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Important Theories and Applications Arise in Physics and Engineering using a New Technique: The Natural Decomposition Method

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ABSTRACT

In the current research, we used the natural Adomian decomposition method (NADM), which is a unique technique, to find new exact solutions to partial and ordinary differential equations, such as the Sawada-Kotera equation, the extended fifth-order Korteweg-de Vries (efKdV) equation, and the Caudrey-Dodd-Gibbon equation. We also provided proofs for newly discovered theorems related to natural transformation. Our solutions to both linear and nonlinear differential equations are provided in series form, and the suggested method has high convergence. To handle differential equations, including nonlinear and linear differential equations, the proposed scheme in this work can be simply viewed as an alternative to those found in the literature.

1. Introduction

A number of physical phenomena, including shallow-water waves around the critical level of surface tension and waves in a nonlinear LC circuit with mutual inductance between surrounding inductors, are best understood using the generalized KdV equation [1-4]. A capacitor and an inductor connected together form an LC circuit, a type of electric circuit. A tuned circuit, tank circuit, or resonant circuit are other names for it. There is no universal solution for this class of models, even though the fifth-order KdV equation's accurate solution was discovered for the special case of solitary waves, as shown in Aski *et al.,* [5]. The extended fifth-order Korteweg-de Vries (efKdV) equation, which is essential in fluid dynamics for the description of nonlinear wave processes, is made up of numerous KdV-type equations, some of which include the Lax equation, Sawada-Kotera equation, Kaup-Kuperschmidt equation, Caudrey-Dodd-Gibbon equation, and Ito equation [6].

The Sawada-Kotera equation is a well-known mathematical model that arises in many physical systems to explain how long waves travel in shallow water while being influenced by gravity and moving through a one-dimensional nonlinear lattice [7,8]. Physics researchers, engineers, computer scientists, and mathematicians frequently employ mathematical models. The social sciences,

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engineering, and natural sciences all greatly benefit from the use of statistical models. For modelling physical phenomena like solid-state physics, plasma physics, fluid mechanics, population models, and chemical kinetics, differential equations are an effective tool. Because of this, finding precise or approximate solutions to various types of equations in physics and applied mathematics continues to be a major problem.

These equations can still be solved using a number of effective mathematical techniques, including the Adomian decomposition method, the natural decomposition method (NDM), the decomposition method, the Variational method and finite element approach, the reduced differential transform method (RDTM), the natural transform method (NTM), Homotopy Variational method, the Sumudu transform method (STM), the Laplace decomposition method (LDM), and the Homotopy perturbation method [1-5,9-23]. In the current technique, the Adomian decomposition and natural transformation are combined. Both of these techniques are relatively new for this century and have been applied in various ways for various goals. Natural transformations' characteristics and uses are described in Omran and Kiliçman [18] under the name N-transform. The team also found a solution for the unstable fluid flow problem over a plane wall and demonstrated how it relates to the Laplace and Sumudu transforms. Mathematical fundamentals include the initial shift, change of scale, transform of the first and second derivatives, and integrals of N- (natural).

In this work, the following ordinary and partial differential equations are considered:

First,

$$\phi_{\zeta}(\tau,\zeta)=\phi_{\tau\tau}(\tau,\zeta)+\phi(\tau,\zeta),\ \ \zeta>0. \tag{1}$$

With I.C:

$$\phi\left(\tau,0\right)=\mathrm{g}(\tau).$$

Second,

$$\phi_{\zeta}\left(\tau,\varsigma,\xi,\zeta\right)=\phi_{\tau\tau}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{\varsigma\varsigma}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{\xi\xi}\left(\tau,\varsigma,\xi,\zeta\right) \tag{2}$$

With I.C:

$$\phi\left(\tau,\varsigma,\xi,0\right)=f\left(\tau,\varsigma,\xi\right).$$

Third,

$$\phi_{\zeta\zeta}(\tau,\zeta)+\phi_{\tau\tau}^{2}(\tau,\zeta)+\phi(\tau,\zeta)-\phi^{2}(\tau,\zeta)=\zeta e^{-\tau}. \tag{3}$$

With I.C:

$$\phi(\tau,0)=h(\tau),\ \phi_{\zeta}(\tau,0)=g(\tau).$$

Fourth,

$$\frac{d^2\psi}{d\zeta^2}+6\psi(\zeta)=7\cosh(\zeta). \tag{4}$$

With the conditions:

$$\psi(0)=\psi_0,\ \ \psi'(0)=\psi_1.$$

The current work is presented as follows: The fundamental ideas of the natural Adomian decomposition transformation, together with their characteristics and definitions, are presented in Section 2. We offered proofs for several theorems pertaining to the natural transformation in Section 3. The analysis of the NADM for nonlinear PDEs is presented in detail in Section 4. Section 5 uses the novel method to tackle certain nonlinear PDEs and linear ODE applications. The sixth section uses the method to tackle certain linear PDE application. In Section 7, we provide the research work's final concluding remarks.

2. Background of Natural Adomian Decomposition Method

An overview of the definitions, research, and connections between the Sumudu and Laplace transformations is given in this section. The natural transformation method (NTM), an integral operator that transforms a function into another function (via integration), was first proposed by Belgacem *et al.,* [21]. For further information, see Omran and Kiliçman [18]. With the natural decomposition method (NDM), we do not have to linearize, discretize, or make any constrictive assumptions, in contrast to the differential transform technique (DTM) and Homotopy perturbation method (HPM). The NDM computational strategy then greatly reduces the size of the computing task and prevents round-off errors. We suggest readers familiarize themselves with the general integral transform's history, the Laplace, Sumudu, and natural transform methods, as well as the accompanying properties of each for every given function; for example, in Omran and Kiliçman [18].

Definition 1. Let $H(t)$ denote the Heaviside function, more precisely $H(t)=1$ for $t>0$ and $H(t)=0$ for $t<0$. We introduce the following class of real-valued function of real variable, on which the Natural Transform $\aleph^{+}$ is well-defined on the half plane $s>au$ for some $a>0$. Let $f(t)$ be a piecewise continuous function on $\mathbb{R}$. For some $M,a>0$, define

$$A=\left\{f(t):\left|f(t)\right|<Me^{a|t|}H(t)+Me^{-a|t|}H(-t)\right\}.$$

Note that for any $f(t)$ in the class A we have

$$\left|\int_{-\infty}^{\infty}e^{-st}f(tu)\,dt\right|\le M\int_{0}^{\infty}e^{-st}e^{a|tu|}dt+M\int_{-\infty}^{0}e^{-st}e^{-a|tu|}dt$$
$$=M\int_{0}^{\infty}e^{-(s-au)t}dt+M\int_{-\infty}^{0}e^{-(s+au)t}dt.$$

which is convergent provided that $s>au$.

If the real function $f(t)>0$ for $t\geq 0$ and $f(t)=0$ for $t<0$ is section-wise-continuous, exponentially order and define in A. Then, the Natural Transform is defined by [18]:

$$\aleph\left[f\left(t\right)\right]=L\left(r,w\right)=\int_{0}^{\infty}e^{-rt}f\left(wt\right)dt,\quad r>0,w>0. \tag{6}$$

where r and w are the transform variables. If $w=1$, then Eq. (6) converges to Laplace Transform, and when $r=1$, Eq. (6) converges to Sumudu transform [18,21].

Thus,

$$\aleph\left[f\left(t\right)\right]=L\left(r,w\right)=\frac{1}{w}\int_{0}^{\infty}e^{-\frac{rt}{w}}f\left(t\right)dt. \tag{7}$$

And

$$\aleph^{-1}\left[L\left(r,w\right)\right]=f\left(t\right)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\frac{rt}{u}}L\left(r,w\right)ds. \tag{8}$$

Thus, Eq. (7) is the natural transform and Eq. (8) is the inverse natural transform.

2.1 Properties of Interest

The following are a few of the fundamental characteristics of N-transforms [9]:

(i) $\aleph^{+}\left[K\right]=\frac{K}{r}.$

(ii) $\aleph^{+}\left[\zeta^{m}\right]=\frac{m!w^{m}}{r^{m+1}}$, where m ³ 0.

(iii) $\aleph^{+}\left[e^{c\zeta}\right]=\frac{1}{\left(r-cw\right)}.$

(iv) If $\psi^{(i)}(\zeta)$ is the i[th] derivative of the function $\psi(\zeta)$, then its N-transformation given by: $\aleph^{+}\left[\psi^{(i)}\left(\zeta\right)\right]=L_{i}\left(r,w\right)=\frac{r^{i}}{w^{i}}L\left(r,w\right)-\sum_{j=0}^{i-1}\frac{r^{i-(j-1)}}{w^{i-j}}\psi^{(j)}\left(0\right).$

(v) $\aleph^{+}\left[\frac{\psi(\zeta)}{\zeta^{n}}\right]=\frac{1}{w^{n}}\int_{s}^{\infty}...\int_{s}^{\infty}L(r,w)\left(ds\right)^{n}.$

2.2 Adomian Polynomials Evaluations

In this section, we provide methodical algebraic computations of Adomian polynomials. For many years, Taylor series based on Adomian polynomials were utilized to readily break down complex nonlinear expressions into smaller, more integrable terms. The following decomposition series can be used to represent the unknown linear function during the process:

$$w = \sum_{j=0}^{\infty} w_j \ . \tag{9}$$

Here the terms $w_j,\ j \geq 0$ can be evaluated as a recursive formula. For the nonlinear components, such as; $w^2,\ w^3,\ \cos(w),\ e^w,\ ww_x,\ w_x^2$ etc. $N(w)$ can easily be presented as the so-called Adomian polynomials C_n, which are:

$$N(w) = \sum_{j=0}^{\infty} C_j\left(w_0, w_1, ..., w_j\right), \tag{10}$$

and the C_j of the nonlinear part $N(w)$ can be easily evaluated [1-3]:

$$C_j = \frac{1}{k!}\frac{d^k}{d\sigma^k}\left[N\left(\sum_{i=0}^{j}\sigma^i w_i\right)\right]_{\sigma=0}, k = 0,1,2,... \ . \tag{11}$$

Now, the general formula (3.8) can easily be simplified as follows:

Let $N(w)$ be the nonlinear function. Then, by using Eq. (10), we obtain the following result from the definition of Adomian polynomials [1-3]:

$$\begin{aligned}
&C_0 = N(w_0).\\
&C_1 = u_1\, N'(w_0).\\
&C_2 = w_2\, N'(w_0) + \frac{1}{2!}w_1^2\, N''(w_0).\\
&C_3 = w_3\, N'(w_0) + w_1 w_2\, N''(w_0) + \frac{1}{3!}w_1^3\, N'''(w_0).\\
&C_4 = w_4\, N'(w_0) + \left(\frac{1}{2!}w_2^2 + w_1 w_3\right)N''(w_0) + \frac{1}{2!}w_1^2 w_2\, N'''(w_0) + \frac{1}{4!}w_1^4 N^{(4)}(w_0).
\end{aligned} \tag{12}$$

The rest of these components can be easily accomplished in the same way. The above polynomials in Eq. (12) give two important observations.

Here C_0 depends only on w_0, C_1 depends only on w_0 and w_1, C_2 depends only on w_0, w_1 and w_2, etc.

Secondly, by substituting Eq. (10) into Eq. (11), we get:

$$\begin{aligned}N(w) &= C_0 + C_1 + C_2 + C_3 + \dots \\ &= N(w_0) + (w_1 + w_2 + w_3 + \dots) N'(w_0) \\ &+ \frac{1}{2!}\left(w_1^2 + 2w_1w_2 + 2w_1w_3 + w_2^2 + \dots\right) N''(w_0) \\ &+ \frac{1}{3!}\left(w_1^3 + 3w_1^2w_2 + 3w_1^2w_3 + 6w_1w_2w_3 + \dots\right) N''(w_0) + \dots \\ &= N(w_0) + (w - w_0) N'(w_0) + \frac{1}{2!}(w - w_0)^2 N''(w_0) + \dots .\end{aligned}$$

Remark: This may be seen as a result of the Taylor series being provided to a function rather than a conventional point in the most recent extension. Thus, it is clearly shown by the Adomian polynomials given in Eq. (11) that for every term of the sum of the subscripts of the components to compute Adomian polynomials for the nonlinear terms, a number of techniques are available. It is important to note that there are several algorithms available for computing Adomian polynomials in nonlinear terms.

3. Derivation of Natural Transform Theories

This section will examine some recent, thorough justifications for key theorems pertaining to natural transformation. Under appropriate initial conditions, we will also use these to find exact solutions to a few ODEs and PDEs.

Theorem 1 Let $\psi(\zeta) = \frac{1}{2\alpha^3}\left(\sinh(\alpha\zeta) - \sin(\alpha\zeta)\right)$. Then its N-transformation is:

$$\aleph^+\left[\frac{1}{2\alpha^3}\left(\sinh(\alpha\zeta) - \sin(\alpha\zeta)\right)\right] = \frac{w^3}{\left(r^4 - w^4\alpha^4\right)}.$$

Proof. Applying the linear property of N-transformation, we have:

$$\aleph^+\left[\frac{1}{2\alpha^3}\left(\sinh(\alpha\zeta) - \sin(\alpha\zeta)\right)\right] = \frac{1}{2\alpha^3}\left(\aleph^+\left[\sinh(\alpha\zeta)\right] - \aleph^+\left[\sin(\alpha\zeta)\right]\right).$$

First, we show that:

$$\aleph^+\left[\sin(\alpha\zeta)\right] = \frac{w\alpha}{r^2 + w^2\alpha^2}.$$

and

$$\aleph^+\left[\sinh(\alpha\zeta)\right] = \frac{w\alpha}{r^2 - w^2\alpha^2}.$$

$$\aleph^{+}\left[\frac{1}{2\alpha^{3}}\left(\sinh(\alpha\zeta)-\sin(\alpha\zeta)\right)\right]=\frac{1}{2\alpha^{3}}\aleph^{+}\left[\sinh(\alpha\zeta)\right]-\aleph^{+}\left[\sin(\alpha\zeta)\right]$$

$$=\frac{1}{2\alpha^{3}}\left(\frac{w\alpha}{r^{2}-w^{2}\alpha^{2}}-\frac{w\alpha}{r^{2}+w^{2}\alpha^{2}}\right)$$

$$=\frac{w^{3}}{\left(r^{4}-w^{4}\alpha^{4}\right)}.$$

Theorem 2 Let $\psi\left(\zeta\right)=\frac{1}{2\alpha}\left(\sinh(\alpha\zeta)+\sin(\alpha\zeta)\right)$. Then its N-transformation is:

$$\aleph^{+}\left[\frac{1}{2\alpha}\left(\sinh(\alpha\zeta)+\sin(\alpha\zeta)\right)\right]=\frac{wr^{2}}{\left(r^{4}-w^{4}\alpha^{4}\right)}.$$

Proof. Employ the linear property of the N –transformation, we have:

$$\aleph^{+}\left[\frac{1}{2\alpha}\left(\sinh(\alpha\zeta)+\sin(\alpha\zeta)\right)\right]=\frac{1}{2\alpha}\left(\aleph^{+}\left[\sinh(\alpha\zeta)\right]+\aleph^{+}\left[\sin(\alpha\zeta)\right]\right)$$

$$=\frac{1}{2\alpha}\left(\frac{w\alpha}{r^{2}-w^{2}\alpha^{2}}+\frac{w\alpha}{r^{2}+w^{2}\alpha^{2}}\right)$$

$$=\frac{2r^{2}w}{\left(r^{4}-w^{4}\alpha^{4}\right)}.$$

Theorem 3 Let $\psi\left(\zeta\right)=\frac{1}{2}\left(\cosh(\alpha\zeta)+\cos(\alpha\zeta)\right)$. Then its N-transformation is:

$$\aleph^{+}\left[\frac{1}{2\alpha^{2}}\left(\cosh(\alpha\zeta)+\cos(\alpha\zeta)\right)\right]=\frac{r^{3}}{\left(r^{4}-w^{4}\alpha^{4}\right)}.$$

Proof. Applying the properties of natural transformation, we arrive at:

$$\aleph^{+}\left[\frac{1}{2}\left(\cosh(\alpha\zeta)+\cos(\alpha\zeta)\right)\right]=\frac{1}{2}\left(\aleph^{+}\left[\cosh(\alpha\zeta)\right]+\aleph^{+}\left[\cos(\alpha\zeta)\right]\right)$$
$$=\frac{1}{2}\left(\frac{r}{r^{2}-w^{2}\alpha^{2}}+\frac{r}{r^{2}+w^{2}\alpha^{2}}\right)$$
$$=\frac{r^{3}}{\left(r^{4}-w^{4}\alpha^{4}\right)}.$$

Theorem 4 Let $\psi\left(\zeta\right)=\dfrac{\sin(\alpha\zeta)}{\zeta}$. Then its N-transformation is:

$$\aleph^{+}\left[\frac{\sin(\alpha\zeta)}{\zeta}\right]=\frac{1}{w}\tan^{-1}\left(\frac{r}{w\alpha}\right).$$

Proof. Applying property 5 with $n=1,$ we have:

$$\aleph^{+}\left[\frac{\sin(\alpha\zeta)}{\zeta}\right]=\frac{1}{w}\int_{s}^{\infty}\aleph^{+}\left[\sin(\alpha\zeta)\right]ds$$

$$=\frac{1}{w}\int_{s}^{\infty}\alpha\left(\frac{w}{r^{2}+w^{2}\alpha^{2}}\right)ds$$

$$=\frac{1}{w}\tan^{-1}\left(\frac{r}{w\alpha}\right).$$

4. Analysis of the NDM for Linear PDEs and ODEs

In this section, we shall discuss the methodology of the NADM for a general linear partial differential equation and then employ the method in some applications.

4.1 Methodology of NADM for LFPDE

Given the linear, nonhomogeneous partial differential equation of the form:

$$F\phi(\tau,\zeta)+L\phi(\tau,\zeta)=\sigma(\tau,\zeta). \tag{13}$$

With I.C:

$$\phi(\tau,0)=\alpha\left(\tau\right);\qquad \phi_{\zeta}(\tau,\zeta)=\beta\left(\tau\right). \tag{14}$$

Note that F represents the operator of the highest derivative, L represents the linear part, and the remainder of the differential operator and $\sigma(\tau,\zeta)$ is an outsource part.

Now employing the N-transformation to Eq. (13) one can arrive at:

$$\frac{r^2\Phi(\tau,r,w)}{w^2}+\frac{r\phi(\tau,0)}{w^2}+\frac{\phi'(\tau,0)}{w}+\aleph^{+}\left[L\phi(\tau,\zeta)\right]=\aleph^{+}\left[\sigma(\tau,\zeta)\right]. \tag{15}$$

Substitute Eq. (14) into Eq. (15) to arrive at:

$$\Phi(\tau,r,w)=\frac{\alpha(\tau)}{w}+\frac{w\beta(\tau)}{r^2}+\frac{w^2\aleph^{+}\left[\sigma(\tau,\zeta)+L\phi(\tau,\zeta)\right]}{r^2}. \tag{16}$$

Implementing the inverse N-transformation in Eq. (16), we can get:

$$\phi(\tau,\zeta)=\chi(\tau,\zeta)-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[L\phi(\tau,\zeta)\right]\right], \tag{17}$$

where the outsource term, $\chi(\tau,\zeta)$, describes both the initial conditions and the nonhomogeneous parts.

Suppose we have a solution of the form:

$$\phi(\tau,\zeta)=\sum_{j=0}^{\infty}\phi_j(\tau,\zeta). \tag{18}$$

If we substitute Eq. (18) into Eq. (17), one can arrive at:

$$\sum_{j=0}^{\infty}\phi_j(\tau,\zeta)=\chi(\tau,\zeta)-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[\sum_{j=0}^{\infty}\phi_j(\tau,\zeta)\right]\right]. \tag{19}$$

Comparing both sides of Eq. (19), we arrive at:

$$\phi_0(\tau,\zeta)=\chi(\tau,\zeta),$$
$$\phi_1(\tau,\zeta)=-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[\phi_0(\tau,\zeta)\right]\right],$$
$$\phi_2(\tau,\zeta)=-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[\phi_1(\tau,\zeta)\right]\right].$$

We proceed as usual to come up with a general formula as follows:

$$\phi_{j+1}(\tau,\zeta)=-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[\phi_j(\tau,\zeta)\right]\right]. \tag{20}$$

Therefore, one can evaluate the rest of the terms using Eq. (20) to arrive at:

$$\phi(\tau,\zeta)=\sum_{j=0}^{\infty}\phi_j(\tau,\zeta).$$

4.2 Numerical Results and Discussion

In this section, we shall apply the NADM to some linear PDEs and ODEs.

Example 1 Given the following first-order linear partial differential equation:

$$\phi_{\zeta}(\tau,\zeta)=\phi_{\tau\tau}(\tau,\zeta)+\phi(\tau,\zeta),\ \ \zeta>0. \tag{21}$$

With I.C:

$$\phi\left(\tau,0\right)=\cos\left(\pi\tau\right). \tag{22}$$

Solution. Employ N-transformation on both sides of Eq. (21) along with the I.C. to arrive at:

$$\begin{aligned}\frac{r\Phi\left(\tau,r,w\right)}{w}-\frac{\phi\left(\tau,0\right)}{w}&=\aleph^{+}\left[\phi_{\tau\tau}(\tau,\zeta)\right]+\aleph^{+}\left[\phi(\tau,\zeta)\right],\\ \frac{r\Phi\left(\tau,r,w\right)}{w}-\frac{\cos\left(\pi\tau\right)}{w}&=\aleph^{+}\left[\phi_{\tau\tau}(\tau,\zeta)+\phi(\tau,\zeta)\right].\end{aligned} \tag{23}$$

Substituting Eq. (22) into Eq. (23), we obtain:

$$\Phi\left(\tau,r,w\right)=\frac{\cos\left(\pi\tau\right)}{w}+\frac{w}{r}\aleph^{+}\left[\phi_{\tau\tau}(\tau,\zeta)+\phi(\tau,\zeta)\right] \tag{24}$$

Employ the inverse N-transformation to Eq. (24) to arrive at:

$$\phi\left(\tau,\zeta\right)=\frac{\cos\left(\pi\tau\right)}{w}+\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{\tau\tau}(\tau,\zeta)+\phi(\tau,\zeta)\right]\right]. \tag{25}$$

Suppose we have a solution given by:

$$\phi(\tau,\zeta)=\sum_{j=0}^{\infty}\phi_j(\tau,\zeta). \tag{26}$$

Substitute Eq. (26) into Eq. (25) to arrive at:

$$\sum_{j=0}^{\infty}\phi_j(\tau,\zeta)=\cos(\pi\tau)+\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{j\tau\tau}(\tau,\zeta)+\phi_j(\tau,\zeta)\right]\right],\qquad j\geq 0. \tag{27}$$

Evaluating term by term in Eq. (27), one can come up with:

$$\begin{aligned}
&\phi_0(\tau,\zeta)=\cos(\pi\tau),\\
&\phi_1(\tau,\zeta)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{0\tau\tau}(\tau,\zeta)+\phi_0(\tau,\zeta)\right]\right],\\
&\phi_2(\tau,\zeta)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{1\tau\tau}(\tau,\zeta)+\phi_1(\tau,\zeta)\right]\right].
\end{aligned} \tag{28}$$

Thus, our general formula is:

$$\phi_{j+1}(\tau,\zeta)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{j\tau\tau}(\tau,\zeta)+\phi_j(\tau,\zeta)\right]\right]. \tag{29}$$

From Eq. (29), one can evaluate the rest of the terms as follows:

$$\begin{aligned}
\phi_1(\tau,\zeta)&=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{0\tau\tau}(\tau,\zeta)+\phi_0(\tau,\zeta)\right]\right]\\
&=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\left(\cos(\pi\tau)\right)+\left(-\pi^2\cos(\pi\tau)\right)\right]\right]\\
&=\aleph^{-1}\left[\frac{w}{r}\left[\frac{\cos(\pi\tau)}{r}-\frac{\pi^2\cos(\pi\tau)}{r}\right]\right]\\
&=\aleph^{-1}\left[\frac{w\cos(\pi x)}{r^2}-\frac{w\pi^2\cos(\pi x)}{r^2}\right]\\
&=\zeta\cos(\pi\tau)-\zeta\pi^2\cos(\pi\tau)\\
&=(1-\pi^2)\zeta\cos(\pi\tau).
\end{aligned}$$

Now,

$$\phi_2(\tau,\zeta)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{1\tau\tau}(\tau,\zeta)+\phi_1(\tau,\zeta)\right]\right].$$

But,

$$\phi_{1\tau\tau}\left(\tau,\zeta\right)=-\zeta\pi^2\cos\left(\pi\tau\right)+\zeta\pi^4\cos\left(\pi\tau\right).$$

Thus,

$$\begin{aligned}\phi_2\left(\tau,\zeta\right)&=\aleph^{-1}\left[\frac{w}{r}\aleph^+\left[-\zeta\pi^2\cos\left(\pi\tau\right)+\zeta\pi^4\cos\left(\pi\tau\right)+\zeta\cos\left(\pi\tau\right)-\zeta\pi^2\cos\left(\pi\tau\right)\right]\right]\\&=\aleph^{-1}\left[\frac{w}{r}\aleph^+\left[\zeta\cos\left(\pi\tau\right)-2\zeta\pi^2\cos\left(\pi\tau\right)+\zeta\pi^4\cos\left(\pi\tau\right)\right]\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\phi_2\left(\tau,\zeta\right)&=\aleph^{-1}\left[\frac{w}{r}\left[\frac{w}{r^2}\cos\pi\tau-\frac{2w}{r^2}\pi^2\cos\pi\tau+\frac{w}{r^2}\pi^4\cos\pi\tau\right]\right]\\&=\frac{\zeta^2}{2!}\cos(\pi\tau)-\zeta^2\pi^2\cos(\pi\tau)+\frac{\zeta^2}{2!}\pi^4\cos(\pi\tau)\\&=\left(\frac{1}{2}-\pi^2+\frac{1}{2}\pi^4\right)\zeta^2\cos(\pi\tau).\end{aligned}$$

We proceed as usual to arrive at the intended solution:

$$\phi\left(\tau,\zeta\right)=\sum_{j=0}^{\infty}\phi_j\left(\tau,\zeta\right).$$

$$\begin{aligned}\sum_{j=0}^{\infty}\phi_j\left(\tau,\zeta\right)&=\phi_0\left(\tau,\zeta\right)+\phi_1\left(\tau,\zeta\right)+\phi_2\left(\tau,\zeta\right)+\ldots\\&=\cos\left(\pi\tau\right)+\left(\zeta\left(1-\pi^2\right)\cos\left(\pi\tau\right)\right)+\left(\zeta^2\left(\frac{1}{2}-\pi^2+\frac{1}{2}\pi^4\right)\cos\left(\pi\tau\right)\right)+\ldots\end{aligned}$$

Thus,

$$\sum_{j=0}^{\infty}\phi_j\left(\tau,\zeta\right)=\cos\left(\pi\tau\right)\sum_{k=0}^{\infty}\frac{\left(\zeta\left(1-\pi^2\right)\right)^k}{k!}.$$

Hence, the intended solution of Eq. (21) is:

$$\phi\left(\tau,\zeta\right)=\cos\left(\pi\tau\right)e^{\zeta\left(1-\pi^2\right)}.$$

The precise answer is in perfect agreement with the outcome produced by ADM [13].

Example 2 Given the following first-order linear partial differential equations of the form:

$$\phi_{\zeta}\left(\tau,\varsigma,\xi,\zeta\right)=\phi_{\tau\tau}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{\varsigma\varsigma}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{\xi\xi}\left(\tau,\varsigma,\xi,\zeta\right) \tag{30}$$

With I.C:

$$\phi\left(\tau,\varsigma,\xi,0\right)=e^{\tau+\varsigma+\xi}. \tag{31}$$

Solution. Employing the N-transformation in Eq. (30), we obtain:

$$\Phi\left(\tau,\varsigma,\xi,r,w\right)=\frac{e^{\tau+\varsigma+\xi}}{r}+\frac{w}{r}\aleph^{+}\left[\phi_{\tau\tau}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{\varsigma\varsigma}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{\xi\xi}\left(\tau,\varsigma,\xi,\zeta\right)\right]. \tag{32}$$

Employ the inverse N-transformation of Eq. (32) to arrive at:

$$\phi\left(\tau,\varsigma,\xi,\zeta\right)=\frac{e^{\tau+\varsigma+\xi}}{r}+\frac{w}{r}\aleph^{+}\left[\phi_{\tau\tau}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{\varsigma\varsigma}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{\xi\xi}\left(\tau,\varsigma,\xi,\zeta\right)\right]. \tag{33}$$

Suppose we have a solution as follows:

$$\phi\left(\tau,\varsigma,\xi,\zeta\right)=\sum_{j=0}^{\infty}\phi_{j}\left(\tau,\varsigma,\xi,\zeta\right). \tag{34}$$

Substitute Eq. (34) into Eq. (33) to arrive at:

$$\sum_{j=0}^{\infty}\phi\left(\tau,\varsigma,\xi,\zeta\right)=e^{\tau+\varsigma+\xi}+\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{j\tau\tau}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{j\varsigma\varsigma}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{j\xi\xi}\left(\tau,\varsigma,\xi,\zeta\right)\right]\right],\quad j\geq 0. \tag{35}$$

Comparing Eq. (35), one can arrive at a formula as follows:

$$\begin{aligned}
&\phi_{0}\left(\tau,\varsigma,\xi,\zeta\right)=e^{\tau+\varsigma+\xi},\\
&\phi_{1}\left(\tau,\varsigma,\xi,\zeta\right)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{0\tau\tau}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{0\varsigma\varsigma}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{0\xi\xi}\left(\tau,\varsigma,\xi,\zeta\right)\right]\right],\\
&\phi_{2}\left(\tau,\varsigma,\xi,\zeta\right)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{1\tau\tau}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{1\varsigma\varsigma}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{1\xi\xi}\left(\tau,\varsigma,\xi,\zeta\right)\right]\right].
\end{aligned}$$

Then finally, we can arrive at:

$$\phi_{j+1}\left(\tau,\varsigma,\xi,\zeta\right)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{j\tau\tau}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{j\varsigma\varsigma}\left(\tau,\varsigma,\xi,\zeta\right)+\phi_{j\xi\xi}\left(\tau,\varsigma,\xi,\zeta\right)\right]\right]. \tag{36}$$

From Eq. (36), one can evaluate the rest of the terms as follows:

$$\phi_1(\tau,\varsigma,\xi,\zeta)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{0\tau\tau}(\tau,\varsigma,\xi,\zeta)+\phi_{0\varsigma\varsigma}(\tau,\varsigma,\xi,\zeta)+\phi_{0\xi\xi}(\tau,\varsigma,\xi,\zeta)\right]\right].$$

But,

$$\phi_0(\tau,\varsigma,\xi,\zeta)=e^{\tau+\varsigma+\xi}.$$

Note that,

$$\phi_{0\tau\tau}(\tau,\varsigma,\xi,\zeta)=e^{\tau+\varsigma+\xi},\ \phi_{0\varsigma\varsigma}(\tau,\varsigma,\xi,\zeta)=e^{\tau+\varsigma+\xi},\ \phi_{0\xi}(\tau,\varsigma,\xi,\zeta)=e^{\tau+\varsigma+\xi}.$$

Thus,

$$\begin{aligned}\phi_1(\tau,\varsigma,\xi,\zeta)&=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[e^{\tau+\varsigma+\xi}+e^{\tau+\varsigma+\xi}+e^{\tau+\varsigma+\xi}\right]\right],\\&=\aleph^{-1}\left[\frac{3we^{\tau+\varsigma+\xi}}{r^2}\right],\\&=3\zeta e^{\tau+\varsigma+\xi}.\end{aligned}$$

And,

$$\phi_2(\tau,\varsigma,\xi,\zeta)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{1\tau\tau}(\tau,\varsigma,\xi,\zeta)+\phi_{1\varsigma\varsigma}(\tau,\varsigma,\xi,\zeta)+\phi_{1\xi\xi}(\tau,\varsigma,\xi,\zeta)\right]\right].$$

But,

$$\phi_{1\tau\tau}(\tau,\varsigma,\xi,\zeta)=3\zeta e^{\tau+\varsigma+\xi},\ \phi_{1\varsigma\varsigma}(\tau,\varsigma,\xi,\zeta)=3\zeta e^{\tau+\varsigma+\xi},\ \phi_{1\xi\xi}(\tau,\varsigma,\xi,\zeta)=3\zeta e^{\tau+\varsigma+\xi}.$$

So,

$$\begin{aligned}\phi_2(\tau,\varsigma,\xi,\zeta)&=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[9\zeta e^{\tau+\varsigma+\xi}\right]\right]\\&=\aleph^{-1}\left[\frac{e^{\tau+\varsigma+\xi}9w^2}{r^3}\right]\\&=\frac{9\zeta^2e^{\tau+\varsigma+\xi}}{2!}\end{aligned}$$

$$\phi_3(\tau,\varsigma,\xi,\zeta)=\aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[\phi_{2\tau\tau}(\tau,\varsigma,\xi,\zeta)+\phi_{2\varsigma\varsigma}(\tau,\varsigma,\xi,\zeta)+\phi_{2\xi\xi}(\tau,\varsigma,\xi,\zeta)\right]\right].$$

where,

$$\phi_{2\tau\tau} = \frac{9}{2}\zeta^2 e^{\tau+\varsigma+\xi},\ \phi_{2\varsigma\varsigma} = \frac{9}{2}\zeta^2 e^{\tau+\varsigma+\xi},\ \phi_{2\xi\xi} = \frac{9}{2}\zeta^2 e^{\tau+\varsigma+\xi}.$$

$$\begin{aligned}\phi_3(\tau,\varsigma,\xi,\zeta) &= \aleph^{-1}\left[\frac{w}{r}\aleph^{+}\left[e^{\tau+\varsigma+\xi}\,\frac{27\zeta^2}{2}\right]\right]\\ &= \aleph^{-1}\left[\frac{e^{\tau+\varsigma+\xi}w}{r}\frac{27}{2}\frac{2!w^2}{r^3}\right]\\ &= \frac{e^{\tau+\varsigma+\xi}27\zeta^3}{3!}.\end{aligned}$$

We proceed as usual to arrive at our intended solution as follows:

$$\phi(\tau,\varsigma,\xi,\zeta) = \sum_{j=0}^{\infty}\phi_j(\tau,\varsigma,\xi,\zeta).$$

Thus,

$$\begin{aligned}\sum_{j=0}^{\infty}\phi_j(\tau,\varsigma,\xi,\zeta) &= \phi_0(\tau,\varsigma,\xi,\zeta)+\phi_1(\tau,\varsigma,\xi,\zeta)+\phi_2(\tau,\varsigma,\xi,\zeta)+\dots\\ &= e^{(\tau+\varsigma+\xi)}\left(1+3\zeta+\frac{9\zeta^2}{2!}+\frac{27\zeta^3}{3!}+\dots\right).\end{aligned}$$

Then the solution of Eq. (30), in implicit from, is:

$$\phi(\tau,\varsigma,\xi,\zeta) = e^{(\tau+\varsigma+\xi+3\zeta)}.$$

The exact solution is in perfect agreement with the outcome produced by ADM [13].

Example 3 Given a second-order linear ordinary differential equation of the form:

$$\frac{d^2\psi}{d\zeta^2}+6\psi(\zeta) = 7\cosh(\zeta). \tag{37}$$

With these conditions:

$$\psi(0)=1,\ \ \psi'(0)=0. \tag{38}$$

Solution. Implement the N-transformation in Eq. (37) to arrive at:

$$\frac{r^2\Psi(r,w)}{w^2}-\frac{r\psi(0)}{w^2}-\frac{\psi'(0)}{w}+\aleph^+\left[6\psi(\zeta)\right]=\aleph^+\left[7\cosh(\zeta)\right],$$
$$\frac{r^2\Psi(r,w)}{w^2}-\frac{1}{w}+\aleph^+\left[6\psi(\zeta)\right]=\frac{7r}{r^2-w^2}. \tag{39}$$

Substitute Eq. (38) into Eq. (39) to conclude:

$$\Psi(r,w)=\frac{-6}{r}+\frac{7r}{r^2-w^2}+\frac{w^2}{r^2}\aleph^+\left[6\psi(\zeta)\right]. \tag{40}$$

Suppose we have a solution as follows:

$$\psi(\zeta)=\sum_{j=0}^{\infty}\psi_j(\zeta). \tag{41}$$

Employ the inverse N-transformation to Eq. (41) to arrive at:

$$\sum_{j=0}^{\infty}\psi_j(\zeta)=-6t+7\cosh(\zeta)+\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^+\left[6\psi(\zeta)\right]\right],\qquad j\geq 0. \tag{42}$$

Comparing both sides of Eq. (42), one can arrive at the formula:

$$\psi_0(\zeta)=-6+7\cosh(\zeta),$$
$$\psi_1(\zeta)=-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^+\left[6\psi_0(\zeta)\right]\right], \tag{43}$$
$$\psi_2(\zeta)=-\square^{-1}\left[\frac{w^2}{r^2}\aleph^+\left[6\psi_1(\zeta)\right]\right].$$

Then, the general formula is of the form:

$$\psi_{j+1}(\zeta)=-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^+\left[6\psi_j(\zeta)\right]\right]. \tag{44}$$

From Eq. (44), one can compute the rest of the terms as:

$$\begin{aligned}\psi_1(\zeta) &= -\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^+\left[6\psi_0(\zeta)\right]\right] \\ &= -\aleph^{-1}\left[\frac{6w^2}{r^2}\aleph^+\left[-6+7\cosh(\zeta)\right]\right] \\ &= -\aleph^{-1}\left[\frac{6w^2}{r^2}\left[\frac{-6}{r}+\frac{7r}{r^2-w^2}\right]\right] \\ &= \frac{-36\zeta^2}{2!}-42+42\cosh(\zeta).\end{aligned}$$

Now, cancelling the same but opposite signs of the terms in both $\psi_0(\zeta)$ and $\psi_1(\zeta)$ one sees that the remaining terms of $\psi_0(\zeta)$ will satisfy Eq. (37), which in this case will lead to an exact solution given as:

$$\psi(\zeta) = \cosh(\zeta) \ .$$

The exact solution is in perfect agreement with the outcome produced by ADM [14].

5. Analysis of the NDM for Nonlinear PDEs

In this section, we shall discuss the methodology of the NADM for a general nonlinear partial differential equation and then employ the method in some applications.

5.1 Methodology

Given the partial differential equation with the general nonlinear nonhomogeneous form:

$$F\phi(\tau,\zeta) + L\phi(\tau,\zeta) + \left[M\phi(\tau,\zeta)\right] = \sigma(\tau,\zeta). \tag{45}$$

With I.C:

$$\phi(\tau,0) = \alpha\left(\tau\right); \qquad \phi_\zeta(\tau,\zeta) = \beta\left(\tau\right). \tag{46}$$

Note that F represents the operator of the highest derivative, L represents the linear part, and the remaining of the operators, M represents the nonlinear term and $\sigma(\tau,\zeta)$ is an outsource part.

Employing the N-transformation in Eq. (45), one can arrive at:

$$\frac{r^2\Phi\left(\tau,r,w\right)}{w^2} + \frac{r\phi\left(\tau,0\right)}{w^2} + \frac{\phi'\left(\tau,0\right)}{w} + \aleph^+\left[M\phi(\tau,\zeta)\right] + \aleph^+\left[N\phi(\tau,\zeta)\right] = \aleph^+\left[\sigma(\tau,\zeta)\right]. \tag{47}$$

Substitute Eq. (46) into Eq. (47) to arrive at:

$$\Phi\left(\tau,r,w\right) = \frac{\alpha\left(\tau\right)}{w} + \frac{w\beta\left(\tau\right)}{r^2} + \frac{w^2\aleph^+\left[\sigma(\tau,\zeta)M\phi(\tau,\zeta)+N\phi(\tau,\zeta)\right]}{r^2}. \tag{48}$$

Implementing the inverse N-transformation to Eq. (48), we can get:

$$\phi_j(\tau,\zeta)=\chi(\tau,\zeta)-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[M\phi_j(\tau,\zeta)+N\phi(\tau,\zeta)\right]\right], \tag{49}$$

where the outsource term, $\chi(\tau,\zeta)$, describes both the initial conditions and the nonhomogeneous parts.

Suppose we have a solution of the form:

$$\phi(\tau,\zeta)=\sum_{j=0}^{\infty}\phi_j(\tau,\zeta). \tag{50}$$

Moreover, one can write the nonlinear as:

$$N\phi(\tau,\zeta)=\sum_{j=0}^{\infty}C_j, \tag{51}$$

where the C_j's are the polynomials of $\phi_0,\phi_1,\phi_2,\phi_3,\ldots,\phi_j$ and one can evaluated as:

$$C_j=\frac{1}{j!}\frac{d^j}{dx^j}\left[N\left(\sum_{i=0}^{j}\lambda^i v_i\right)\right],\ \ j=0,1,2,\ldots \tag{52}$$

From the formula above in Eq. (52), we can simplify it as:

$$\begin{aligned}
C_0&=N\left(\phi_0\right),\\
C_1&=v_1N'\left(\phi_0\right),\\
C_2&=v_2N'\left(\phi_0\right)+\frac{1}{2!}\phi_1^2N''\left(\phi_0\right).
\end{aligned}$$

If we substitute Eq. (50) and Eq. (52) into Eq. (51), one can arrive at:

$$\sum_{j=0}^{\infty}\phi_j(\tau,\zeta)=\chi(\tau,\zeta)-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[M\sum_{j=0}^{\infty}\phi_j(\tau,\zeta)+\sum_{j=0}^{\infty}C_j\right]\right]. \tag{53}$$

Comparing both sides of Eq. (53), we arrive at:

$$\begin{aligned}
\phi_0(\tau,\zeta)&=\chi(\tau,\zeta),\\
\phi_1(\tau,\zeta)&=-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[M\phi_0(\tau,\zeta)+C_0\right]\right],\\
\phi_2(\tau,\zeta)&=-\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[M\phi_1(\tau,\zeta)+C_1\right]\right].
\end{aligned}$$

We proceed as usual to come up with a general formula as follows:

$$\phi_{j+1}(\tau,\zeta) = -\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[M\phi_j(\tau,\zeta)+C_j\right]\right]. \tag{54}$$

Therefore, one can evaluate the rest of the terms using Eq. (30) to arrive at:

$$\phi(\tau,\zeta) = \sum_{j=0}^{\infty}\phi_j(\tau,\zeta).$$

Example 4 Given the second-order linear partial differential equations of the form:

$$\phi_{\zeta\zeta}(\tau,\zeta)+\phi_{\tau\tau}^2(\tau,\zeta)+\phi(\tau,\zeta)-\phi^2(\tau,\zeta) = \zeta e^{-\tau}. \tag{55}$$

With I.C:

$$\phi(\tau,0)=0,\ \phi_{\zeta}(\tau,0)=e^{-\tau}. \tag{56}$$

Solution. Employing the N-transformation of Eq. (55) to arrive at:

$$\frac{r^2\Phi(\tau,r,w)}{w^2}+\frac{r\phi(\tau,0)}{w^2}+\frac{\phi'(\tau,0)}{w}+\aleph^{+}\left[\phi_{\tau\tau}^2(\tau,\zeta)+\phi(\tau,\zeta)-\phi^2(\tau,\zeta)\right]=\aleph^{+}\left[\zeta e^{-\tau}\right].$$

Thus,

$$\frac{r^2\Phi(\tau,r,w)}{w^2}-\frac{e^{-\tau}}{w}+\aleph^{+}\left[\phi_{\tau\tau}^2(\tau,\zeta)+\phi(\tau,\zeta)-\phi^2(\tau,\zeta)\right]=\frac{e^{-\tau}w}{r^2}. \tag{57}$$

Substitute Eq. (56) into Eq. (57) to arrive at:

$$\Phi(\tau,r,w)=\frac{e^{-\tau}w}{r^2}+\frac{e^{-\tau}w^3}{r^4}+\frac{w^2}{r^2}\aleph^{+}\left[-\phi_{\tau\tau}^2(\tau,\zeta)-\phi(\tau,\zeta)+\phi^2(\tau,\zeta)\right]. \tag{58}$$

Implementing the inverse N-transformation of Eq. (58) to arrive at:

$$\phi(\tau,\zeta)=\zeta e^{-\tau}+\frac{e^{-\tau}\zeta^3}{3!}+\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[-\phi_{j\tau\tau}^2(\tau,\zeta)-\phi_j(\tau,\zeta)+\phi_j^2(\tau,\zeta)\right]\right]. \tag{59}$$

Suppose we have a solution as follows:

$$\phi(\tau,\zeta)=\sum_{j=0}^{\infty}\phi_j(\tau,\zeta). \tag{60}$$

From Eq. (59) and Eq. (60), one can rewrite,

$$\sum_{j=0}^{\infty}\phi_j\left(\tau,\zeta\right)=\frac{e^{-\tau}w}{r^2}+\frac{e^{-\tau}w^3}{r^4}+\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[\sum_{j=0}^{\infty}A_j-\phi_j\left(\tau,\zeta\right)+\sum_{j=0}^{\infty}B_j\right]\right]. \quad (61)$$

Where the Adomian polynomials represented by A_j and B_j are for ϕ_τ^2 and ϕ^2, respectively.

Using Eq. (61), one can come up with these formulas:

$$\phi_0\left(\tau,\zeta\right)=\zeta e^{-\tau}+\frac{e^{-\tau}\zeta^3}{3!},$$
$$\phi_1\left(\tau,\zeta\right)=\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[-\phi_{0\tau\tau}^2(\tau,\zeta)-\phi_0(\tau,\zeta)+\phi_0^2(\tau,\zeta)\right]\right],$$
$$\phi_2\left(\tau,\zeta\right)=\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[-\phi_{1\tau\tau}^2(\tau,\zeta)-\phi_1(\tau,\zeta)+\phi_1^2(\tau,\zeta)\right]\right],$$
$$\phi_3\left(\tau,\zeta\right)=\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[-\phi_{2\tau\tau}^2(\tau,\zeta)-\phi_2(\tau,\zeta)+\phi_2^2(\tau,\zeta)\right]\right].$$

We proceed as usual to arrive at:

$$\phi_{j+1}\left(\tau,\zeta\right)=\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[-\phi_{j\tau\tau}^2(\tau,\zeta)-\phi_j(\tau,\zeta)+\phi_j^2(\tau,\zeta)\right]\right],\ j\geq 0. \quad (62)$$

From Eq. (62), one can evaluate the rest of the terms as follows:

$$\phi_1\left(\tau,\zeta\right)=\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[-\phi_{0\tau\tau}^2(\tau,\zeta)-\phi_0(\tau,\zeta)+\phi_0^2(\tau,\zeta)\right]\right]$$
$$A_0=(\phi_0^2)$$
$$=e^{-2\tau}\zeta^2+\frac{e^{-2\tau}\zeta^4}{3}+\frac{e^{-2\tau}\zeta^6}{36}.$$

But,

$$B_0=(\phi_0^2)_\tau$$
$$=-2e^{-2\tau}\zeta^2-\frac{2e^{-2\tau}\zeta^4}{3}-\frac{2e^{-2\tau}\zeta^6}{36}.$$

$$\phi_1(\tau,\zeta)=\aleph^{-1}\left[\frac{w^2}{r^2}\aleph^{+}\left[3e^{-2\tau}\zeta^2+e^{-2\tau}\zeta^4+\frac{3e^{-2\tau}\zeta^6}{36}-\zeta e^{-\tau}-\frac{e^{-\tau}\zeta^3}{6}\right]\right]$$

$$=\aleph^{-1}\left[\frac{w^2}{r^2}\left[3e^{-2\tau}\frac{2!w^2}{r^3}+e^{-2\tau}\frac{4!w^4}{r^5}+\frac{3e^{-2\tau}}{36}\frac{6!w^6}{r^7}-e^{-\tau}\frac{w}{r^2}-\frac{e^{-\tau}}{6}\frac{3!w^3}{r^4}\right]\right]$$

$$=3e^{-2\tau}\frac{2!\zeta^4}{4!}+e^{-2\tau}\frac{4!\zeta^6}{6!}+\frac{3e^{-2\tau}}{36}\frac{6!\zeta^8}{8!}-e^{-\tau}\frac{\zeta^3}{3!}-\frac{e^{-\tau}}{6}\frac{3!\zeta^5}{5!}.$$

Finally, by eliminating similar but with opposite signs terms for $\phi_0(\tau,\zeta)$ and $\phi_1(x,\zeta)$, one can see that the remaining terms will satisfy Eq. (55) and our intended solution is:

$$\phi(\tau,\zeta)=\zeta e^{-\tau}.$$

The exact solution is in perfect agreement with the outcome produced by ADM [13].

6. Conclusion

The nonlinear dispersive equation, a higher-dimensional heat flow problem, has been successfully solved in this paper using a unique technique using the NADM. The results show that the NADM converges at a faster rate than other techniques that have been discussed in the literature. The NADM's relevance in the fields of engineering and applied science was shown. Additionally, the effectiveness and applicability of the suggested technique were demonstrated when we applied it to various situations. According to the study mentioned above, the NADM can be used to precisely solve additional non-linear ODEs and PDEs, such as systems of ODEs and PDEs, which are frequently encountered in science and engineering Therefore, a deeper knowledge of the real-world applications represented by these modelling issues will emerge from the NADM solutions to numerous situations.

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