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The Topological Indices of the *p*-Subgroup Graph of Dihedral Groups

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ABSTRACT

A topological index is a number generated from a molecular structure that signifies the molecule's fundamental structural characteristics. This correlation between the index and various physical attributes is based on an algebraic quantity related to the chemical structure. Many topological indices, such as the Wiener index, first and second Zagreb indices, can be employed to determine various properties, including chemical activity, thermodynamic properties, physicochemical activity, and biological activity. Meanwhile, the p-subgroup graph of a group G is defined as a graph whose vertices represent the elements of the group, and two vertices are adjacent if and only if the order of the subgroup is a prime power. The main objective of this paper is to establish the general formula for certain topological indices, specifically the Wiener index, first Zagreb index, and second Zagreb index for the p-subgroup graph associated with dihedral groups.

1. Introduction

Topological indices are numerical values derived for graphs associated with groups, serving as tools for modelling chemical characteristics and other aspects of molecules. Problems in biology and chemistry have been solved using various topological indices. Descriptors like the Wiener and Zagreb indices establish connections between molecular structures and potential physical properties. The nature and applications of topological indices have been extensively studied by Basak *et al.,* [1] in relation to chemistry. Additionally, several studies regarding graphs from groups have been widely reported. The topological index of a graph can be classified into three major categories based on degree, distance, and the eccentricity of a vertex in the graph. For instance, Sarmin *et al.,* [2] determined the general formula of the Wiener index, the first Zagreb index and the second Zagreb index for the non-commuting graph associated with generalised quaternion groups. Recently, Alimon *et al.,* [3] generalized the Wiener index and the Zagreb index of the conjugacy class graph of dihedral groups. Additionally, Bello *et al.,* [4] generalized the topological indices of the order product prime graph on dihedral groups.

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The Wiener index was established by Wiener in 1947 [5], is calculated as half the sum of the distances between every pair of vertices in a graph. Behtoei et al., [6] introduce new topological indices, which include the Wiener index, hyper-Wiener, Schultz and modified Schultz indices as special cases for trees. This method takes into account the distance between vertices when computing the Schultz and modified Schultz indices for trees. Furthermore, Yu et al., [7] extended the research on the Wiener index, providing some of the sufficient conditions for a nearly balanced bipartite graph with a given minimum degree to be traceable. Meanwhile, the Zagreb index has two types: the first Zagreb index and the second Zagreb index. These Zagreb indices were introduced by Gutman and Trinajsti´c [8] in 1972. The first Zagreb index is calculated as the sum of the squares of the degrees of each vertex in a graph, while the second Zagreb index is the sum of the products of the degrees of the two vertices connected by each edge. There are numerous types of topological indices, including the Szeged index, the Harary index, the Degree-distance index, the Kirchhoff index, and many more. Recent results related to the topological indices of graphs can also be found in previous studies [9-11].

In this paper, we introduce a new graph associated with groups, specifically the p-subgroup graph of dihedral groups. Additionally, this study explores the formulas for the Wiener index, the first Zagreb index, and the second Zagreb index for the p-subgroup graph of dihedral groups.

2. Preliminaries

This section introduces fundamental concepts utilized in this research. The group under consideration is finite, specifically the dihedral groups.

Definition 1 [12] Dihedral Groups

For $n \in \mathbb{Z}$ and $n \geq 3$, the dihedral group, D_{2n}, is the set of symmetries of a regular n-gon. Furthermore, the order of D_{2n} is $2n$ or equivalently | D_{2n}| = $2n$. The dihedral groups can also be represented in a form of generators and relations as:

$$D_{2n} = \langle a, b \mid a^n = b^2 = e, bab = a^{-1} \rangle$$

Definition 2 [13] The Wiener Index

Let Γ be a connected graph, and $V(\Gamma) = \{1,2,\ldots,n\}$ be a vertex set. The Wiener index of Γ, $W(\Gamma)$, is defined as half of the sum of the distances between every pair of vertices of Γ, written as

$$W(\Gamma) = \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}$$

Definition 3 [14] The First Zagreb Index

Let Γ be a connected graph, and $V(\Gamma) = \{1,2,\ldots,n\}$ be a vertex set. The first Zagreb index, denoted by $M_1(\Gamma)$, is defined as the sum of square of the degree of each vertex in Γ, written as

$$M_1(\Gamma) = \sum_{v \in v(\Gamma)}^{n} (\deg(v))^2$$

Definition 4 [14] The Second Zagreb Index

Let Γ be a connected graph, and $V(\Gamma) = \{1,2,\ldots,n\}$ be a vertex set and $E(\Gamma)$ be an edge of the graph. The second Zagreb index, denoted by $M_2(\Gamma)$, is defined as the sum of the product of the degree of two vertices for each edge, respectively, written as

$$M_2(\Gamma) = \sum_{v,u\in E(\Gamma)}^{n} \deg(v)\deg(u)$$

On the other hand, we consider simple undirected graphs without loop or multiple edges. The sets of vertices and edges of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. We denote the adjacency of vertices a, b by $a \sim b$, number of vertices of the graph Γ by $|V(\Gamma)|$. A graph Γ is is considered connected if there exists a path between every pair of its vertices. A graph is termed complete if there is an edge between every pair of its vertices. In a connected graph Γ, the distance from vertex u to vertex v is defined as the length, represented by the number of edges, of the shortest $u - v$ path in Γ. This distance is denoted as $d(u,v)$, and in contexts where clarity is crucial, it can be written as $d_G(u,v)$. Consider simple graphs, the disjoint union $G \cup H$ is defined as $V_{G\cup H} = V_G \cup V_H$ and $E_{G\cup H} = E_G \cup E_H$, the join $G + H$ is defined as $V_{G+H} = V_G \cup V_H$ and $E_{G+H} = E_G \cup E_H\{\{u,v\}|\, u \in V_G, v \in V_H\}$.

3. Results

To begin, the definition and general presentation of a p-subgroup graph are provided. Furthermore, the topological indices, namely the Wiener index, the first Zagreb index, and the second Zagreb index, are generalized for the p-subgroup graph of dihedral groups.

The definition of the p-subgroup graph of finite groups is given in Definition 5.

Definition 5 p-Subgroup Graph of a Group

Let G be a finite group, then the p-subgroup graph of G, denoted by $\Gamma_p(G)$, is a graph whose vertices are the elements of G and two vertices x and y are adjacent if and only if $\langle x,y\rangle$ is a p-group.

General presentations of graphs on dihedral groups are provided. These presentations are instrumental in classifying the graphs as connected, complete, regular, or planar. Meanwhile, an example demonstrating the p-subgroup graph on dihedral groups is presented as follows.

Example 1

Consider the dihedral group of order 12, $D_{12} = \{e, a, a^2, \ldots, a^5, b, ab, \ldots, a^5b\}$. Then $|\langle e,a^2\rangle| = |\langle e,a^4\rangle| = |\langle a^2,a^4\rangle| = 3$, $|\langle e,b\rangle| = |\langle e,ab\rangle| = |\langle e,a^2b\rangle| = |\langle e,a^3b\rangle| = |\langle e,a^4b\rangle| = |\langle e,a^5b\rangle| = |\langle e,a^3\rangle| = 2$, $|\langle a^3,b\rangle| = |\langle a^3,ab\rangle| = |\langle a^3,a^2b\rangle| = |\langle a^3,a^4b\rangle| = |\langle a^3,a^5b\rangle| = 3$, and $|\langle b,a^3b\rangle| = |\langle ab,a^4b\rangle| = |\langle a^2b,a^5b\rangle| = 4$ are the p-groups and the nontrivial cliques in D_{12}. The subgraphs generated by these cliques are given in Figure 1 and the graph $\Gamma_P(D_{12})$ is given in Figure 2.

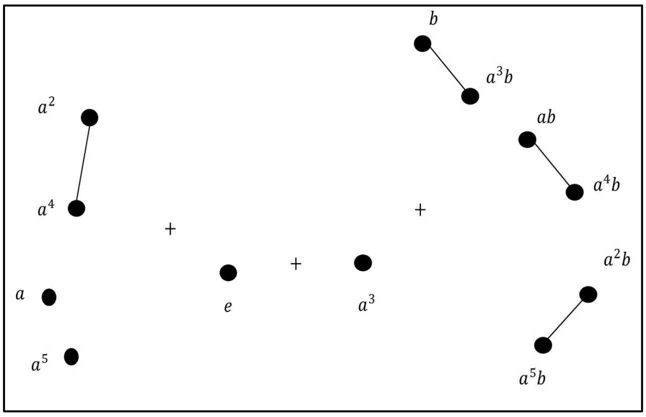

Fig. 1. Subgraph of p-subgroup graph of D_{12}

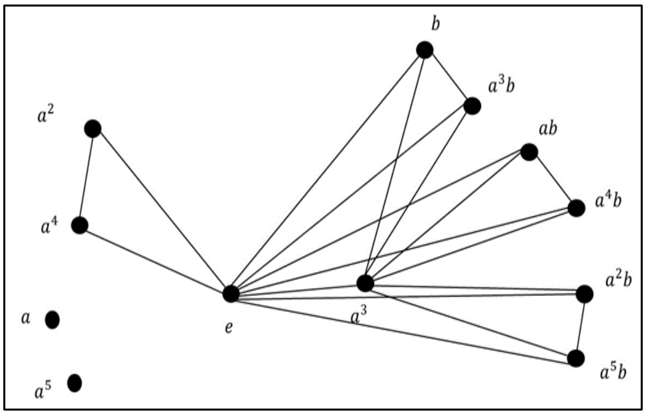

Fig. 2. p-subgroup graph of D_{12}

3.1 General Presentations of p-subgroup Graph of Dihedral Groups

The general presentations for the p-subgroup graph of dihedral groups are presented in the following theorems. The general presentation of the p-subgroup graph of D_{2n} for $n = p^r$ and p prime is given in Theorem 1.

Theorem 1 Let D_{2n} be the dihedral group of order $2n$, where $n = p^r$, p is a prime number and $r \in \mathbb{N}$. Then

$$\Gamma_p(D_{2n}) = \begin{cases} K_{2n} & \text{if} \quad p = 2 \\ K_1 + (K_{n-1} \cup \overline{K}_n) & \text{if} \quad p \neq 2. \end{cases}$$

Proof: Suppose D_{2n} is the dihedral group of order 2n, then $D_{2n} = \{e, a, a^2, a^3, \dots, a^{n-1}, b, ab, a^2b, a^3b, \dots, a^{n-1}b\}$. Let $A = \{e, a, a^2, \dots, a^{n-1}\}$ and $B = \{b, ab, a^2b, a^3b, \dots, a^{n-1}b\}$ be the sets of the rotations and reflections of D_{2n}, then $A \cong \mathbb{Z}_{p^r}$. Pick arbitrary $A_i, A_j \in A$ such that $A_j = \langle a^j \rangle$, for $i \neq j$. Then $|\langle a^i, a^j \rangle| = p^t$, for $t \in \mathbb{N}$ since $|A| = p^r$. Thus A is a clique of size p^r. If $p = 2$, then B is also a clique of size n since $\langle a^i b, a^j b \rangle = D_{2n}$ and $|\langle a^i b, a^j b \rangle| = |D_{2n}| = 2p^r = 2^{r+1}$. Thus, $\langle a^i b, a^j b \rangle$ is a p-group. Moreover, $|\langle a^i, a^j b \rangle| = p^t$. Meaning that there are edge linking every pair of the vertices of D_{2n}. Therefore, $\Gamma_p(D_{2n}) = K_{2n}$. If $p \neq 2$, then $|\langle a^i b, a^j b \rangle| = |D_{2n}| = 2p^r$. Thus, in this case $\langle a^i b, a^j b \rangle$ is not p-subgroup. Hence there are no edge linking $a^i b$ and $a^j b$. But $|\langle e, a^i \rangle| = |\langle a^i b \rangle| = 2$. Thus,

$$\begin{aligned} \Gamma_p(D_{2n}) &= K_{|\langle e \rangle|} + \left(K_{|A \setminus \{e\}|} \cup \overline{K}_{|B|}\right) \\ &= K_1 + (K_{n-1} \cup \overline{K}_n) \end{aligned}$$

The general presentation of the p-subgroup graph of D_{2n} for $n = p^r q$ where p and q are distinct primes is given in Theorem 2.

Theorem 2 Let D_{2n} be the dihedral group of order $2n$, for $n = p^r q$, where p and q are distinct prime numbers and $r \in \mathbb{N}$. Then

$$\Gamma_p(D_{2n}) = \begin{cases} K_1 + \left(K_{q-1} \cup \left(K_{p^r-1} + qK_{p^r}\right)\right) \cup \overline{K}_{n+1-(p^r+q)} & \text{if} \quad p = 2, \\ K_1 + \left(K_{p^r-1} \cup \left(K_{q-1} + p^r K_q\right)\right) \cup \overline{K}_{n+1-(p^r+q)} & \text{if} \quad q = 2 \\ K_1 + \left(K_{p^r-1} \cup K_{q-1} \cup \overline{K}_{p^r q}\right) \cup \overline{K}_{n+1-(p^r+q)} & \text{if} \quad p \neq 2 \text{ and } q \neq 2. \end{cases}$$

Proof: Let $D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$ be the dihedral group of order $2n$. To determine the vertex adjacencies. We need to find the order of the elements. Let $A = \{e, a, a^2, \dots, a^{n-1}\}$ and $B = \{b, ab, a^2b, \dots, a^{n-1}b\}$ be the sets of the rotations and the reflections of D_{2n}, respectively. Since $a^i b$ are reflections, then $|\langle a^i b\rangle| = 2$. For the rotations $a^i, \{e, a, a^2, \dots, a^{n-1}\} \cong \mathbb{Z}_n$. Thus, $|a^i| = \frac{n}{\gcd(i,n)}$. Let $A_p = \langle a^{p'}\rangle$ and $A_q = \langle a^q\rangle$, then $A_p \subset A$ and $A_q \subset A$. Thus, A_p, A_q are independent clique of size $q-1$ and $p^r - 1$, respectively. Moreover, since the $|\langle a^i b\rangle| = 2$, then there exist edges between B and A_q or A_p if $p = 2$ or $q = 2$, respectively. If $p \neq 2$, $q \neq 2$, then there no edges between B and A_p or A_q. Now, the element of set B form cliques of size p^r and q if $p = 2$ and $q = 2$, respectively. Let M and N be these cliques. If $p = 2$, then $M = \{a^i b, a^{i+q}b, a^{i+2q}b, \dots, a^{i+(p^r-1)q}b\}$ for $0 \le i \le (q-1)$ and if $q = 2$, then $N = \{a^i b, a^{i+p'}b\}$ for $0 \le i \le \left(\frac{n}{2} - 1\right)$. Then $|M| = p^r$ and $|N| = q$. For the situation when $p \neq 2, q \neq 2$, then B is an independent set. Let L be the set containing the remaining vertices of the graph other than A_p, A_q and B. Therefore,

$$\begin{aligned}
|L| &= |G| - |\{e\}| - |A_p| - |A_q| - |B| \\
&= (2n-1) - (q-1) - (p^r - 1) - n \\
&= n + 1 - (p^r + q).
\end{aligned}$$

Clearly the vertex e is adjacent to A_p, A_q, M and N, while e is not adjacent to L since the order of $\langle e, L\rangle$ is not p-group. Now, if $p = 2$, then

$$\begin{aligned}
\Gamma_p(D_{2n}) &= K_{|e|} + \left(K_{|A_p|} \cup \left(K_{|A_q|} + qK_{|M|}\right)\right) \cup \overline{K}_{|L|} \\
&= K_1 + \left(K_{q-1} \cup \left(K_{p^r-1} + qK_{p^r}\right)\right) \cup \overline{K}_{n+1-(p^r+q)}.
\end{aligned}$$

If $q = 2$, then

$$\begin{aligned}
\Gamma_p(D_{2n}) &= K_{|e|} + \left(K_{|A_q|}\right) \cup \left(K_{|A_p|} + p^r K_{|N|}\right) \cup \overline{K}_{|L|} \\
&= K_1 + \left(K_{p^r-1} \cup \left(K_{q-1} + p^r K_q\right)\right) \cup \overline{K}_{n+1-(p^r+q)}.
\end{aligned}$$

If $p \neq 2, q \neq 2$, then

$$\begin{aligned}
\Gamma_p(D_{2n}) &= K_{|e|} + \left(K_{|A_q|} \cup K_{|A_p|} \cup \overline{K}_{|B|}\right) \cup \overline{K}_{|L|} \\
&= K_1 + \left(K_{p^r-1} \cup K_{q-1} \cup \overline{K}_n\right) \cup \overline{K}_{n+1-(p^r+q)}. \\
&= K_1 + \left(K_{p^r-1} \cup K_{q-1} \cup \overline{K}_{p^r q}\right) \cup \overline{K}_{n+1-(p^r+q)}.
\end{aligned}$$

Next, the general presentation of the p-subgroup graph of D_{2n} for $n = pqh$ where p, q and h primes, is given in Theorem 3.

Theorem 3 Let D_{2n} be the dihedral group of order $2n$, for $n = pqh$ where p, q and h are distinct prime numbers. Then

$$\Gamma_p(D_{2n}) = \begin{cases} K_1 + \left(K_{h-1} \cup K_{q-1} \cup \left(K_{p-1} + \frac{n}{2}K_2\right)\right) \cup \overline{K}_{n-(p+q+h-2)} & \text{if} \quad p = 2, \\ K_1 + \left(K_{h-1} \cup K_{q-1} \cup K_{p-1} \cup \overline{K}_n\right) \cup \overline{K}_{n-(p+q+h-2)} & \text{if} \quad n \text{ odd}. \end{cases}$$

Proof: Suppose $D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$ be the dihedral group of order $2n$. Let $A = \{e, a, a^2, \dots, a^{n-1}\}$ and $B = \{b, ab, a^2b, \dots, a^{n-1}b\}$ be the sets of the rotations and the reflections of D_{2n}, respectively. Since a^ib are reflections, then $|\langle a^ib\rangle| = 2$. For the rotations a^i, since $\{e, a, a^2, \dots, a^{n-1}\} \cong \mathbb{Z}_n$, then the order of an element $a^i \in A$, is $|a^i| = \frac{n}{\gcd(i,n)}$. Let $A_1 = \langle a^{pq}\rangle, A_2 = \langle a^{ph}\rangle$ and $A_3 = \langle a^{qh}\rangle$ be the subsets of A. Thus, A_1, A_2 and A_3 are independent clique of size $h-1, q-1$ and $p-1$ respectively. Since the order of $|A_3| = 2$ and the and the $|\langle a^ib\rangle| = 2$ then there exist edges between B and A_3 if $p = 2$. Now, the elements of set B form cliques of size qh if $p = 2$. Let M be this clique, then $M = \{a^ib, a^{i+qh}b\}$ for $0 \le i \le \left(\frac{n}{2} - 1\right)$ and $|M| = 2$. If n odd, then B is an independent set. Let L be the set containing the remaining vertices of the graph other than A_1, A_2 and A_3 and B. Then

$$\begin{aligned}|L| &= |G| - |\{e\}| - |A_1| - |A_2| - |A_3| - |B| \\ &= (2n-1) - (h-1) - (q-1) - (p-1) - n \\ &= n - (p+q+h-2).\end{aligned}$$

Clearly, the vertex e is adjacent to A_1, A_2 and A_3, and M, while e is not adjacent to L since the order of $\langle e, L\rangle$ is not p-group. Now, if $p = 2$, then

$$\begin{aligned}\Gamma_p(D_{2n}) &= K_{|e|} + \left(K_{|A_1|} \cup K_{|A_2|} \cup \left(K_{|A_3|} + qhK_{|M|}\right)\right) \cup \overline{K}_{|L|} \\ &= K_1 + \left(K_{h-1} \cup K_{q-1} \cup \left(K_{p-1} + \frac{n}{2}K_2\right)\right) \cup \overline{K}_{n-(p+q+h-2)}.\end{aligned}$$

If n odd, then

$$\begin{aligned}\Gamma_p(D_{2n}) &= K_{|e|} + \left(K_{|A_1|} \cup K_{|A_2|} \cup K_{|A_3|} \cup \overline{K}_{|B|}\right)\big) \cup \overline{K}_{|L|} \\ &= K_1 + \left(K_{h-1} \cup K_{q-1} \cup K_{p-1} \cup \overline{K}_n\right) \cup \overline{K}_{n-(p+q+h-2)}.\end{aligned}$$

To start, we establish a connected and distance of vertices that simplify the study of various topological indices. These results are given in Proposition 1, Proposition 2, and Proposition 3.

Proposition 1 Let D_{2n} be the dihedral group of order $2n$. Then $\Gamma_p(D_{2n})$ is connected only if $n = p^r$, where $r \in \mathbb{N}$ and p is a prime number.

Proof: By definition 5, the vertices of $\Gamma_p(D_{2n})$ are the sets of the rotations and the reflections. Let $A = \{e, a, a^2, \dots, a^{n-1}\}$ and $B = \{b, ab, a^2b, \dots, a^{n-1}b\}$ be the sets of the rotations and the reflections of D_{2n}, respectively. Pick arbitrary nontrivial element $a \in A, b \in B$, then $|\langle e, a\rangle| = p^r$ and $|\langle e, b\rangle| = 2$. That is $a \sim e$ and $e \sim b$ by the vertex adjacency. Thus, all the vertices of $\Gamma_p(D_{2n})$ are reachable through e. Hence, $\Gamma_p(D_{2n})$ is connected if $n = p^r$.

Proposition 2 Let D_{2n} be the dihedral group of order $2n$, for $n = 2^r$, $r \in \mathbb{N}$. Then $d(v_k, v_l) = 1$, where $k \ne l$.

Proof: $\Gamma_p(D_{2n})$ is a complete graph by Theorem 1. Then the distance for each pair of vertices in this graph is one. Therefore, $d(v_k, v_l) = 1$, where $k \ne l$.

Proposition 3 Let D_{2n} be the dihedral group of order $2n$, for $n = p^r$, $p \ne 2$. Then

$$d(v_k, v_l) = \begin{cases} 1 & \text{if } v_k \text{ and } v_l \text{ in the same complete subgraph,} \\ 2 & \text{otherwise,} \end{cases}$$

where p is a prime and $r \in \mathbb{N}$.

Proof: If $p \neq 2$, then by Theorem 1, $\Gamma_p(D_{2n}) = K_1 + (k_{n-1} \cup \overline{K}_n)$. If v_k , v_l in the same complete subgraph, then by using the same proof in Proposition 2, $d(v_k, v_l) = 1$. If v_k, v_l are not belong the same clique or belong to $\overline{K}_n$, then by Theorem 1, v_k can be reached from v_l through e. Thus, $d(v_k, v_l) = 2$.

The topological indices of the p-subgroup graph associated with dihedral groups for $n = p^r$ which is a connected graph are determined. Then, the Wiener index of $\Gamma_p(D_{2n})$ for $n = p^r$ is given in Theorem 4.

Theorem 4 LetD_{2n} be the dihedral group of order $2n$. Then, the Wiener index of the p-subgroup graph, Γ_p of dihedral group is stated as follows:

$$W\left(\Gamma_p(D_{2n})\right) = \begin{cases} n(2n-1) \text{ if } p = 2, \\ \frac{1}{2}n(7n-5) \text{ otherwise.} \end{cases}$$

Proof: Suppose p = 2, then by Proposition 2, $d(v_k, v_l) = 1$. By Theorem 1, $n\left(V\left(\Gamma_p(D_{2^r}\right)\right) = V(K_{2n}) = 2n$. Hence,

$$\begin{aligned} W\left(\Gamma_p(D_{2n})\right) &= \frac{1}{2}\sum_{k=0}^{2n-1}\sum_{l=0}^{2n-1} d(v_k, v_l) = \frac{1}{2}\sum_{k=0}^{2n-1} [d(v_k, v_0) + d(v_k, v_1) + d(v_k, v_2) + \cdots + d(v_k, v_{2n-1})] \\ &= \frac{1}{2}[d(v_0, v_0) + d(v_1, v_0) + d(v_2, v_0) + \cdots + d(v_{2n-1}, v_0) + d(v_0, v_1) + d(v_1, v_1) + \cdots \\ &\quad + d(v_{2n-1}, v_1) + d(v_0, v_2) + d(v_1, v_2) + \cdots + d(v_{2n-1}, v_2) + \cdots \\ &\quad + d(v_0, v_{2n-1}) + d(v_1, v_{2n-1}) + \cdots + d(v_{2n-1}, v_{2n-1})] \\ &= \frac{1}{2}[2n-1 + 2n-1 + 2n-1 + \underbrace{\cdots}_{2n \text{ times}}] \\ &= \frac{1}{2}[2n(2n-1)] = n(2n-1). \end{aligned}$$

If $p \neq 2$, then by Theorem 1, there are two subgraphs K_{n-1} which is a clique of size $n-1$ and K_n. Let A and B be the sets of rotations and reflections of D_{2p} where $p \neq 2$, then

$$W\left(\Gamma_p(D_{2n})\right) = \frac{1}{2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} d(v_k, v_l) = \frac{1}{2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} [d(d^k, a^k) + d(d^k, a^k b) + d(d^k b, d^k b)]$$
$$= \frac{1}{2}\sum_{k=0}^{n-1} [d(d^k, e) + d(d^k, a) + d(a^k, a^2) + \cdots + d(d^k, a^{n-1})$$
$$+d(a^k, b) + d(d^k, ab) + \cdots + d(d^k, a^{n-1}b) + \cdots + d(a^k b, b) + d(d^k b, ab) + \cdots + d(a^k b, d^{n-1}b)]$$
$$= \frac{1}{2}[d(e,e) + d(a,e) + \cdots + d(a^{n-1},e) + d(e,a) + d(a,a) + \cdots + d(a^{n-1},a) + d(e,a^2) + d(a,a^2)$$
$$+ \cdots + d(a^{n-1},a^2) + \cdots + d(e,d^{n-1}) + d(a,a^{n-1}) + \cdots d(d^{n-1},d^{n-1}) + d(e,b) + d(a,b) + \cdots$$
$$+d(a^{n-1},b) + d(e,ab) + d(a,ab) + \cdots + d(a^{n-1},ab) + \cdots + d(e,a^{n-1}b) + d(a,a^{n-1}b) + \cdots$$
$$+d(a^{n-1},d^{n-1}b) + \cdots + d(b,b) + d(ab,b) + \cdots + d(a^{n-1}b,b) + d(b,ab) + d(ab,ab) + \cdots$$
$$+d(a^{n-1}b,ab) + \cdots + d(b,d^{n-1}b) + d(ab,a^{n-1}b) + \cdots + d(a^{n-1}b,a^{n-1}b)]$$
$$= \frac{1}{2}[d(e,c) + d(e,a) + \cdots + d(e,a^{n-1}), +d(e,b) + d(e,ab) + \cdots + d(e,a^{n-1}b) + d(a,e)$$
$$+d(a,a) + \cdots + d(a,a^{n-1}) + d(a,b) + d(a,ab) + \cdots + d(a,a^{n-1}b) + d(a^2,e) + d(a^2,a)$$
$$+ \cdots + d(a^2,a^{n-1}) + d(a^2,b) + d(a^2,ab) + \cdots + d(a^2,a^{n-1}b) + \cdots + d(a^{e-1},e)$$
$$+d(a^{n-1},a) + \cdots + +d(a^{n-1},d^{n-1}) + d(a^{n-1},b) + d(a^{n-1},ab) + \cdots + d(d^{n-1},a^{n-1}b)$$
$$+d(b,e) + d(b,a) + \cdots + d(b,a^{n-1}) + d(b,b) + d(b,ab) + \cdots + d(b,a^{n-1}b) + d(ab,e)$$
$$+d(ab,a) + \cdots + d(ab,b) + d(ab,ab) + \cdots + d(ab,,^{n-1}b) + \cdots + d(a^{n-1}b,e)$$
$$+d(a^{n-1}b,a) + \cdots + d(a^{n-1}b,b) + d(a^{n-1}b,ab) + \cdots + d(a^{n-1}b,a^{n-1}b)]$$
$$= \frac{1}{2}[(2n-1) + |A| - e(1(|A|-e) + 2|B|) + |B|(2(|A|-e) + 2|B| - 1 + 1\{e\})]$$
$$= \frac{1}{2}[(2n-1) + (n-1)((n-1) + 2n) + n(2(n-1) + 2(n-1) + 1)] = \frac{1}{2}n(7n-5).$$

Subsequently, the Zagreb index of the p-subgroup graph of the dihedral groups are presented. Theorem 5 and Theorem 6 present the results of the first Zagreb index and the second Zagreb index of the p-subgroup graph for dihedral groups, respectively.

Theorem 5 Let D_{2n} be the dihedral group of order $2n$. Then, the first Zagreb index of the p-subgroup graph, Γ_p of dihedral group is stated as follows:

$$M_1\left(\Gamma_p(D_{2n})\right) = \begin{cases} 2n(2n-1)^2 & \text{if} \quad p = 2, \\ n^2(n+1) & \text{otherwise.} \end{cases}$$

Proof: If $p = 2$, then by Theorem 1, $\Gamma_p(D_{2n}) = K_{2n}$ and the degree of all vertices in $\Gamma_p(D_{2n})$ is $2n - 1$, then

$$M_1\left(\Gamma_p(D_{2n})\right) = \sum_{k=0}^{2n-1} \deg(v_k)^2 = (2n-1)^2 + (2n-1)^2 + (2n-1)^2 + \cdots + (2n-1)^2$$
$$= 2n(2n-1)^2.$$

If $p \neq 2$, then by Theorem 1, $\Gamma_p(D_{2n}) = K_1 + (K_{n-1} \cup \overline{K}_n)$ and the degree of the non-trivial subgraphs is $n - 1$, $2n - 1$ and 1 with multiples $n - 1$, 1 and n respectively. Therefore,

$$M_1\left(\Gamma_p(D_{2n})\right)=\sum_{k=0}^{2n-1}\deg\,(v_k)^2=n-1(n-1)^2+(2n-1)^2+n(1)^2=n^3+n^2=n^2(n+1).$$

Theorem 6 Let D_{2n} be the dihedral group of order $2n$. Then, the second Zagreb index of the p-subgroup graph, Γ_p of dihedral group is stated as follows:

$$M_2\left(\Gamma_p(D_{2n})\right)=\begin{cases} n(2n-1)^3 & \text{if} \quad p=2, \\ \dfrac{n[n^3-n^2+3n-1]}{2} & \text{if} \quad p\neq 2. \end{cases}$$

Proof: Suppose D_{2n} is a dihedral group of order $2n$. By Theorem 1, $\Gamma_p(D_{2n})=K_{2n}$ if $p=2$ and $\Gamma_p(D_{2n})=K_1+(K_{n-1}\cup\overline{K}_n)$ if $p\neq 2$. For the second Zagreb index, the number of edge and the degree of each vertex of the graph need to be connected. If $p=2$, the $\Gamma_p(D_{2n})$ is a complete graph with $2n$ vertices. Hence the number of edges is $\frac{2n(2n-1)}{2}=n(2n-1)$ edges, and the degree of each vertex is $2n-1$. Thus,

$$M_2\left(\Gamma_p(D_{2n})\right)=\sum_{u,v\in E\left(\Gamma_p(D_{2n})\right)}\deg\,(u)\deg\,(v)=(2n-1)(2n-1)(n(2n-1))=n(2n-1)^3.$$

If $p\neq 2$, then in$\Gamma_p(D_{2n})$, all the vertices of K_{n-1} are not adjacent to any vertices of $\overline{K}_n$, but all vertices of Kn−1 are adjacent to any vertex of K_1, and also all vertices of $\overline{K}_n$ are adjacent the vertex of K_1. Pick arbitrary vertices $u\in V(K_{n-1}), v\in V(\overline{K}_n)$ and let $V(K_1)=e$. Then $deg(u)=n-1$, $deg(v)=1$ and $deg(e)=n+(n-1)=2n-1$. Now

$$\begin{aligned}M_2\left(\Gamma_p(D_{2n})\right)&=\sum_{u,v\in E\left(\Gamma_p(D_{2n})\right)}\deg(u)\deg(v)\\&=\frac{(n-1)(n-2)}{2}(n-1)^2+(n-1)^2(2n-1)+n(2n-1)\\&=\frac{n^4-n^3+3n^2-n}{2}=\frac{n[n^3-n^2+3n-1]}{2}\end{aligned}$$

4. Conclusions

The computations of Wiener and Zagreb indices of the p-subgroup graph of dihedral groups are presented in this paper. This computation involves only a p-subgroup graph on dihedral groups of prime power degree, which is connected graph. The findings of this study provide a new perspective on the study of chemical or biological characteristics of a particular molecular structure. Future research will cover the graphs of every finite group.

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