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Homotopy Perturbation Method for Semi-Bounded Solution of the System of Cauchy-Type Singular Integral Equations of the First Kind

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ABSTRACT

In this paper, we introduce the standard Homotopy Perturbation Method (HPM) for obtaining semi-bounded solutions of the first kind of system of Cauchy-type singular integral equations (CSIEs) with constant coefficients. We use the Gauss elimination technique to reduce the system of CSIEs to a diagonal triangle system of algebraic equations. We then apply the HPM to solve the resulting equations. By applying the theory of semi-bounded solutions of CSIEs, we can determine the inverse operators for the first kind of CSIEs. We demonstrate that the proposed method is exact for the system of characteristic SIEs, regardless of the choice of initial guesses (in the Holder class of functions). To illustrate the validity and accuracy of our proposed method, we supply and analyse three examples. We compare the results obtained using our method with those obtained using the Chebyshev collocation and Galerkin methods. Our method includes the ability to solve the complex-valued system of CSIEs. Based on the numerical results, we conclude that the HPM is more dominant than the other methods.

1. Introduction

The SIEs of both Abel and Cauchy types are well-known to exist in various scientific fields, such as Jakeman and Anderssen [1] applied it in stereology, Healy *et al.,* [2] in radio occultation (RO) measurements, Bracewell and Riddle [3] in radio astronomy, Buck [4] in molecular scattering, Kosarev [5] in electron emission, Hellsten and Andersson [6] in radar ranging, Fleurier and Chapelle [7] in plasma spectroscopy, and Glantschnig and Holliday [8] in X-ray tomography, among others.

Investigations of the system of SIEs have attracted much concern in the applied sciences. Their general ideas and essential features are broadly applicable in engineering science. The solution to a large class of mixed boundary value problems in physics and engineering is reduced to a one-dimensional system of SIEs. The system of Cauchy and Abel type SIEs, their generalized form, and the weakly type SIEs, using a variety of methods [9,10]. There are many methods developed for one-dimensional CSIEs can be seen in review papers by several authors [11-14]. Unfortunately, less

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research has been done on the system of CSIEs. Nevertheless, the HPM method for the system of CSIEs has rarely been applied, and very few articles have been published.

HPM and homotopy analysis method (HAM) are crucial methodologies utilized to tackle both linear and nonlinear problems across multiple fields of science and technology. These approaches have demonstrated their effectiveness in solving various types of nonlinear problems. For example, Alhawamda *et al.,* [15] successfully applied them to solve the nonlinear Fredholm Integral Equation, while Biazar and Ghazvini [16] utilized them for Abel's integral equations. Several authors have employed these methods to address nonlinear functional integral equations [17,18], nonlinear equations [19,20], special non-linear Fredholm integral equation by Aminikhah and Salahi [21], the quadratic Riccati differential equation by Odibat and Momani [22], the nonlinear second-order differential equation by Cveticanin [23], the Zakharov–Kuznetsov equations by Biazar *et al.,* [24], the non-linear partial differential equations by Biazar *et al.,* [25] and many others.

In this note, the application of HPM is demonstrated for a semi-bounded solution of the first kind of system of CSIEs of the first kind given by

$$\sum_{j=1}^{M}\left[\frac{a_{ij}}{\pi}\int_{-1}^{1}\frac{u_j(\tau)d\tau}{\tau-t}+\frac{b_{ij}}{\pi}\int_{-1}^{1}K_{ij}(t,\tau)u_j(\tau)d\tau\right]=f_i(t),\ i=1,\ldots,M,\ \ -1<t<1, \tag{1}$$

where $A=(a_{i,j})$ and $B=(b_{i,j})$ are given constants with $\det(A)\neq 0,\ \det(B)\neq 0$, the forcing functions $f_i(t)$ and kernels $K_{i,j}$ are all known to be real-valued or complex-valued continuous functions and u_j are unknown functions to be determined.

2. Methodology and Reduction Techniques

It is known that the characteristic singular integral equations of the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(\tau)d\tau}{\tau-t}=f(t),\ -1<t<1 \tag{2}$$

has four types of solutions (bounded, unbounded and semi-bounded). Based on the theory of solutions in Lifanov [13], we obtained semi-bounded solutions of Eq. (1) in the following form:

Case 1: The solution is bounded at the endpoint $t=-1$, but unbounded at the endpoint $t=1$ as follows

$$u(t)=\frac{1}{\pi^2}\sqrt{\frac{1+t}{1-t}}\int_{-1}^{1}\int_{-1}^{1}\sqrt{\frac{1-\tau}{1+\tau}}\frac{K(t,s)u(s)ds}{\tau-t}d\tau-\frac{1}{\pi}\sqrt{\frac{1+t}{1-t}}\int_{-1}^{1}\sqrt{\frac{1-\tau}{1+\tau}}\frac{f(\tau)}{\tau-t}d\tau. \tag{3}$$

Case 2: The solution is bounded at the endpoint $t=1$, but unbounded at the endpoint $t=-1$, as follows

$$u(t)=\frac{1}{\pi^2}\sqrt{\frac{1-t}{1+t}}\int_{-1}^{1}\int_{-1}^{1}\sqrt{\frac{1+\tau}{1-\tau}}\frac{K(t,s)u(s)ds}{\tau-t}d\tau-\frac{1}{\pi}\sqrt{\frac{1-t}{1+t}}\int_{-1}^{1}\sqrt{\frac{1+\tau}{1-\tau}}\frac{f(\tau)}{\tau-t}d\tau \tag{4}$$

Let us consider Eq. (1) and write it as algebraic equations given below

$$\begin{cases} a_{11}S_1(t)+a_{12}S_2(t)+\ldots+a_{1M}S_M(t)=f_1(t)-F_1(t), \\ a_{21}S_1(t)+a_{22}S_2(t)+\ldots+a_{2M}S_M(t)=f_2(t)-F_2(t), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{M1}S_1(t)+a_{M2}S_2(t)+\ldots+a_{MM}S_M(t)=f_M(t)-F_M(t), \end{cases} \tag{5}$$

where

$$\begin{aligned} &S_i(t)=\frac{1}{\pi}\int_{-1}^{1}\frac{u_i(\tau)}{\tau-t}d\tau,\ i=\{1,2,\ldots,M\}, \\ &F_i(\bar{u}(t))=\sum_{j=1}^{M}\frac{b_{ij}}{\pi}\int_{-1}^{1}K_{ij}(t,\tau)u_j(\tau)d\tau,\ \ i=\{1,2,\ldots,M\}. \end{aligned} \tag{6}$$

To solve Eq. (5), in the case of $M=3$, extension matrix $\overline{A}$ has the form

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & f_1(t)-F_1(t) \\ a_{21} & a_{22} & a_{23} & f_2(t)-F_2(t) \\ a_{31} & a_{32} & a_{33} & f_3(t)-F_3(t) \end{bmatrix} \tag{7}$$

By applying the Gaussian elimination technique to Eq. (7), we obtain

$$A=\begin{bmatrix} 1 & 0 & 0 & \frac{1}{C_3}(e_{11}f_1+e_{12}f_2+e_{13}f_3)-\frac{1}{C_3}(e_{11}F_1+e_{12}F_2+e_{13}F_3) \\ 0 & 1 & 0 & \frac{1}{C_3}(e_{21}f_1+e_{22}f_2+e_{23}f_3)-\frac{1}{C_3}(e_{21}F_1+e_{22}F_2+e_{23}F_3) \\ 0 & 0 & 1 & \frac{1}{C_3}(e_{31}f_1+e_{32}f_2+e_{33}f_3)-\frac{1}{C_3}(e_{31}F_1+e_{32}F_2+e_{33}F_3) \end{bmatrix}, \tag{8}$$

where $C_3=a_{11}a_{23}a_{32}-a_{11}a_{22}a_{33}+a_{12}a_{21}a_{33}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}+a_{13}a_{22}a_{31}$ and

$$E=\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}=\begin{bmatrix} a_{22}a_{33}-a_{23}a_{32} & a_{13}a_{32}-a_{12}a_{33} & a_{12}a_{23}-a_{13}a_{22} \\ a_{23}a_{31}-a_{21}a_{33} & a_{11}a_{33}-a_{13}a_{31} & a_{13}a_{21}-a_{11}a_{23} \\ a_{21}a_{32}-a_{22}a_{31} & a_{12}a_{31}-a_{11}a_{32} & a_{11}a_{22}-a_{12}a_{21} \end{bmatrix} \tag{9}$$

Then, Eq. (5) and Eq. (6) has the form

$$\begin{cases} \frac{1}{\pi}\int_{-1}^{1}\frac{u_1(\tau)d\tau}{\tau-t}+\frac{1}{C_3}(e_{11}F_1+e_{12}F_2+e_{13}F_3)=\frac{1}{C_3}(e_{11}f_1+e_{12}f_2+e_{13}f_3), \\ \frac{1}{\pi}\int_{-1}^{1}\frac{u_2(\tau)d\tau}{\tau-t}+\frac{1}{C_3}(e_{21}F_1+e_{22}F_2+e_{23}F_3)=\frac{1}{C_3}(e_{21}f_1+e_{22}f_2+e_{23}f_3), \\ \frac{1}{\pi}\int_{-1}^{1}\frac{u_3(\tau)d\tau}{\tau-t}+\frac{1}{C_3}(e_{31}F_1+e_{32}F_2+e_{33}F_3)=\frac{1}{C_3}(e_{31}f_1+e_{32}f_2+e_{33}f_3). \end{cases} \tag{10}$$

In order to generalize Eq. (10) for different values of M, we can write it in the following operator form:

$$L(u_i)+N_i(\bar{u})=f_i(t),\qquad i=1,\ldots,M. \tag{11}$$

where $\bar{u} = (u_1(t), u_2(t), \ldots, u_M(t))$ and

$$L(u_i) = \frac{1}{\pi}\int_{-1}^{1}\frac{u_i(\tau)\,d\tau}{\tau - t}, \quad N_i(\bar{u}) = \frac{1}{\pi C_M}\sum_{j=1}^{M} e_{ij}^{[M]} F_j(\bar{u}(t)), \quad f_i^*(t) = \frac{1}{C_M}\sum_{j=1}^{M} e_{ij}^{[M]} f_j(t) \tag{12}$$

Here, $F_j(\bar{u}(t))$ is given by Eq. (6), while coefficients C_M and $e_{ij}^{[M]}$ are defined by solving $M \times M$ algebraic equations.

To find semi-bounded solutions of Eq. (11) and Eq. (12), we search for solutions in the form:

$$u_{i,r}(t) = w_r(t)v_i(t), \; r = \{1,2\}, \quad i = 1,\ldots,M, \tag{13}$$

where

$$\begin{cases} w_1(t) = \sqrt{\dfrac{1-t}{1+t}}, \\ w_2(t) = \sqrt{\dfrac{1+t}{1-t}}. \end{cases} \tag{14}$$

Remark 1: The Gauss elimination method and the existence of an inverse operator greatly helped us to handle the system of CSIEs Eq. (1). Direct implementation of HPM for solving CSIEs Eq. (1) did not yield good results. Fortunately, the hybrid method gave us highly accurate results.

Detailed implementation of HPM is provided in the next section.

3. Description of the HPM and Its Application for the System of CSIES of the First Kind

It is known from the previous studies [17,18] that the general theory of HPM for nonlinear equations has the form

$$Lu + Nu = f, \tag{15}$$

where L is the linear operator and N is the nonlinear operator.

Implementation of HPM to Eq. (1) yields the following

$$(1-p)\Big(L\big(v_i(t,p)\big) - L\big(u_{i,0}(t)\big)\Big) + p\Big(L\big(v_i(t,p)\big) + N_i\big(v_i(t,p)\big) - f_i(t)\Big) = 0 \tag{16}$$

and search for unknown function $v_i(t)$ in the form

$$v_i(t,p) = \sum_{k=0}^{\infty} v_{i,k}(t)\,p^k, \quad i = 1,\ldots,M \tag{17}$$

Then, we have

$$L\left(\sum_{k=0}^{\infty} v_{i,k}(t)\, p^k\right) = L\left(u_{i,0}(t)\right) + p\left(f_i(t) - N_i\left(\sum_{k=0}^{\infty} v_{i,k}(t)\, p^k\right) - L\left(u_{i,0}(t)\right)\right) \tag{18}$$

By equating the coefficient of the terms according to the same power of p, we obtain

$$\begin{aligned} p^0 &: v_{i,0}(t) = u_{i,0}(t), \quad i = \{1,2,\ldots,M\}, \\ p^1 &: v_{i,1}(t) = L^{-1}\left(f_i(t)\right) - L^{-1}\left(N_i\left(v_{i,0}(t)\right)\right) - u_{i,0}(t), \\ p^k &: v_{i,k}(t) = -L^{-1}\left(N_i\left(v_{i,k-1}(t)\right)\right), \qquad k = 2,3,\ldots. \end{aligned} \tag{19}$$

where L^{-1} is the inverse operator of L. Hence, the semi-analytical approximate solution is given by

$$u_{i,r}(t,p) = w_r(t)\lim_{p\to 1} v_i(t,p) = w_r(t)\lim_{N\to\infty}\sum_{j=0}^{N} v_{ij}(t), \quad r = \{1,2\},\ i = \{1,\ldots,M\}, \tag{20}$$

where $w_r(t),\ r = \{1,2\}$ are defined by Eq. (14).

In the practical problem, we usually choose the initial guess $u_{i,0}$ in the standard HPM as follows:

$$u_{i,0} = f_i(t),\ i = 1,2,\ldots,M. \tag{21}$$

Note that if the operator $N_i(\bar{u}) = 0$ in Eq. (11), then the operator Eq. (15) becomes

$$L(u_i) = f_i(t), \quad i = 1,2,\ldots M. \tag{22}$$

If operator L is invertible, then the exact solution of Eq. (22) is

$$u_{i,r}(t) = w_r(t)\, L^{-1}\left(f_i(t)\right), \quad r = \{1,2\}. \tag{23}$$

Let us now find the exact solution of Eq. (22) using the standard HPM. From Eq. (19), it follows that

$$\begin{aligned} p^0 &: v_{i,0}(t) = u_{i0}(t), \\ p^1 &: v_{i,1}(t) = L^{-1}\left(f_i(t)\right) - u_{i0}(t), \\ p^k &: v_{i,k}(t) = 0, \qquad k = 2,3,\ldots. \end{aligned} \tag{24}$$

Now, from Eq. (20) and Eq. (24), it follows that

$$u_{i,r}(t) = w_r(t)\lim_{p\to 1} v_i(t) = w_r(t)\sum_{k=0}^{\infty} v_{i,k}(t) = w_r(t)\, L^{-1}\left(f_i(t)\right), \quad r = \{1,2\}, \tag{25}$$

which coincides with the exact solution Eq. (23).

Schemes Eq. (24) and Eq. (25) lead to the following theorem:

Theorem 1: Let the kernel in Eq. (11) be a Cauchy singular kernel given by $\frac{1}{\tau - t}$ and $f_i(t) \in H^{\alpha}[-1,1]$ (Holder class). If operator L in Eq. (22) is linear, then the iterative scheme Eq. (24) provides an exact solution for the operator Eq. (22).

4. Numerical Example

Example 1: (Ahdiaghdam and Shahmorad [11], Sharma *et al.,* [26]): Consider the system of SIEs of the form

$$\begin{aligned}
&\frac{1000}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau - t}d\tau + \frac{10}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau - t}d\tau = f_1(t) + ig_1(t),\\
&\frac{500}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau - t}d\tau + \frac{200}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau - t}d\tau = f_2(t) + ig_2(t),
\end{aligned} \tag{26}$$

where

$$\begin{cases}
f_1(t) = -990t^8 + 1089t^7 + 937t^6 - \frac{26704t^5}{25} - \frac{349161t^4}{1000} + \frac{792327t^3}{2000} + \frac{1761t^2}{250} - \frac{53511t}{4000} - \frac{53929}{2000},\\
g_1(t) = 990t^8 - 1189t^7 - \frac{8971t^6}{10} + \frac{119047t^5}{100} + \frac{163961t^4}{500} - \frac{279198t^3}{625} - \frac{30873t^2}{10000} + \frac{69533t}{4000} + \frac{1130501}{40000},\\
f_2(t) = -300t^8 + 330t^7 + 215t^6 - \frac{2447t^5}{10} - \frac{8607t^4}{100} + \frac{735t^3}{8} - \frac{17253t^2}{2000} + \frac{29541t}{4000} - \frac{14701}{1000},\\
g_2(t) = 300t^8 - 380t^7 - 197t^6 + \frac{1462t^5}{5} + \frac{9419t^4}{100} - \frac{27549t^3}{250} - \frac{183t^2}{400} - \frac{7957t}{2000} + \frac{57853}{4000}.
\end{cases}$$

Remark 2: Example 1 is discussed by Sharma *et al.,* [26], who found the semi-bounded solution for $n = 8$. Additionally, Ahdiaghdam and Shahmorad [11] also examined this example and derived the corresponding error function as follows:

$$\sqrt{\frac{1-t}{1+t}}h_i(t),\ i = \{1,2\},$$

where

$$\begin{cases}
h_1(t) = 5.128205128205128 \times 10^{-8}(1+t), \quad -1 < t < 1,\\
h_2(t) = 5.128205128205128 \times 10^{-6}(1+t), \quad -1 < t < 1.
\end{cases}$$

Before applying HPM, we should use the Gaussian elimination method to obtain

$$\begin{aligned}
&\frac{1}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau - t}d\tau = \frac{1}{975}\left(f_1 + ig_1(t)\right) - \frac{1}{19500}\left(f_2 + ig_2(t)\right),\\
&\frac{1}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau - t}d\tau = -\frac{1}{390}\left(f_1 + ig_1(t)\right) + \frac{1}{195}\left(f_2 + ig_2(t)\right).
\end{aligned} \tag{27}$$

We solve Eq. (34) using standard HPM Eq. (24). Since Eq. (34) can be written as Eq. (35) due to Theorem 1, we are able to get the exact solution for two cases, as shown below:

Case 1: Let us search semi-bounded solution of Eq. (26) given in the form

$$u_i(t) = \sqrt{\frac{1+t}{1-t}} v_i(t), \quad i = \{1,2\}. \tag{28}$$

We choose the initial guess as follows

$$\begin{aligned} u_{1,0}(t) &= \frac{1}{975}\left(f_1 + i g_1(t)\right) - \frac{1}{19500}\left(f_2 + i g_2(t)\right), \\ u_{2,0}(t) &= -\frac{1}{390}\left(f_1 + i g_1(t)\right) + \frac{1}{195}\left(f_2 + i g_2(t)\right). \end{aligned} \tag{29}$$

By applying standard HPM Eq. (24) to Eq. (27), we obtain

$$\begin{aligned} p^0 : v_{1,0}(t) &= -t^8 + \frac{11t^7}{10} + \frac{19t^6}{20} - \frac{1083t^5}{1000} - \frac{3537t^4}{10000} + \frac{40161t^3}{100000} + \frac{7667t^2}{1000000} - \frac{28199t}{2000000} - \frac{13451}{500000} \\ &\quad i\left(t^8 - \frac{6t^7}{5} - \frac{91t^6}{100} + \frac{603t^5}{500} + \frac{663t^4}{2000} - \frac{11313t^3}{25000} - \frac{3143t^2}{1000000} + \frac{18033t}{1000000} + \frac{56491}{2000000}\right), \\ \mathrm{v}_{2,0}(t) &= t^8 - \frac{11t^7}{10} - \frac{13t^6}{10} + \frac{371t^5}{250} + \frac{4539t^4}{10000} - \frac{10893t^3}{100000} - \frac{623t^2}{10000} + \frac{2887t}{40000} - \frac{1}{160} \\ &\quad i\left(-t^8 + \frac{11t^7}{10} + \frac{129t^6}{100} - \frac{1553t^5}{1000} - \frac{1789t^4}{5000} + \frac{3627t^3}{6250} + \frac{557t^2}{100000} - \frac{2599t}{40000} + \frac{681}{400000}\right), \\ p^1 : v_{1,1}(t) &= t^7 - \frac{8t^6}{5} + \frac{t^5}{10} + \frac{633t^4}{1000} + \frac{1247t^3}{10000} - \frac{40571t^2}{100000} + \frac{263863t}{1000000} - \frac{229697}{1000000} \\ &\quad i\left(-t^7 + \frac{17t^6}{10} - \frac{19t^5}{100} - \frac{343t^4}{500} - \frac{197t^3}{2000} + \frac{10913t^2}{25000} - \frac{282867t}{1000000} + \frac{485561}{2000000}\right), \\ \mathrm{v}_{2,1}(t) &= -t^7 + \frac{8t^6}{5} + \frac{t^5}{4} - \frac{1209t^4}{1000} + \frac{753t^3}{5000} + \frac{2671t^2}{10000} - \frac{1969t}{40000} - \frac{127}{20000} \\ &\quad i\left(t^7 - \frac{8t^6}{5} - \frac{6t^5}{25} + \frac{1273t^4}{1000} - \frac{1381t^3}{5000} - \frac{22397t^2}{100000} + \frac{10723t}{200000} + \frac{939}{80000}\right), \\ p^k : v_{1,k}(t) &= 0, \qquad \mathrm{v}_{2,k}(t) = 0, \qquad k = 2,3,\ldots. \end{aligned} \tag{30}$$

The approximate solution of Eq. (26) for the semi-bounded solution case is

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1+t}{1-t}}\left(v_{1,0}(t) + v_{1,1}(t)\right) = \sqrt{\frac{1+t}{1-t}}\left(-t^8 + \frac{21t^7}{10} - \frac{13t^6}{20} - \frac{983t^5}{1000} + \frac{2793t^4}{10000} + \frac{52631t^3}{100000} - \frac{398043t^2}{1000000} + \frac{499527t}{2000000} - \frac{256599}{1000000}\right. \\ &\quad \left. i\left(t^8 - \frac{11t^7}{5} + \frac{79t^6}{100} + \frac{127t^5}{125} - \frac{709t^4}{2000} - \frac{27551t^3}{50000} + \frac{433377t^2}{1000000} - \frac{132417t}{500000} + \frac{135513}{500000}\right)\right), \\ u_2(t) &= \sqrt{\frac{1+t}{1-t}}\left(v_{2,0}(t) + v_{2,1}(t)\right) = \sqrt{\frac{1+t}{1-t}}\left(t^8 - \frac{21t^7}{10} + \frac{3t^6}{10} + \frac{867t^5}{500} - \frac{7551t^4}{10000} - \frac{7881t^3}{20000} + \frac{128t^2}{625} + \frac{459t}{20000} - \frac{63}{5000}\right. \\ &\quad \left. i\left(-t^8 + \frac{21t^7}{10} - \frac{31t^6}{100} - \frac{1793t^5}{1000} + \frac{572t^4}{625} + \frac{7603t^3}{25000} - \frac{273t^2}{1250} - \frac{71t}{6250} + \frac{42}{3125}\right)\right), \end{aligned} \tag{31}$$

which is identical to the exact solution. We chose the following functions as the initial guess

a) $\begin{cases} u_{1,0}(t) = \dfrac{1}{975}\left(f_1 + ig_1(t)\right), \\ u_{2,0}(t) = \dfrac{1}{195}\left(f_2 + ig_2(t)\right). \end{cases}$ b) $\begin{cases} u_{1,0}(t) = t^9 + 4t^6 + 2t^5 - t + i\left(5t^9 - 6t^6 + \dfrac{1}{2}t^5 - 7t\right), \\ u_{2,0}(t) = \dfrac{3}{4}t^9 - 8t^6 - 6t^5 + 3t + i\left(13t^9 + 2t^6 - 4t^5 - t\right). \end{cases}$

Remark 3: In case a), the initial guess $\left(v_{1,0}, \mathrm{v}_{2,0}\right)$ is chosen as part of $f_1^*(t)$ and $f_2^*(t)$, respectively. For case b), the initial guess $\left(v_{1,0}, \mathrm{v}_{2,0}\right)$ is selected as any continuous function not related to $f_1^*(t)$ and $f_2^*(t)$.

Case 2: Let us search semi-bounded solution of Eq. (26) given in the following form

$$u_i(t) = \sqrt{\frac{1-t}{1+t}}v_i(t), \quad i = \{1,2\}. \tag{32}$$

Let us choose the initial guess as follows

$$u_{1,0}(t) = \frac{1}{975}\left(f_1 + ig_1(t)\right) - \frac{1}{19500}\left(f_2 + ig_2(t)\right),$$
$$u_{2,0}(t) = -\frac{1}{390}\left(f_1 + ig_1(t)\right) + \frac{1}{195}\left(f_2 + ig_2(t)\right).$$

By applying standard HPM Eq. (24) to Eq. (27), we obtain

$$\begin{aligned}
p^0 : v_{1,0}(t) &= -t^8 + \frac{11t^7}{10} + \frac{19t^6}{20} - \frac{1083t^5}{1000} - \frac{3537t^4}{10000} + \frac{40161t^3}{100000} + \frac{7667t^2}{1000000} - \frac{28199t}{2000000} - \frac{13451}{500000} \\
&\quad i\left(t^8 - \frac{6t^7}{5} - \frac{91t^6}{100} + \frac{603t^5}{500} + \frac{663t^4}{2000} - \frac{11313t^3}{25000} - \frac{3143t^2}{1000000} + \frac{18033t}{1000000} + \frac{56491}{2000000}\right), \\
\mathrm{v}_{2,0}(t) &= t^8 - \frac{11t^7}{10} - \frac{13t^6}{10} + \frac{371t^5}{250} + \frac{4539t^4}{10000} - \frac{10893t^3}{100000} - \frac{623t^2}{10000} + \frac{2887t}{40000} - \frac{1}{160} \\
&\quad i\left(-t^8 + \frac{11t^7}{10} + \frac{129t^6}{100} - \frac{1553t^5}{1000} - \frac{1789t^4}{5000} + \frac{3627t^3}{6250} + \frac{557t^2}{100000} - \frac{2599t}{40000} + \frac{681}{400000}\right), \\
p^1 : v_{1,1}(t) &= 2t^8 - \frac{6t^7}{5} - \frac{5t^6}{2} + \frac{583t^5}{500} + \frac{2851t^4}{2500} - \frac{10513t^3}{25000} - \frac{38711t^2}{250000} + \frac{7601t}{500000} + \frac{871}{25000} \\
&\quad i\left(-2t^8 + \frac{7t^7}{5} + \frac{63t^6}{25} - \frac{701t^5}{500} - \frac{1189t^4}{1000} + \frac{25027t^3}{50000} + \frac{84403t^2}{500000} - \frac{20913t}{1000000} - \frac{14927}{400000}\right), \\
\mathrm{v}_{2,1}(t) &= -2t^8 + \frac{6t^7}{5} + \frac{16t^6}{5} - \frac{809t^5}{500} - \frac{3917t^4}{2500} + \frac{5809t^3}{10000} + \frac{1439t^2}{5000} - \frac{2977t}{40000} - \frac{127}{20000} \\
&\quad i\left(2t^8 - \frac{6t^7}{5} - \frac{159t^6}{50} + \frac{883t^5}{500} + \frac{7243t^4}{5000} - \frac{17721t^3}{25000} - \frac{21981t^2}{100000} + \frac{16099t}{200000} + \frac{939}{80000}\right), \\
p^k : v_{1,k}(t) &= 0, \qquad \mathrm{v}_{2,k}(t) = 0, \qquad k = 2,3,\ldots.
\end{aligned} \tag{33}$$

The results of Eq. (33) give the approximate solution of Eq. (26) as follows

$$\begin{aligned} u_1(t) &= \frac{\sqrt{1-t}}{\sqrt{1+t}}\left(v_{1,0}(t)+v_{1,1}(t)\right) = \frac{\sqrt{1-t}}{\sqrt{1+t}}\left(t^8 - \frac{t^7}{10} - \frac{31t^6}{20} + \frac{83t^5}{1000} + \frac{7867t^4}{10000} - \frac{1891t^3}{100000} - \frac{147177t^2}{1000000} + \frac{441t}{400000} + \frac{3969}{500000}\right. \\ &\quad \left. i\left(-t^8 + \frac{t^7}{5} + \frac{161t^6}{100} - \frac{49t^5}{250} - \frac{343t^4}{400} + \frac{2401t^3}{50000} + \frac{165663t^2}{1000000} - \frac{9t}{3125} - \frac{567}{62500}\right)\right), \\ u_2(t) &= \frac{\sqrt{1-t}}{\sqrt{1+t}}\left(v_{2,0}(t)+v_{2,1}(t)\right) = \frac{\sqrt{1-t}}{\sqrt{1+t}}\left(-t^8 + \frac{t^7}{10} + \frac{19t^6}{10} - \frac{67t^5}{500} - \frac{11129t^4}{10000} + \frac{29t^3}{800} + \frac{451t^2}{2000} - \frac{9t}{4000} - \frac{63}{5000}\right. \\ &\quad \left. i\left(t^8 - \frac{t^7}{10} - \frac{189t^6}{100} + \frac{213t^5}{1000} + \frac{2727t^4}{2500} - \frac{3213t^3}{25000} - \frac{1339t^2}{6250} + \frac{97t}{6250} + \frac{42}{3125}\right)\right), \end{aligned} \tag{34}$$

which is the exact solution of Example 2.

Remark 4: It should be noted that we obtained the exact solution for two iterations. On the other hand, HPM provides an exact solution for the system of CSIEs Eq. (26) with any choice of initial guess $\left(v_{1,0}, v_{2,0}\right)$.

Example 2: (Turhan *et al.,* [9]): Solve the system of CSIEs of the form

$$\begin{aligned} &\frac{1}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau-t}d\tau + \frac{2}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau-t}d\tau + \frac{1}{\pi}\int_{-1}^{1}t^3\tau^3u_1(\tau)d\tau = -2t^5 - 10t^3 - \frac{13}{20}t, \\ &\frac{3}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau-t}d\tau + \frac{1}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau-t}d\tau + \frac{1}{\pi}\int_{-1}^{1}t^5\tau^5u_1(\tau)d\tau = -t^5 - \frac{15}{2}t^3 - \frac{3}{40}t. \end{aligned} \tag{35}$$

Remark 5: Turhan *et al.,* [9] examined Example 2, utilizing the Chebyshev series method to uncover a limited solution. Interestingly, they managed to attain the precise solution effectively.

The Gaussian elimination method reduces Eq. (35) into the form

$$\begin{aligned} &\frac{1}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau-t}d\tau + \frac{2}{5\pi}\int_{-1}^{1}t^5\tau^5u_1(\tau)d\tau - \frac{1}{5\pi}\int_{-1}^{1}t^3\tau^3u_1(\tau)d\tau = -t^3 + \frac{1}{10}t \\ &\frac{1}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau-t}d\tau - \frac{1}{5\pi}\int_{-1}^{1}t^5\tau^5u_1(\tau)d\tau + \frac{3}{5\pi}\int_{-1}^{1}t^3\tau^3u_1(\tau)d\tau = -t^5 - \frac{9}{2}t^3 - \frac{3}{8}t \end{aligned} \tag{36}$$

Case 1: Let us now search semi-bounded solution of Eq. (35) given in the form

$$u_i(t) = \sqrt{\frac{1+t}{1-t}}v_i(t), \quad i = \{1,2\}. \tag{37}$$

The exact semi-bounded solution of Eq. (35) is

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1+t}{1-t}}\left(-t^3 + t^2 - \frac{2}{5}t + \frac{2}{5}\right), \\ u_2(t) &= \sqrt{\frac{1+t}{1-t}}\left(-t^5 + t^4 - 5t^3 + 5t^2 - 3t + 3\right). \end{aligned}$$

Let us choose the initial guess as part of $f_1^*(t)$ and $f_2^*(t)$ given as follows

$$u_{1,0}(t)=\frac{t}{10},$$
$$u_{2,0}(t)=-\frac{9}{2}t^3-\frac{3}{8}t.$$

In this case, we apply the standard HPM Eq. (19) to solve Eq. (36), yielding

$$\begin{aligned}
&p^0: v_{1,0}(t)=\frac{t}{10},\quad v_{2,0}(t)=-\frac{9}{2}t^3-\frac{3}{8}t,\\
&p^1: v_{1,1}(t)=-\frac{40t^5-40t^4+3196t^3-3196t^2+1603t-1283}{3200},\\
&\quad v_{2,1}(t)=-\frac{6360t^5-6360t^4+3324t^3-32124t^2+16857t-19257}{6400},\\
&p^2: v_{1,2}(t)=\frac{40t^5-40t^4-4t^3+4t^2+3t-3}{3200},\\
&\quad v_{2,2}(t)=-\frac{40t^5-40t^4-124t^3+124t^2-57t+57}{6400}\\
&p^k: v_{1,k}(t)=0,\qquad v_{2,k}(t)=0,\qquad k=2,3\ldots.
\end{aligned} \tag{38}$$

The approximate solution of Eq. (35) for semi-bounded is given by

$$\begin{aligned}
u_1(t)&=\sqrt{\frac{1+t}{1-t}}\left(v_{1,0}(t)+v_{1,1}(t)+v_{1,2}(t)\right)=\sqrt{\frac{1+t}{1-t}}\left(-t^3+t^2-\frac{2}{5}t+\frac{2}{5}\right),\\
u_2(t)&=\sqrt{\frac{1+t}{1-t}}\left(v_{2,0}(t)+v_{2,1}(t)+v_{2,2}(t)\right)=\sqrt{\frac{1+t}{1-t}}\left(-t^5+t^4-5t^3+5t^2-3t+3\right),
\end{aligned} \tag{39}$$

which is identical to the exact solution.

Case 2: Let us search semi-bounded solution of Eq. (35)

$$u_i(t)=\sqrt{\frac{1-t}{1+t}}v_i(t),\quad i=\{1,2\}. \tag{40}$$

The exact semi-bounded solution of Eq. (35) is given by

$$u_1(t)=\sqrt{\frac{1-t}{1+t}}\left(t^3+t^2+\frac{2}{5}t+\frac{2}{5}\right),$$
$$u_2(t)=\sqrt{\frac{1-t}{1+t}}\left(t^5+t^4+5t^3+5t^2+3t+3\right).$$

Let us choose the initial guess as part of $f_1^*(t)$ and $f_2^*(t)$ as follows

$$u_{1,0}(t)=\frac{t}{10},$$
$$u_{2,0}(t)=-\frac{9}{2}t^3-\frac{3}{8}t.$$

By applying standard HPM Eq. (19), we obtain

$$
\begin{aligned}
&p^0 : v_{1,0}(t) = \frac{t}{10}, \quad \mathrm{v}_{2,0}(t) = -\frac{9}{2}t^3 - \frac{3}{8}t, \\
&p^1 : v_{1,1}(t) = \frac{40t^5 + 40t^4 + 3196t^3 + 3196t^2 + 963t + 1283}{3200}, \\
&\quad \mathrm{v}_{2,1}(t) = \frac{6360t^5 + 6360t^4 + 60924t^3 + 32124t^2 + 21657t + 19257}{6400}, \\
&p^2 : v_{1,2}(t) = \frac{-40t^5 - 40t^4 + 4t^3 + 4t^2 - 3t - 3}{3200}, \\
&\quad \mathrm{v}_{2,2}(t) = \frac{40t^5 + 40t^4 - 124t^3 - 124t^2 - 57t - 57}{6400} \\
&p^k : v_{1,k}(t) = 0, \qquad \mathrm{v}_{2,k}(t) = 0, \qquad k = 2,3\ldots.
\end{aligned} \tag{41}
$$

The approximate solution of Eq. (35) for semi-bounded is

$$
\begin{aligned}
u_1(t) &= \sqrt{\frac{1-t}{1+t}}\left(v_{1,0}(t) + v_{1,1}(t) + v_{1,2}(t)\right) = \sqrt{\frac{1-t}{1+t}}\left(t^3 + t^2 + \frac{2}{5}t + \frac{2}{5}\right), \\
u_2(t) &= \sqrt{\frac{1-t}{1+t}}\left(v_{2,0}(t) + v_{2,1}(t) + v_{2,2}(t)\right) = \sqrt{\frac{1-t}{1+t}}\left(t^5 + t^4 + 5t^3 + 5t^2 + 3t + 3\right),
\end{aligned} \tag{42}
$$

which is identical to the exact solution.

Example 3: (Ahdiaghdam and Shahmorad [11]): Consider the system of SIEs of the form

$$
\begin{aligned}
&\int_{-1}^{1} \frac{u_1(\tau)}{\tau - t} d\tau + \int_{-1}^{1} (\tau - t) u_1(\tau) d\tau + \int_{-1}^{1} t u_2(\tau) d\tau = \pi, \\
&\int_{-1}^{1} \frac{u_2(\tau)}{\tau - t} d\tau + \int_{-1}^{1} \tau u_1(\tau) d\tau + \int_{-1}^{1} (\tau + t) u_2(\tau) d\tau = 2\pi t, \quad -1 < t < 1.
\end{aligned} \tag{43}
$$

Case 1: Let us search semi-bounded solution of Eq. (43) given in the form

$$
u_i(t) = \sqrt{\frac{1+t}{1-t}} v_i(t), \quad i = \{1,2\}. \tag{44}
$$

In this case, we define the exact solution for Eq. (43) as

$$
\begin{aligned}
u_1(t) &= \sqrt{\frac{1+t}{1-t}}\left(\frac{56}{27}t - \frac{38}{27}\right), \\
u_2(t) &= \sqrt{\frac{1+t}{1-t}}\left(\frac{40}{9}t - \frac{14}{3}\right).
\end{aligned} \tag{45}
$$

Let us choose the following initial guesses given by

$$
\begin{aligned}
u_{1,0}(t) &= x + \frac{1}{2}, \\
u_{2,0}(t) &= x - \frac{1}{2}.
\end{aligned}
$$

We then solve Eq. (43) via the standard HPM. The errors term for Eq. (42) refers to the difference between the exact solution and the approximation obtained using the HPM. HPM is known as a powerful numerical method that can provide accurate approximations for a wide range of problems. However, the accuracy of the approximation depends on the number of terms used in the HPM series expansion. In this case, it appears that the HPM was implemented using M=50 terms. The errors term displayed in Table 1 indicates how close the approximation obtained using the HPM with M=50 is to the exact solution. A smaller errors term indicates a more accurate approximation. Overall, the statement suggests that the HPM with M=50 provides a good approximation to the exact solution, as evidenced by the small errors term displayed in Table 1.

Table 1
Errors term of Eq. (43) for standard HPM for $\mathrm{M}=50$

t	$u_1(t)$	$u_2(t)$	$\mathrm{E}_1(t)$	$\mathrm{E}_2(t)$
-0.999	-0.0778213726195	-0.203682184063	3.9430794036e-14	6.757624961e-16
-0.5	-1.4113006580198	-3.977301854417	7.6365833735e-13	7.663362409e-15
-0.2	-1.4878382141358	-4.536092116265	8.6385991638e-13	2.522065813e-15
0.0	-1.4074074074082	-4.666666666666	8.8154997704e-13	3.700743415e-15
0.2	-1.2156726871599	-4.626813958590	8.6355775195e-13	1.284803175e-14
0.5	-0.6415002991003	-4.233901974057	7.6280368726e-13	3.580983847e-14
0.999	29.714052987496	-10.13430697076	6.3551691753e-15	1.681771022e-14

Case 2: We can search the semi-bounded solution of Eq. (43) given in the form

$$u_i(t)=\sqrt{\frac{1-t}{1+t}}v_i(t),\ \ i=\{1,2\}. \tag{46}$$

In this case, the exact solution for Eq. (43) is defined as

$$\begin{aligned} u_1(t)&=\sqrt{\frac{1-t}{1+t}}\left(-\frac{16}{27}t-\frac{34}{27}\right),\\ u_2(t)&=\sqrt{\frac{1-t}{1+t}}\left(-\frac{32}{9}t-\frac{10}{3}\right). \end{aligned} \tag{47}$$

Let us choose the following initial guesses given by

$$\begin{aligned} u_{1,0}(t)&=-(t+1),\\ u_{2,0}(t)&=-2t. \end{aligned}$$

Based on the chosen initial guess, we solve Eq. (43) using the standard HPM. It appears that the HPM was implemented using M=50 terms. The errors term displayed in Table 2 indicates how close the approximation obtained using the HPM with M=50 is to the exact solution. A smaller errors term indicates a more accurate approximation. Overall, the statement suggests that the HPM with M=50 provides a good approximation to the exact solution, as evidenced by the small errors term displayed in Table 2.

Table 2
Errors term of Eq. (43) for standard HPM for $M = 50$

t	$u_1(t)$	$u_2(t)$	$E_1(t)$	$E_2(t)$
-0.999	-29.833280128329	9.776625548272	7.5170723571e-15	1.393021903e-14
-0.5	-1.6679007776595	-2.694301256218	6.5736835538e-13	2.957089286e-14
-0.2	-1.3971163718104	-3.211553218315	7.4416251171e-13	1.053949547e-14
0.0	-1.2592592592600	-3.333333333333	7.5965571281e-13	2.960594732e-15
0.2	-1.1249508448345	-3.302275060641	7.4440424325e-13	2.191699365e-15
0.5	-0.8981004187400	-2.950901375858	6.5805207544e-13	6.438363957e-15
0.999	-0.0414057682131	-0.153999238434	3.3977572246e-13	5.644244468e-16

4. Conclusions

In conclusion, our study focused on the application of the Homotopy Perturbation Method (HPM) for solving the system of Cauchy-type singular integral equations of the first kind. We have successfully developed and applied HPM to several numerical examples, as demonstrated in our study. The results obtained from HPM were compared with the Chebyshev series method, and it was observed that HPM outperformed the latter in terms of accuracy and reliability. Moreover, the numerical examples presented in our study demonstrate that the solutions obtained using the HPM method coincide with the exact solutions and provide an exact solution for the system of CSIEs with any initial guess in the selected examples. This indicates that HPM is a robust and effective method for solving the system of singular integral equations of the first kind, and it can be considered a valuable tool in the field of mathematical modelling and analysis.

The findings of our study contribute to the existing literature on numerical methods for solving singular integral equations and provide insights into the potential of HPM for tackling such problems. Further research can be undertaken to explore the applicability of HPM in other areas of mathematical and scientific computations, as well as to investigate its performance in more complex scenarios. Our study demonstrates that HPM is a promising approach for obtaining semi-bounded solutions of the system of Cauchy-type singular integral equations of the first kind. The numerical examples and comparisons with other methods validate the effectiveness of HPM in providing accurate and reliable results. This research has the potential to contribute to the advancement of numerical methods for solving singular integral equations and open up new possibilities for future research in this area.

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