SEMARAK ILMU
PUBLISHING
202103268166(003316878-P)

Journal of Advanced Research in Applied Sciences and Engineering Technology

Journal homepage:
https://semarakilmu.com.my/journals/index.php/applied_sciences_eng_tech/index
ISSN: 2462-1943

Power Cayley Graphs of Dihedral Groups with Certain Order

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ARTICLE INFO

Article history:
Received 2 August 2023
Received in revised form 16 October 2023
Accepted 20 November 2023
Available online 28 February 2024

Keywords:

Power graph; Cayley graph; Union power Cayley graph; Intersection power Cayley graph; Dihedral group

ABSTRACT

Combination of the concepts of power graph and Cayley graph associated to groups has led to the introduction to two new variations of Cayley graph known as the union power Cayley graph and the intersection power Cayley graph. The set of vertices for both graphs consist of the elements of a finite group G. Consider any inverse-closed subset S of G, two vertices x and y are adjacent in the union power Cayley graph if $xy^{-1} \in S$ or if either one is an integral power of the other. Furthermore, x and y are adjacent in the intersection power Cayley graph if $xy^{-1} \in S$ and if either one is an integral power of the other. In this paper, the generalization of the union power Cayley graphs and the intersection power Cayley graphs of the dihedral groups with order $2n$, for $n \geq 3$ and $n = P^m$; P is prime and m is a natural number, relative to a specific subset containing rotation elements in the groups is found. In addition, properties of these graphs including the clique numbers, vertex chromatic numbers, girths and diameters are computed. Finally, the characteristics of the graphs, whether they are connected, regular, complete, and planar are also determined.

1. Introduction

Groups linked with graphs have been an exciting research topic in the last few decades, leading to various algebraic properties using graph theory. The Cayley graph visualizes the structures of a group by representing it in terms of graphs. This graph was first established by Cayley [1] in 1878. A Cayley graph of a group G relative to $S \subseteq G$, where $S^{-1} \subseteq S$, denoted as $\mathrm{Cay}(\mathrm{G}, \mathrm{S})$, is a graph with elements of G as its vertices and two vertices x and y are adjacent if there exist $s \in S$ such that $x = ys$ or $y = xs$. Recent results related to the Cayley graph can be referred in the previous studies [2-6].

A directed or in directed graph can be related to an algebraic structure G in a variety of ways, and the algebraic properties of G are examined in terms of the characteristics of the associated graph. In 2000, Kelarev and Quinn [7] introduced the directed power graph of a group G, which is denoted by

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https://doi.org/10.37934/araset.40.2.212220

$P(G)$, as a graph with the elements of G as the vertices and $x, y \in G$ are adjacent in $P(G)$ if $x = y^n$ or $y = x^m$ for $m, n \in \mathbb{N}$. Since then, many studies related to the power graph of groups have been conducted, for instance, see the previous studies [8-12].

By combining the concept of Cayley graph with power graph, the union power Cayley graphs, $Pow\text{-}\ Cay^+(G,S)$ and the intersection power Cayley graph, $Pow\text{ - }\ Cay^-(G,S)$ of finite groups have been introduced in the papers [13,14], respectively. In this study, the generalization of these two graphs is governed for the dihedral groups of order $2n$, D_{2n}, for $n \geq 3$ and $n = P^m$ where P is a prime and $m \in \mathbb{N}$. The graphs $Pow\text{-}\ Cay^+(G,S)$ and $Pow\text{ - }\ Cay^-(G,S)$ with respect to the subsets $S^{(n-1)} = \{a^1, a^2, a^3, \dots, a^{(n-1)}\}$ are constructed based on the structures of power graph and Cayley graph of the groups, in terms of their union and their intersection. Many structures on the groups can be observed from these two graphs together with their properties on finite groups. Hence, the general presentations for $Pow\text{-}\ Cay^+(G,S)$ and $Pow\text{ - }\ Cay^-(G,S)$ on D_{2P^m} together with some properties of the graphs which include clique numbers, vertex chromatic numbers, girth and diameters are determined in this study. Moreover, the connectivity, regularity, completeness, and planarity of these graphs are also found.

2. Notations and Preliminaries

All of the standard notations throughout the paper come from the books written by Gallian [15] for groups and by Rahman *et al.,* [16] as well as Chartrand and Zhang [17] for graphs. Important definitions and related basic concepts are provided in this section, together with notations and results from previous studies.

In this study, all groups taken into account are finite, and the investigation is limited to dihedral groups of order $2P^m$, with prime p and $m \in \mathbb{N}$. Furthermore, undirected simple graphs without multiple edge or loop are considered in this paper.

The set containing of all vertices of a graph Γ is denoted by $V(\Gamma)$ while $E(\Gamma)$ denotes the set of edges of Γ. The adjacency of vertex x with vertex y is labelled as $x \sim y$. Meanwhile, the notation $|V(\Gamma)|$ represents the number of vertices of Γ and $deg(x)$ represents the degree of the vertex x in Γ. If $deg(x) = n$ for all $x \in V(\Gamma)$, then Γ is n −regular.

A graph is said to be a star graph S_n if it contains one central vertex with edges to other vertices in it. Moreover, if there exists $(x, y) -$ path which connects each pair of vertices x and y in $V(\Gamma)$, then the graph Γ is a connected graph. Otherwise, Γ is called a disconnected graph. A complete graph K_n is a simple graph with n vertices and each vertex in a graph is adjacent to all the others while a graph that can be drawn in the plane with its edges only intersecting at their ends is said to be planar.

A clique is an induced subgraph of Γ that is complete. The clique number of Γ, which is represented by $\omega(\Gamma)$, is the largest size of a clique of the graph Γ. The vertex chromatic number of $\Gamma, \chi(\Gamma)$, is the minimum number of colours needed to colour the vertices of Γ so that no two adjacent vertices have the same colour. A perfect graph Γ is a graph in which $\chi(\Gamma) = \omega(\Gamma)$, both for the graph itself and for every induced subgraph.

The largest distance between any two vertices in a graph Γ is the diameter of that graph, $diam(\Gamma)$ while the size of the shortest cycle in Γ is its girth, denoted by $girth(\Gamma)$. The union of two simple graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a graph $\Gamma_1 \cup \Gamma_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. Meanwhile, we denote the join of $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ as the graph $\Gamma_1 + \Gamma_2$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ together with edges joining all vertices in V_1 with vertices in V_2. Throughout this paper, $S^{(n-1)}$ denotes a subset of size $n - 1$ such that $n \in \mathbb{N}$. In addition, $\varphi(n)$ denotes Euler's function, $n \in \mathbb{N}$.

The relevant definitions and theorems for this study are given next.

Theorem 1 [4] Every Cayley graph $Cay\ (G, S)$ is $|S| -$ regular.

Theorem 2 [2] Let $a_k \in S$. Then, in $C_n(S)$, the length of a cycle of period a_k is $\frac{n}{\gcd(n, a_k)}$ and the number of disjoint periodic cycles of period a_k is $gcd(n, a_k)$.

Proposition 1 [2] $\Gamma(n: r_1, r_2, \ldots, r_k) \cong C_n(r_1, r_2, \ldots, r_k)$, where $C_n(r_1, r_2, \ldots, r_k)$ is a circulant graph and $1 \leq r_1 < r_2 < \cdots < r_k \leq \left[\frac{n}{2}\right]$.

Then, $C_n(r_1, r_2, \ldots, r_k)$ is a connected graph if and only if $\gcd\ (n: r_1, r_2, \ldots, r_k) = 1$.

Theorem 3 [18] The power graph of a finite group $G, P(G)$ is complete if and only if G is a cyclic group of order 1 or P^m, for some primes p and for some $m \in \mathbb{N}$.

Corollary 1 [18] If a group G is finite, then $P(G)$ is always connected.

Theorem 4 [19] $P(D_n)$ is non-planar if and only if $n \geq 5$.

Corollary 2 [10] $\chi(P(D_n)) = \omega(P(D_n)) = \chi(P(\mathbb{Z}_n))$.

Theorem 5 [11] If a group G is finite, then $P(G)$ is perfect.

Definition 1 [13] Union power Cayley Graph

The union power Cayley graph of a group G relative to an inverse-closed subset S of $G \setminus \{e\}$, denoted by *Pow* - $Cay^+(G, S)$ is a simple undirected graph with $V($*Pow* - $Cay^+(G, S)) = G$ and two vertices a and b are adjacent if and only if at least one of the following two conditions is satisfied:

i. $ab^{-1} \in S$
ii. $a = b^n\ or\ b = a^m$ for some $m, n \in \mathbb{N}$.

In other words, *Pow* - $Cay^+(G, S) = \{\{a, b\} \mid \{a, b\} \in E(P(G)) \cup E(\text{Cay}\ (G, S))\}$

Definition 2 [14] Intersection power Cayley Graph

The intersection power Cayley graph of a group G relative to an inverse-closed subset S of $G \setminus \{e\}$, denoted by *Pow* - $Cay^-(\text{G}, \text{S})$ is a simple undirected graph with $V($*Pow* - $Cay^-(G, S)) = G$ and two vertices a and b are adjacent if and only if both of the following two conditions are satisfied:

i. $ab^{-1} \in S$
ii. $a = b^n\ or\ b = a^m$ for some $m, n \in \mathbb{N}$.

In other words, *Pow* - $Cay^-(G, S) = \Big\{\{a, b\} \,\Big|\, \{a, b\} \in E\big(P(G)\big) \cap E(\text{Cay}(G, S))\Big\}$.

3. Results and Discussions

Without loss of generality, in this section, the notation D_{2n} represents the dihedral groups with order $2n$ for $n \geq 3$ and, n = P^m where p is a prime and $m \in \mathbb{N}$. In addition, $S^{(n-1)} = \{a^1, a^2, a^3, \ldots, a^{(n-1)}\}$. Hence, the general structures of the Cayley graphs and power graphs of D_{2n} relative to $S^{(n-1)}$ is given in Theorem 6 and Theorem 7, respectively. These two theorems are applied in obtaining the *Pow* - $Cay^+(D_{2n}, S^{(n-1)})$ and *Pow* - $Cay^-\big(D_{2n}, S^{(n-1)}\big)$ which are given in Theorem 8 and Theorem 9, respectively.

Theorem 6 The Cayley graph of D_{2n} with respect to $S^{(n-1)}$ is
$\text{Cay}\ \big(D_{2n}, S^{(n-1)}\big) \cong 2K_{p^m}$,
where K_{p^m} is the complete graph of order p^m.

Proof: Let D_{2n} be a dihedral group with order $2n$: $n = p^m$. Then, $D_{2n} = \{e, a, a^2, \ldots, a^{(n-1)}, b, ab, a^2b, \ldots, a^{(n-1)}b\}$ can be partitioned into sets, $A = \{e, a, a^2, \ldots, a^{(n-1)}\}$ and $B = \{\,b, ab, a^2b, \ldots, a^{(n-1)}b\}$. Then, $A \cong \mathbb{Z}_n$. We shall consider the Cayley graph of each set separately. Firstly, for the set A, by using Proposition 1 and Theorem 2, there are $n-1$ connected cycles of length $\frac{n}{\gcd(|A|,g)}$, where $g \in A$. Recall that by definition of Cayley graph, $x \sim y$ if $x = gy$ for $g \in S^{(n-1)}$,.

i. For $g \in S^{(n-1)}$, if $g = a^k \in \varphi(n)$, then $e \sim a^k \sim a^{2k} \sim \cdots \sim a^{n-k} \sim e$.
ii. For $g \in S^{(n-1)}$, if $g = a^{\frac{n}{2}}$, then $e \sim a^{\frac{n}{2}}, a \sim a^{\frac{n}{2}+1}, \ldots, a^{\frac{n}{2}-1} \sim a^{(n-1)}$.
iii. $g \in S^{(n-1)}$, if $g = a^k \notin \varphi(n)$, then $e \sim a^k \sim a^{2k} \sim \cdots \sim e, a \sim a^{k+1} \sim a^{2k+1} \sim \cdots \sim a, \ldots, a^{k-1} \sim a^{2(k-1)+1} \sim \cdots \sim a^{k-1}$.

From the equations above, since $a^{i+k} = a^k a^i$, then $a^{i+k} \sim a^i$ for any $a^k \in S^{(n-1)}$. By using Theorem 1 the Cayley graph $Cay\left(A, S^{(n-1)}\right)$ is $\left|S^{(n-1)}\right|$ -regular. In other words, $Cay\left(A, S^{(n-1)}\right)$ is $(n-1)$- regular. Hence,

$$\text{Cay}\left(A, S^{(n-1)}\right) = K_n. \tag{1}$$

Applying the same procedure for the set B gives the following:

i. For $g \in S^{(n-1)}$, if $g = a^k \in \varphi(n)$, then $b \sim a^k b \sim a^{2k} b \sim \cdots \sim a^{n-k} b \sim b$.
ii. For $g \in S^{(n-1)}$, if $g = a^{\frac{n}{2}}$, then $b \sim a^{\frac{n}{2}} b, a \sim a^{\frac{n}{2}+1} b, \ldots, a^{\frac{n}{2}-1} b \sim a^{(n-1)} b$.
iii. $g \in S^{(n-1)}$, if $g = a^k \notin \varphi(n)$, then $b \sim a^k b \sim a^{2k} b \sim \cdots \sim b, ab \sim a^{k+1} b \sim a^{2k+1} b \sim \cdots \sim ab, \ldots, a^{k-1} b \sim a^{2(k-1)+1} b \sim \cdots \sim a^{k-1} b$.

Also, again by definition of Cayley graph, $x \sim y$ if $x = gy$ for $g \in S^{(n-1)}$ and $x, y \in V\left(\text{Cay}\left(D_{2n}, S^{(n-1)}\right)\right)$. From the equation above, since $a^{(i+k)} b = a^{(k)} a^{(i)} b$ then $a^{i+k} b \sim a^i b$ for any $a^k \in S^{(n-1)}$, which shows that

$$\text{Cay}\left(B, S^{(n-1)}\right) = K_n. \tag{2}$$

Now combining Eq. (1) and Eq. (2) gives Cay $\left(D_{2n}, S^{(n-1)}\right) \cong K_n \cup K_n = 2K_n$.

Theorem 7 The power graph of D_{2n} is

$$Pow\,(D_{2n}) \cong S_1 + \left(K_{(n-1)} \cup nK_1\right),$$

where S_1 is the star graph.

Proof: Let D_{2n} be a dihedral group with order $2n$: $n = p^m$. Then, $D_{2n} = \left\{e, a, a^2, \ldots, a^{(n-1)}, b, ab, a^2 b, \ldots, a^{(n-1)} b\right\}$ can be partitioned into sets, $A = \left\{e, a, a^2, \ldots, a^{(n-1)}\right\}$ and $B = \left\{b, ab, a^2 b, \ldots, a^{(n-1)} b\right\}$. Then, $A \cong \mathbb{Z}_n$, meaning that $A = \langle a \rangle$ is a cyclic subgroup of D_{2n} with $|A| = n$. By Theorem 3, $Pow\,(A) = K_n$ and $Pow\,(A \setminus \{e\}) = K_{(n-1)}$, so, we can write that $Pow\,(A) = S_1 + K_{(n-1)}$. By the group presentation, $a^n = b^2 = e$, so $(ab)^2 = a(ba)b = a\left(a^{(n-1)} b\right) b = e$ also $\left(a^{(n-1)} b\right)^2 = a^{(n-1)}(ba) a^{(n-2)} b = a^{(n-1)}\left(a^{(n-1)} b\right) a^{(n-2)} b = \left(a^{(n-2)} b\right)^2$. Continuing the process in this manner, it is obtained that $\left(a^{(n-1)} b\right)^2 = \left(a^{(n-2)} b\right)^2 = \cdots = (a^2 b)^2 = (ab)^2 = e$. Thus, the square of any $a^i b$ is e, showing that $\left(a^i b\right)^i$ are transpositions. This implies that, the power of any $a^i b$ is either itself or e for $0 \leq k \leq n-1$ meaning that the subgroup generated by any $a^i b$ contains only itself and e, that is $\langle a^i b \rangle \supset \{e\}$. Hence, Pow $(\{e\} \cup B) = S_{|e|} + \overline{K}_{|B|}$. Combining Pow (A) and Pow (B) gives Pow $(D_{2n}) \cong S_1 + \left(K_{(n-1)} \cup \overline{K}_n\right) = S_1 + \left(K_{(n-1)} \cup nK_1\right)$.

Theorem 8 The union power Cayley graph of D_{2n} with respect to $S^{(n-1)}$ is

$$Pow - Cay^{+}\left(D_{2n}, S^{(n-1)}\right) \cong S_1 + \left(K_{(n-1)} \cup nK_n\right),$$

where K_n is the complete graph of order $n = p^m$.

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. By Definition 1, " $E\left(Pow - \text{Cay}^{-}\left(D_{2n}, S^{(n-1)}\right)\right) = \left\{\{x, y\} \mid \{x, y\} \in E\left(P(D_{2n})\right) \cup E\left(\text{Cay}\left(D_{2n}, S^{(n-1)}\right)\right)\right\}$. From Theorem 7, the power graph for D_{2p^m} is isomorphic to $S_1 + \left(K_{(n-1)} \cup \overline{K}_n\right)$ and and from Theorem 6, the Cayley graph for D_{2p^m} with respect to a subset of D_{2n} of size $(n-1)$, $S^{(n-1)} = \left\{a^1, a^2, a^3, \dots, a^{(n-1)}\right\}$ is isomorphic to $2K_{\text{n}}$. It now remains to take the union of the edges in $Pow\left(D_{2p^m}\right)$ and in Cay $\left(D_{2n}, S^{(n-1)}\right)$. The equation Pow $\left(D_{2p^m}\right) = S_1 + \left(K_{(n-1)} \cup \overline{K}_n\right)$ shows that B contains isolated vertices $\overline{K}_n$but the equation $\text{Cay}\left(D_{2n}, S^{(n-1)}\right) = 2K_{p^m}$ shows that all the elements of B form a complete subgraph K_n. This means that in the union with power graph, these edges would be considered as the new edges, forming $S_1 + \left(K_{(n-1)} \cup K_n\right)$. Then, Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right) \cong S_1 + \left(K_{(n-1)} \cup K_n\right)$.

Some invariants of Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right)$ which are clique number, vertex chromatic number, diameter and girth are given in Proposition 2, Proposition 3, Proposition 4 and Proposition 5, respectively.

Proposition 2 The clique number of Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right)$ is

$$\omega\left(\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = n + 1.$$

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. Then, by Theorem 8, Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right) \cong S_1 + \left(K_{(n-1)} \cup K_n\right)$. Thus, Pow - Cay $^{+}\left(D_{2n}, S^{(n-1)}\right)$ is a join union of $\left(K_{(n-1)} \cup K_n\right)$ with S_1, that is $S_1 + K_n$ is the maximum complete component of the graph. Hence the clique number is $|V(S_1 + K_n)|$. Therefore, $\omega\left(\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = n + 1$.

Proposition 3 The chromatic number of Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right)$ is

$$\chi\left(\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = n + 1.$$

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. By Proposition 2 $\omega\left(\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = n + 1$. This shows that the minimum numbers of colours required to colour the vertices of Pow - Cay$^{+}\left(D_{2n}, S^{(n-1)}\right)$ is $n + 1$. By Theorem 8, Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right) \cong S_1 + \left(K_{(n-1)} \cup K_n\right)$, that is Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right)$ can be considered as a disjoint union of $K_{(n-1)}$ with K_n whose each vertex is adjacent with $V(S_1)$. Thus, the colours for the vertices of K_n can be shared with $V\left(K_{(n-1)}\right)$. Therefore, $\chi\left(\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = \chi(S_1 + K_n) = \chi(K_{n+1}) = n + 1$.

Proposition 4 The diameter of Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right)$ is

$$diam\left(Pow - Cay^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = 2.$$

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. Then, by Theorem 8, Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right) \cong S_1 + \left(K_{(n-1)} \cup K_n\right)$. Let $V(S_1) = \{e\}, V(K_{n-1}) = \{u_1, u_2, \dots, u_{n-1}\}$ and $V(K_n) = \{v_1, v_2, \dots, v_n\}$, then $d\left(u_i, u_j\right) = 1, d\left(v_i, v_j\right) = 1, d(u_i, e) = 1, d(v_i, e) = 1$. But $d(u, v) = 2$, since $u \sim v$ for any arbitrary vertices $u \in V(K_{n-1})$ and $v \in V(K_n)$ and $i \neq j$. Thus u can only be reachable from v through e, that is $u \sim e$ and $e \sim v$. Therefore, the maximum distance to reach any vertex to reach any vertex to another is 2. Hence, $diam\left(Pow - Cay^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = 2$.

Proposition 4 The girth of Pow $-$ Cay $^{+}\left(D_{2n}, S^{(n-1)}\right)$ is

$$girth\left(\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = 3.$$

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. Then, by Theorem 8, $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right) \cong S_1 + \left(K_{(n-1)} \cup K_n\right)$. Since $n = p^m$ where $n \geq 3$, then the graph contains a triangle, that is a circle of length 3. Therefore, $girth\left(\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = 3$.

The connectivity, regularity, and completeness of $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)$ is given in Proposition 6. Meanwhile, the planarity of this graph is given in Proposition 7.

Proposition 6 The graph $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)$ is connected, not regular, hence not complete.

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. By Corollary 1, $P\left(D_{p^m}\right)$ is connected and from Definition 1, $\text{Pow} - \text{Cay}^{+}\left(D_{p^m}, S^{(n-1)}\right) = \left\{\{x, y\} \mid \{x, y\} \in E\left(P\left(D_{p^m}\right) \cup E\left(D_{p^m}, S^{(n-1)}\right)\right\}\right.$. This means that the graph has union edges of two graphs which are power graph and Cayley graph and since the union of a connected graph with any graph is a connected graph, hence, $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)$ is a connected graph. For regularity, by Theorem 8, $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right) \cong S_1 + \left(K_{(n-1)} \cup K_n\right)$. Thus, the graph $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)$ has two major components $K_{(n-1)}$ and K_n, where the vertices of each component is adjacent to $V(S_1)$. This means that every vertex of the graph is reachable from one another through $V(S_1)$. Pick arbitrary vertices $u \in V(K_{(n-1)})$ and $v \in V(K_n)$, then $deg(u) = n - 1$ and $deg(v) = n$, showing that $deg(v) > deg(u)$. Therefore, $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)$ is not regular. For completeness, since $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)$ is not regular, hence it is not complete.

Proposition 7 The graph $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)$ is planar only if $n \leq 3$.

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. By Proposition 2, $\chi\left(\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)\right) = n + 1$. This means that $\text{Pow} - \text{Cay}^{+}\left(D_{2n}, S^{(n-1)}\right)$ can only be drawn on a plane without edge crossing if $n \leq 3$.

Next, the computations are continued for the intersection power Cayley graphs of D_{2n} with respect to the subset $S^{(n-1)}$, $\text{Pow} - \text{Cay}^{-}\left(D_{2n}, S^{(n-1)}\right)$. Note again that the notation D_{2n} represents the dihedral groups of order $2n$ for $n \geq 3$ and $n = p^m$ where p is a prime and $m \in \mathbb{N}$. In addition, $S^{(n-1)} = \left\{a^1, a^2, a^3, \ldots, a^{(n-1)}\right\}$.

Theorem 9 The intersection power Cayley graphs of D_{2n} with respect to the subset $S^{(n-1)}$ is $\text{Pow} - \text{Cay}^{-}\left(D_{2n}, S^{(n-1)}\right) \cong K_n \cup nK_1$,

where K_n, is the complete graph of order $n = p^m$.

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. By Definition 2, $E\left(Pow - \text{Cay}^{-}\left(D_{2n}, S^{(n-1)}\right)\right) = \left\{\{x, y\} \mid \{x, y\} \in E\left(P(D_{2n})\right) \cap E\left(\text{Cay}\left(D_{2n}, S^{(n-1)}\right)\right)\right\}$. From Theorem 7, the power graph of D_{2p^m} is isomorphic to $S_1 + \left(K_{(n-1)} \cup nK_1\right)$ and from Theorem 6, the Cayley graph for D_{2n} with respect to a subset of D_{2n} of size $(n-1)$, $S^{(n-1)} = \left\{a^1, a^2, a^3, \ldots, a^{(n-1)}\right\}$ is isomorphic to $2K_n$, and can be rewritten as $\text{Cay}\left(D_{2n}, S^{(n-1)}\right) \cong K_n \cup K_n$. In $\text{Cay}\left(D_{2n}, S^{(n-1)}\right)$, the first K_n is formed by the rotation elements of D_{2n}, while the second K_n is formed by the reflection elements of D_{2n}. Also, in $P(D_{2n})$, the rotation elements form $S_1 + K_{(n-1)} = K_n$, and the reflection elements form nK_1. Thus, the intersection of the rotation elements in power graph and Cayley graph is K_n, while the intersection of the elements of the reflection's forms nK_1. Therefore, $\text{Pow} - \text{Cay}^{-}\left(D_{2n}, S^{(n-1)}\right) \cong K_n \cup nK_1$.

Some invariants of $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ which are clique number, vertex chromatic number, diameter and girth are given in Proposition 8, Proposition 9, Proposition 10 and Proposition 11, respectively.

Proposition 8 The clique number of $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is

$$\omega\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = \omega(P(\mathbb{Z}_n)) = n.$$

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. Then, by Theorem 9, $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)}) \cong K_n \cup nK_1$. Thus, the maximum component of the graph is K_n. Regarding to this structure and using the same arguments as in the proof for Theorem 9, Theorem 6, and Theorem 7 K_n is formed by the intersection the rotation elements $A \cong \mathbb{Z}_n$of $\text{Cay}(A, S^{(n-1)})$ and $P(A)$. Also, from Corollary 2 and Theorem 5, $\chi(P(D_n)) = \omega(P(D_n)) = \chi(P(\mathbb{Z}_n)) = \omega(P(\mathbb{Z}_n))$. Hence, $\omega\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = \omega(P(\mathbb{Z}_n)) = n$.

Therefore, $\omega\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = n$.

Proposition 9 The chromatic number of $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is

$$\chi\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = n.$$

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. Then, by Theorem 9, $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)}) \cong K_n \cup nK_1$. Each vertex of K_n must be assigned with distinct colours, while all the n vertices of nK_1 can be coloured with a single colour of K_n. Thus, n colours are sufficient to properly colour the vertices of the graph. Therefore, $\chi\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = n$.

Proposition 10 The diameter of $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is

$$diam\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = \infty.$$

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. By Proposition 12, $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is disconnected. Hence, the distance between some vertices does not exist. Therefore, $diam\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = \infty$.

Proposition 11 The girth of $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is

$$girth\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = 3.$$

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. Then, by Proposition 9, $\chi\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = p^m$, since $n = p^m$ and $n \geq 3$, then $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ contains a triangle which is a cycle of length 3. Hence, $girth\left(\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})\right) = 3$.

The connectivity, regularity, and completeness of $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is given in Proposition 12. Meanwhile, the planarity of this graph is given in Proposition 13.

Proposition 12 The intersection power Cayley graph of D_{2n} with respect to $S^{(n-1)}$, $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$, is disconnected, not regular, hence not complete.

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. Then, by Theorem 9, $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is the disjoint union of K_n with n isolated vertices. This shows that $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is disconnected since the isolated vertices cannot be reached from the other vertices. Hence, the graph is disconnected. For regularity, pick arbitrary isolated vertex u and $v \in V(K_n)$, then $deg(u) = 0 < deg(v) = n - 1$. Hence, $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is not regular. For completeness, since $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$ is not regular, hence it is not complete.

Proposition 13 The intersection power Cayley graph of D_{2n} with respect to $S^{(n-1)}$, $\text{Pow} - \text{Cay}^{-}(D_{2n}, S^{(n-1)})$, is planar only if $n \leq 4$.

Proof: Let D_{2n} be a dihedral group with order 2n: $n = p^m$. By Proposition 8, $\omega\Big(\text{Pow} - \text{Cay}^{-}\big(D_{2n}, S^{(n-1)}\big)\Big) = p^m$. Using a similar argument as before, $\omega\Big(\text{Pow} - \text{Cay}^{-}\big(D_{2n}, S^{(n-1)}\big)\Big) = \omega\big(P(\mathbb{Z}_n)\big) = n$, since $n = p^m$ such that $n \geq 3$, and from Theorem 4, then the $\text{Pow} - \text{Cay}^{-}\big(D_{2n}, S^{(n-1)}\big)$ graph can be drawn on a plane without edge crossing if $n \leq 4$. Therefore, $\text{Pow} - \text{Cay}^{-}\big(D_{2n}, S^{(n-1)}\big)$ is planar if $n \leq 4$.

4. Conclusions

In this research, the generalization of the union power Cayley graphs and the intersection power Cayley graphs for the dihedral groups of order $2n$, with $n \geq 3$ and $n = p^m$; p is a prime number and m is a natural number relative to the subset $S^{(n-1)} = \{a^1, a^2, a^3, \dots, a^{(n-1)}\}$. Moreover, some properties for the general presentation of these dihedral groups have been evaluated in terms of connectivity, regularity, completeness and planarity. Additionally, the clique numbers, vertex chromatic numbers, girths and diameters of these graphs have been computed. The result of this research can contribute to adding a new dimension to the theoretical results provided, which are significant in the development of algebraic graph theory. In the future, the research will be extended to cover the graphs of all finite groups.

Acknowledgement
This research is funded under Others Grant Scheme (R.J130000.7354.4B711) and Matching Grant Scheme (Q.J130000.3054.03M65) from Universiti Teknologi Malaysia.

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