CG formula for optimization problem and image restoration in section 2. In section 3, we discuss the convergence of our formula under mild assumptions and present results for both unconstrained optimization problems and restored images are given in section 4. The conclusion is discussed in section 5.

2. Motivation and New CG Coefficient

Recently, Liu-Storey (LS) [30] present a modification of HS method [15] by expanding the term in the denominator as follows:

$$\beta_k^{LS} = -\frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T g_{k-1}} \tag{11}$$

The authors showed that their formula possesses descent properties and establish the convergence under mild assumptions. Results from computational experiment show that the formula Eq. (11) is very competitive.

Motivated by the efficiency and convergence analysis of the LS formula, this study proposed a CG coefficient β_k based on the modification of the LS method. This method was constructed by adding a new term $\frac{\|g_k\|}{\|g_{k-1}\|}$ to the numerator of LS method. This new CG parameter is known as β_k^{SFA}, where SFA denotes Saleh Alsuliman, Nur Fadhilah Ibrahim and Nur Aidya Hanum Aizam as below:

$$\beta_k^{SFA} = -\frac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1}\right)}{d_{k-1}^T g_{k-1}} \tag{12}$$

The proposed CG algorithm for the proposed formula β_k^{SFA} is presented as follows:

Algorithm 1.

Step 1: Initialize. Consider the initial guess x_k, set $k = 0$.
Step 2: Calculate β_k formula as Eq. (12).
Step 3: Obtain d_k using Eq. (4). If $\| g_k \| = 0$, terminate. Else, proceed to the next step
Step 4: Compute α_k using Eq. (3).
Step 5: Update the iterates using Eq. (2).
Step 6: Re-evaluate the stopping criteria and convergence test.
If $\|g_k\| \leq \varepsilon$ or $f(x_{k+1}) < f(x_k)$, terminate.
Else go back to Step 1 by replacing $k = k + 1$.

3. Convergence Analysis

This section will discuss the descent condition and global convergence of β_k^{SFA}.

3.1. Sufficient Descent Condition (SDC)

The SDC is defined such that the proposed formula satisfies:

$$g_k^T d_k \leq -c\|g_k\|^2 \; for \; k \geq 0, c > 0 \tag{13}$$

The theorem that follows will be used to show that our formula possesses the SDC under exact minimization conditions.

Theorem 1.
Let x_k and d_k follows from Eq. (2), Eq. (4) and Eq. (12), and $\alpha_k > 0$ computed using Eq. (3), then, condition Eq. (13) will hold $\forall \, \mathrm{k} \geq 0$.

Proof:
Using Induction, we let $k = 0$, then it is clear that $g_0^T d_0 = -c \parallel g_0 \parallel^2$. Thus, Eq. (13) is true. Next, we show that Eq. (13) is also true for $k \geq 1$.

Multiplying Eq. (4), by g_{k+1}^T will give,

$$g_{k+1}^T d_{k+1} = g_{k+1}^T(-g_{k+1} + \beta_{k+1}^{SFA} d_k)$$
$$= - \parallel g_{k+1} \parallel^2 + \beta_{k+1}^{SFA} g_{k+1}^T d_k$$

But $g_{k+1}^T d_k = 0$ based on exact minimization. This implies:

$$g_{k+1}^T d_{k+1} = - \parallel g_{k+1} \parallel^2.$$

which shows that Eq. (13) is true for $k + 1$ and thus, completes the proof.

3.2. Global Convergence Properties

The following assumptions would be needed for the convergence of our formula.

Assumption 1. [13]

i. For an initial point x_0, we have $\ell = \{ x \,| f(x) \leq f(x_0)\}$, the level set which is bounded.

ii. In some neighbourhoods N of ℓ, $f(x)$ is a smooth function with a Lipschitz continuous gradient. This implies there exist a constant $L > 0$ with

$$\|g(x) - g(y)\| \leq L \|x - y\| \text{ for any } x, y \in N.$$

Zoutendijk [31], presents the following lemma based on the above assumption.

Lemma 1.
Assume Assumption 1 is true, let x_k follows from Algorithm 1 and d_k satisfies Eq. (13), then:

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty \tag{14}$$

which can be written as

$$\sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} < \infty \tag{15}$$

The proposed formula β_k^{SFA} is simplified as follows for ease of convergence analysis.

$$\beta_k^{SFA} = -\frac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1}\right)}{d_{k-1}^T g_{k-1}} = \frac{g_k^T\left(\frac{g_k\|g_{k-1}\| - \|g_k\| g_{k-1}}{\|g_{k-1}\|}\right)}{-d_{k-1}^T g_{k-1}}$$

From Theorem 1 we know that $g_{k+1}^T d_{k+1} = -\| g_{k+1} \|^2$, that yields

$$\beta_k^{SFA} = \frac{g_k^T\left(\frac{g_k\|g_{k-1}\| - \|g_k\| g_{k-1}}{\|g_{k-1}\|}\right)}{\| g_{k-1} \|^2} = \frac{\|g_k\|^2\|g_{k-1}\| - \|g_k\| g_k^T g_{k-1}}{\| g_{k-1} \|^3}$$

by using Cauchy-Schwartz inequality, we have

$$\beta_k^{SFA} = \frac{\|g_k\|^2\|g_{k-1}\| - \|g_k\| g_k^T g_{k-1}}{\| g_{k-1} \|^3} \geq \frac{\|g_k\|^2\|g_{k-1}\| - \|g_k\|\|g_k\|\|g_{k-1}\|}{\| g_{k-1} \|^3} = 0$$

Thus, we get

$$\beta_k^{SFA} \geq 0 \tag{16}$$

Also

$$\beta_k^{SFA} = \frac{\|g_k\|^2\|g_{k-1}\| - \|g_k\| g_k^T g_{k-1}}{\| g_{k-1} \|^3} \leq \frac{\|g_k\|^2\|g_{k-1}\| + \|g_k\|^2\|g_{k-1}\|}{\| g_{k-1} \|^3} = \frac{2\|g_k\|^2}{\| g_{k-1} \|^2}$$

which implies

$$\beta_k^{SFA} \leq \frac{2\|g_k\|^2}{\|g_{k-1}\|^2} \tag{17}$$

Also, based on the above lemma, we have the following theorem.

Theorem 2.

Assume that Assumption 1 is true, x_k be obtained by Algorithm 1, α_k follows from Eq. (3), the SDC Eq. (13) is true and the coefficient β_k obtained by Eq. (12). Then

$$\lim_{k\to\infty} \|g_k\| = 0 \tag{18}$$

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Proof:

This prove is by contradiction. i.e., let's assume Theorem 2 is not true and supposed a constant say Ø exists satisfying

$$\|g_k\| \geq \emptyset \tag{19}$$

From Eq. (4), d_k is rewritten as:

$$d_k = -g_k + \beta_k^{SFA} d_{k-1}$$

Squaring both sides, obtained

$$\|d_k\|^2 = \|g_k\|^2 + (\beta_k^{SFA})^2 \|d_{k-1}\|^2 - 2\beta_k^{SFA} g_k^T d_{k-1}$$

For exact minimization conditions $g_k^T d_{k-1} = 0$, and thus, we have

$$\|d_k\|^2 = \|g_k\|^2 + (\beta_k^{SFA})^2 \|d_{k-1}\|^2$$

From Eq. (17) yields

$$\|d_k\|^2 \leq \|g_k\|^2 + \frac{4\|g_k\|^4}{\| g_{k-1} \|^4} \|d_{k-1}\|^2$$

divided both sides by $\|g_k\|^4$ we obtain

$$\frac{\|d_k\|^2}{\|g_k\|^4} \leq \frac{1}{\|g_k\|^2} + \frac{4\|d_{k-1}\|^2}{\| g_{k-1} \|^4}$$

Then we have

$$\frac{\|d_k\|^2}{\|g_k\|^4} \leq \sum_{i=0}^{k} \frac{4}{\|g_i\|^2}$$

by Eq. (15), yields

$$\frac{\|g_k\|^4}{\|d_k\|^2} \geq \frac{\emptyset^2}{4k}$$

So,

$$\sum_{i=0}^{k} \frac{\|g_k\|^4}{\|d_k\|^2} = \infty$$

which contradicts Zoutendijk condition, thus we have

$$\lim_{k\to\infty}\|g_k\| = 0$$

and this completes the proof.

4. Results

This section will evaluate the efficiency of our proposed formula by comparing their performance with other existing formulas with similar characteristics such as LS, HS, DY and RMIL based on iteration numbers and CPU time. Most of the problems used for these computations as seen in Table 1 follows from Andrei [32]. For every problem, four different initial guesses are selected which includes points close to the optimum points to those further away. The algorithms for all the formulas are written on MATLAB R2022a software and run on Ryzen5 Windows 11 Professional operating system. All computations will be terminated provided the condition $\|g_k\| \leq 10^{-6}$ is satisfied.

Table 1
List of Test Problems

N	Functions	Dimensions	Initial Points
1	Extended Rosenbrock	2, 100, 500, 1000	(3,..., 3), (7,…,7), (11,..., 11), (19,..., 19)
2	Fletcher	10, 20, 50, 80	(2,…,2), (-2,…,-2), (9,…,9), (-9,…,-9)
3	Perturbed Quadratic	2, 10, 100, 1000	(2,…,2), (6,…,6), (-2,…,-2), (-6,…,-6)
4	Dixon and Price	2, 4	(6,…,6), (12,…,12), (16,…,16), (22,…,22)
5	Extended Powell	4, 40, 100, 1000	(5,…,5), (-5,…,-5), (2,…,2), (-2,…,-2)
6	Non Scomp	4, 40, 60, 80	(3,…,3), (13,…,13), (20,…,20), (23,…,23)
7	Extended Quadratic Penalty	50	(-2,…,-2), (2,…,2), (-12,…,-12), (12,…,12)
8	Hager	10, 20, 50, 100	(-2,…,-2), (2,…,2), (15,…,15), (20,…,20)
9	Generalized Quadratic	4, 10, 50, 500	(2,…,2), (6,…,6), (18,…,18), (25,…,25)
10	Quadratic 2	100, 500, 1000, 5000	(7,…,7), (17,…,17), (-7,…,-7), (-17,…,-17)
11	Generalized Tridiagonal 2	2, 10	(5,…,5), (2,…,2), (12,…,12), (22,…,22)
12	Extended Trigonometric	4, 10, 50, 100	(-16,…,-16), (-4,…,-4), (4,…,4), (16,…,16)
13	Quadratic 1	4, 10, 100, 1000	(3,…,3), (6,…,6), (12,…,12), (24,…,24)
14	Non Dia	20	(5,…,5), (14,…,14), (19,…,19), (23,…,23)
15	Non Dquar	4	(-10,…,-10), (-7,…,-7), (7,…,7), (10,…,10)
16	DQDRTIC	10, 100, 500, 1000	(5,…,5), (10,…,10), (15,…,15), (20,…,20)
17	Quartic	4	(-6,…,-6), (2,…,2), (6,…,6), (12,…,12)

All these results are obtained under exact minimization conditions. The performance plot in Figure 1 is for iteration numbers while Figure 2 represents the CPU time. This plots are obtained using performance profile tool presented by [33].

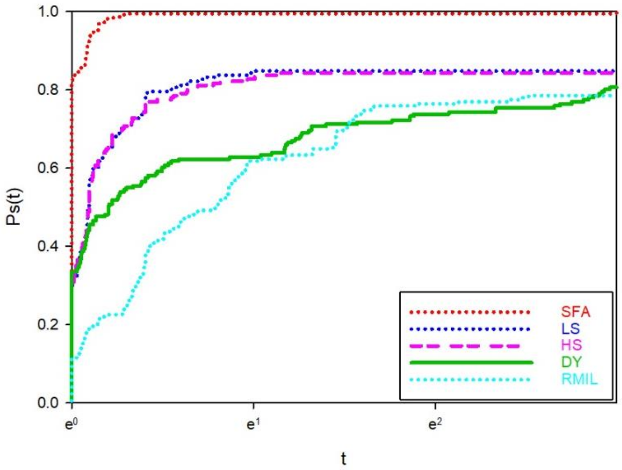

Fig. 1. Performance metric of iteration number

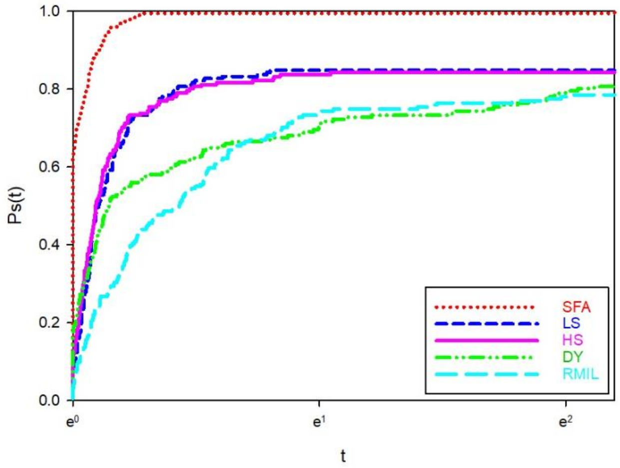

Fig. 2. Performance metric for CPU time

By observing these figures, we can see that the curve of our proposed formula lies above that of LS, HS, DY and RMIL because it was able to solve most of the test problems with better accuracy. This shows that the method is very competitive.

5. Application to Image Restoration

Image restoration is an important topic in the area of image processing which has recently gained attention from the research world because of its numerous applications in security and health sectors [10,34-36]. The process of image restoration involves restoring images from a degraded version; typically, the blurred and noisy images. Some of the important components that needs to be considered when restoring corrupted images includes: an algorithm's experimental efficiency, quality of the restored images, and parameter estimate.

In this study, the performance of the proposed algorithm will be investigated on image restoration problems. The following images: LENA (512 × 512) and CAMERA (512 × 512) corrupted by salt-and-pepper impulse noise would be restored using the proposed algorithm and the restored image quality would be assessed based on relative error (RelErr) and peak signal-to-noise ratio (PSNR). The PSNR measures the quality between the corrupted and restored images and its computed as follows:

$$PNSR = 20 \cdot \log_{10}(MAX_1) - 10 \cdot \log_{10}(MSE)$$

where MSE denotes the mean square error apply to assess the differences of pixels for complete images and MAX_1 defines the image possible maximum pixel value.

In this study, the image restoration problem is transformed into the following optimization problem [10]:

$$\min \chi(u)$$

and

$$\chi(u) = \sum_{(i,j)\in G} \left\{ \sum_{(m,n)\in T_{i,j}/G} \phi_\alpha\left(u_{i,j} - \xi_{m.n}\right) + \frac{1}{2} \sum_{(m,n)\in T_{i,j}\cap G} \phi_\alpha\left(u_{i,j} - u_{m.n}\right) \right\}. \quad (20)$$

where $\phi_\alpha(t) = \sqrt{t^2 + \alpha}$ defines an edge-preserving potential function with the constant value of $\alpha = 1$ and ξ is the observed noisy image corrupted by salt-and-pepper impulse noise.

From Eq. (20), G is the index set of noise candidates x and its computed as:

$$G = \left\{(i,j) \in Q/\xi_{ij} = s_{max} \; or \; s_{min}, \bar{\xi}_{ij} \neq \; \xi_{ij} \right\}$$

where $\bar{\xi}$ defines the adaptive median filter of ξ and $(i,j) \in Q = \{1,2,3,\dots,M\} \times \{1,2,3,\dots,N\}$ with its neighbourhood given as $T_{i,j} = \{(i,j-1),(i,j+1),(i-1,j),(i+1,j)\}$. Also, s_{max} and s_{min} represents the maximum and minimum of the noisy pixel.

Next, we present the experiment results for all the methods as follows:

Tables 2 and 3 demonstrate the numerical performance of the proposed SFA methods and other existing methods based on RelErr and PSNR.

Table 2
Results of Restored Images using SFA, LS, HS, DY, and RMIL methods based on RelErr

Methods		SFA	LS	HS	DY	RMIL
Image	Noise Degree	PSNR	PSNR	PSNR	PSNR	PSNR
Lena	30%	0.8739	0.9457	0.9543	0.9609	0.9426
	60%	1.5654	1.4386	1.7738	1.4853	1.4562
Camera	30%	1.1044	1.1957	1.1896	1.2815	1.1491
	60%	2.1120	2.1201	2.1512	1.7949	2.4673

Table 3
Results of Restored Images using SFA, LS, HS, DY, and RMIL methods based on PSNR

Methods		SFA	LS	HS	DY	RMIL
Image	Noise Degree	PSNR	PSNR	PSNR	PSNR	PSNR
Lena	30%	33.7906	33.5643	33.4291	33.5539	33.6432
	60%	29.5473	29.2987	28.4836	29.7479	29.4587
Camera	30%	30.9399	30.6974	30.8129	30.6671	30.8022
	60%	26.5854	26.7705	25.9684	26.8705	26.8722

On the other hand, Figure 3 and Figure 4 presents the corrupted images and all restored images using the proposed SFA and the classical LS, HS, DY, and RMIL algorithms. A method with higher PSNR values is said to produce better quality of the output images. By observing the results presented above, it is obvious that the proposed SFA algorithm produces the least relative error and higher PSNR values for most of the noise degrees. Also, a close observation of the images show that the new method produces the better quality of the output images when compared to those obtained using the other existing methods. With the above results, we can conclude that our proposed SFA algorithm has the best performance because it generates more efficient results compare to the existing methods.

Fig. 3. Lena and Camera images corrupted by 30 % salt-and-pepper noise: (A1, B1), the restored images using SFA: (A2, B2), LS: (A3, B3), HS: (A4, B4), DY: (A5, B5), and RMIL: (A6, B6)

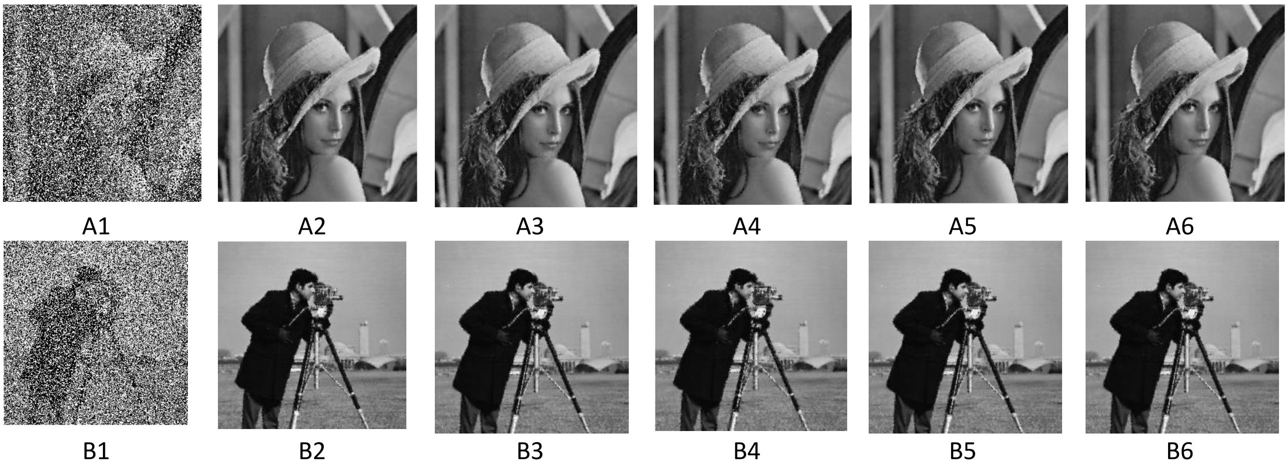

Fig. 4. Lena and Camera images corrupted by 60 % salt-and-pepper noise: (A1, B1), the restored images using SFA: (A2, B2), LS: (A3, B3), HS: (A4, B4), DY: (A5, B5), and RMIL: (A6, B6)

6. Conclusions

This study defines a new conjugate gradient formula by modifying the classical LS formula for optimization problems and image restoration. An important property of our method is that it satisfies the descent condition under suitable condition and the convergence results was established under exact minimization conditions. The efficiency of the new formula was evaluated by comparing its numerical performance with that of other methods with similar characteristics. The comparison is done based on iteration number and CPU time and all results show that the proposed algorithm produced best performance on all metrics. To further demonstrate the efficacy of our proposed formula, we applied the method to restore images corrupted by impulse noise and the results also showed that the new method was able to restore images with better accuracy which further confirmed the efficiency and robustness of our method.

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