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Multistep Reduced Differential Transform Method in Solving Nonlinear Schrodinger Equations

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ABSTRACT

This paper obtains semi-analytical solutions for the nonlinear Schrodinger equations (NLSEs) using the multistep reduced differential transform method (MsRDTM). The implemented method yields an analytical approximate solution over a longer time frame, in which the method applied is treated as an algorithm in a sequence of small sub-division of intervals of identical length compared to the traditional reduced differential transform method (RDTM). Excluding the need of perturbation, linearization, or discretization, this method offers the benefit and reliability of the multistep algorithm. The outcomes show that the MsRDTM generated highly accurate solutions of NLSEs than the RDTM. In addition, the results show that the suggested method is straightforward to use, saves a significant amount of computing work when solving NLSEs, and has potential for broad application in other complex partial differential equations (PDEs) in the fields of engineering and science. The accuracy of the method is shown through the tables and graphical illustrations provided.

1. Introduction

The concept of differential equations and its applications has been one of the most important areas of pure and modern mathematics [1-3]. Scientific phenomena in optics, acoustics, fluid mechanics, hydrodynamics, and astronomy [4-6] is explained through differential equations which are modelled as nonlinear ordinary equation (ODE) or partial differential equation (PDE). This knowledge leads to new and various approximate accurate efficient solutions for these issues [7]. A few analytical techniques used to solve these applied models since their structural complexity is typically high [8,9]. Effectual methods in solving PDEs include the linear superposition principle [10], symmetry reduction strategy [11], and Hirota bilinear forms [12].

One of the most famous equations, the NLSEs appear in various fields, including hydrodynamics, plasma waves, solitary waves in semiconductors (thin plates), fluid-filled viscoelastic tubes, and nonlinear optical waves, biology, elastic media, quantum mechanics, magneto-static rotating waves, oceanography, and other disciplines [13]. As a model for other scientific processes, such as optics,

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the physics of optical waves, and light emission in cables of fibre optics, this kind of equation has also been utilized. It has also been used to fluids, deep surface water waves, and ocean roughness. Also characterised by the NLSE are optical solitons [14]. This equation has been solved numerically and analytically using various methods, such as the Bernoulli sub-ODE and the (G'/G)-expansion method [15], and the modified exponential Jacobi method [16], and several other methods [17-20]. Then, a method called inverse scattering transform method was introduced in 2023 by Adukov and Mishuris [21] for solving discrete NLSEs. In 2022, Jawad and Al-Fayadh [22] applied the variational homotopy transform method (VHTM) to solve Schrodinger equations.

The origin of the method used in this paper called the differential transform method (DTM) and its improved variants have been brought into light by scientists. An improved version of the DTM, the RDTM was initially introduced by Keskin *et al.,* [23,24]. This semi-analytical method has been used in solving time fractional nonlinear Schrodinger-Korteweg-de Vries equation by Owyed *et al.,* [25], and fifth order of KdV equation also known as damped Kawahara Equation (KE) by Aljahdaly and Alharbi [26]. It gained plenty attention since it solved a wide variety of problems by many researchers. In 2021, RDTM was implemented in solving many types of differential equations with great effectiveness [27].

The multistep DTM (MsDTM) was first introduced in 2010 by Odibat *et al.,* [28] and applied to various systems. It produces a solution where its convergent series rapidly converges in a large time frame which then improves the convergence of the series solution. Other researchers also used the multistep scheme by applying it on RDTM, called the multistep RDTM (MsRDTM) [29-31]. In 2018 and 2019, Hussin *et al.,* [32,33] proposed and implemented the Multistep Modified RDTM (MMRDTM) in solving NLSEs and fractional NLSEs (FNLSEs) respectively. Furthermore, Che Hussin *et al.,* [34,35] used MMRDTM then used in solving forced nonlinear Korteweg-de Vries (fNkdVE) and KdV equations with compact support. Recently, the same method, MMRDTM is also used in [36,37] by Sabdin *et al.,* to solve nonlinear telegraph equations (NLTEs) and time-fractional NLTEs with source term.

This paper aims in implementing the MsRDTM to solve NLSE as its main goal. The answers are compared with exact solutions and RDTM solutions. The finding indicates accuracy of the approach in tackling the considered problems. The remaining portions of this work are organized as follows. The definitions, solution formulations, fundamental operations related to the MsRDTM discussed in Section 2. Section 3 illustrates the application of the RDTM in several NLSEs with analytical answers are presented in tables and graphical illustrations. Lastly, Section 5 provides concluding observations.

2. Multistep Reduced Differential Transform Method

2.1 Reduced Differential Transform Method

Two-variable function $u(x,t)$ is considered that may be written as the product of two single-variable functions: $v(x,t) = f(x)g(t)$. On the foundational properties of the one-dimensional differential transform, the function $u(x,t)$ may be written as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k \tag{1}$$

where the t-dimensional span function of $u(x,t)$ is denoted by $U_k(x)$. The following [23,24] are the fundamental definitions of RDTM:

Definition 1. If the domain of interest's function $u(x,t)$ is analytical and continuously differentiable with regard to time t and space x, then letting

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} \tag{2}$$

where the transformed function is the t-dimension span function $U_k(x)$. In this paper, the primary function is denoted by the small letter $u(x,t)$, while the altered function is symbolized by the capital letter $U_k(x)$.

Definition 2. Given the following for the differential inverse transform of $U_k(x)$:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k, \tag{3}$$

Then, by fusing Eq. (2) and Eq. (3), we obtain

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} t^k. \tag{4}$$

According to the preceding definitions, the RDTM concept is obtained from the expanded power series. Consider the following operator-form nonlinear PDE to explain the fundamental RDTM concepts as in Eq. (5)

$$\mathcal{L}u(x,t) + \mathcal{R}u(x,t) + \mathcal{N}u(x,t) = g(x,t), \tag{5}$$

with initial condition

$$u(x,0) = f(x) \tag{6}$$

where $\mathcal{L} = \frac{\partial}{\partial t}$, $\mathcal{R}$ is a partial derivatives linear operator, $\mathcal{N}u(x,t)$ is a nonlinear operator and $g(x,t)$ is an inhomogeneous term.

Based on MsRDTM, the iteration formula shown below may be formed:

$$(k + 1)U_{k+1}(x) = \mathcal{G}_k(x) - \mathcal{R}U_k(x) - \mathcal{N}U_k(x), \tag{7}$$

where $U_k(x), \mathcal{R}U_k(x), \mathcal{N}U_k(x)\ and\ \mathcal{G}_k(x)$ are the transformations of the functions $\mathcal{L}u(x,t), \mathcal{R}u(x,t),\ \mathcal{N}u(x,t), and\ g(x,t)$ respectively.

Based on initial condition Eq. (6), we write

$$U_0(x) = f(x), \tag{8}$$

The following $U_k(x)$ values are obtained by substituting Eq. (8) into Eq. (6) and doing a simple computation. The n-terms approximation solution is then obtained as follows by applying inverse transformation on the set of values $\{U_k(x)\}_{k=0}^{n}$:

$$u_n(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k\ , t \in [0,T] \tag{9}$$

2.2 Multistep Scheme

The multistep scheme is as follows. The interval $[0,T]$ is divided by $s=\frac{T}{R}$ to generate R equal size subintervals $[t_{r-1},t_r]$ and nodes $t_r=rs$ such that for $r=1,2,\dots,R$. The upcoming procedures are used to compute MsRDTM. Firstly, RDTM is applied to the initial value problem of interval $[0,t_1]$. Then by using the initial conditions

$$u(x,0)=f_0(x). \tag{10}$$

We obtain the result

$$u_1(x,t)=\sum_{k=0}^{\mathrm{k}}U_{k,1}(x)t^k, t\in[0,t_1]. \tag{11}$$

At each subinterval $[t_{r-1},t_r]$, the initial conditions

$$u_r(x,t_{r-1})=u_{r-1}(x,t_{r-1}) \tag{12}$$

are used for $r\geq 2$ and the implementation of MsRDTM to the initial value problem on $[t_{r-1},t_r]$, where t_{r-1} replaces t_0. For $r=1,2,\dots,R$, the repetition of the process is performed and carried out to construct an approximate solutions sequence $u_r(x,t)$ such as,

$$u_r(x,t)=\sum_{k=0}^{K}U_{k,r}(x)(t-t_{r-1})^k, t\in[t_{r-1},t_r]. \tag{13}$$

Finally, the MsRDTM proposes the following solutions:

$$u(x,t)=\begin{cases} u_1(x,t), & for\ t\in[0,t_1] \\ u_2(x,t), & for\ t\in,[t_1,t_2] \\ \vdots & \\ u_R(x,t), & for\ t\in,[t_{R-1},t_R] \end{cases} \tag{14}$$

It is crucial to note that when the step size $s=T$, the RDTM is derived from MsRDTM.

3. Results

Consider the three numerical examples to show the reliability of the MsRDTM and its benefit for solving NLSE.

Example 1. Cubic NLSE of the form [38]

$$iu_t+u_{xx}+2|u|^2u=0 \tag{15}$$

is considered with initial condition

$$u(x,0)=e^{ix}. \tag{16}$$

$e^{i(x+t)}$ is the exact solution of this equation.

By applying the MsRDTM to Eq. (15) and using fundamental properties of MsRDTM, we have

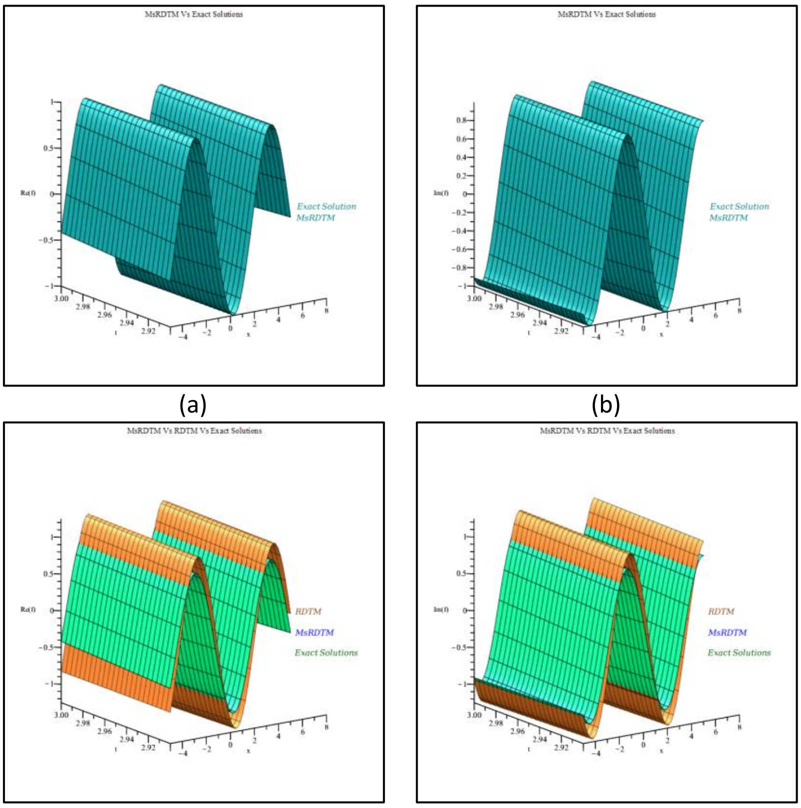
$$U_{k+1,\mathrm{r}}(x) = \left(\frac{I}{k+1}\right)\left(\frac{\partial^2}{\partial x^2}U_{k,r}(x) + 2\sum_{l=0}^{k}\overline{U}_{k-l,r}(x)\sum_{m=0}^{l}U_{m.r}(x)U_{l-m.r}(x)\right). \quad (17)$$

We write the transformed initial condition Eq. (16) as

$$U_0(x) = e^{ix}. \quad (18)$$

The multistep algorithm is then implemented to get approximate solution.

The comparison through graphical illustrations of the approximate solution MsRDTM, RDTM and exact solution for $t \in [2.9, 3.0]$ *and* $x \in [-5, 5]$, which involves the real and imaginary part, are shown in Figure 1(a), Figure 1(b), Figure 1(c), and Figure 1(d) respectively. Figure 1(a) and Figure 1(b) show that the graphs of the MsRDTM have similar shape and size with their exact solutions than the graph of RDTM as shown in Figure 1(c) and 1(d). The MsRDTM solutions for this sort of NLSE are therefore proved to be quite near to the exact solutions.



(a) (b) (c) (d)

Fig. 1. The graphs shown in (a) and (b) are the comparison between exact solutions, and the MsRDTM, while (c) and (d) are the comparison between exact solutions, the RDTM and the MsRDTM involving the real and imaginary part, respectively.

In Table 1, the performance error analysis is presented. Based on the table, numerical results for absolute error and error norms, L_2 and L_∞ from MsRDTM are smaller which mirrors that it is accurate than RDTM. MsRDTM give better results in solving the NLS equation compared to the RDTM.

Table 1
Error analysis of semi-analytic solution for RDTM and MsRDTM

T	Exact Solutions	Absolute Error RDTM	Absolute Error MsRDTM
0	0.5403023059 + 0.8414709848I	0	0
0.1	0.4535961214 + 0.8912073601I	$2.996664813 \times 10^{-10}$	$3.162277660 \times 10^{-10}$
0.2	0.3623577545 + 0.9320390860I	$2.580697580 \times 10^{-9}$	$3.605551275 \times 10^{-10}$
0.3	0.2674988286 + 0.9635581854I	$4.330415684 \times 10^{-8}$	$3.000000000 \times 10^{-10}$
0.4	0.1699671429 + 0.9854497300I	$3.246369357 \times 10^{-7}$	$5.385164807 \times 10^{-10}$
0.5	0.07073720167 + 0.9974949866I	$1.547822690 \times 10^{-6}$	$6.660330322 \times 10^{-10}$
0.6	-0.02919952230 + 0.9995736030I	$5.542108513 \times 10^{-6}$	$6.797793760 \times 10^{-10}$
0.7	-0.1288444943 + 0.9916648105I	$1.629145204 \times 10^{-5}$	$8.944271910 \times 10^{-10}$
0.8	-0.2272020947 + 0.9738476309I	$4.144872897 \times 10^{-5}$	$8.246211251 \times 10^{-10}$
0.9	-0.3232895669 + 0.9463000877I	$9.443486255 \times 10^{-5}$	$1.104536102 \times 10^{-9}$
1.0	-0.4161468365 + 0.9092974268I	$1.972130480 \times 10^{-4}$	$1.200000000 \times 10^{-9}$
	L_2	$2.232208684 \times 10^{-4}$	$2.056720691 \times 10^{-9}$
	L_∞	$1.972130480 \times 10^{-4}$	$1.200000000 \times 10^{-9}$

Example 2. NLSE with zero trapping potential was taken consideration [38]

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u = 0 \tag{19}$$

with initial condition

$$u(x,0) = e^{ix}. \tag{20}$$

$e^{i(x+\frac{t}{2})}$ is this equation's exact solution. By applying the MsRDTM to Eq. (19) and using fundamental properties of MsRDTM, we have

$$U_{k+1,\mathrm{r}}(x) = \left(\frac{I}{k+1}\right)\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}U_{k,r}(x) + \sum_{l=0}^{k}\overline{U}_{k-l,r}(x)\sum_{m=0}^{l}U_{m,r}(x)U_{l-m,r}(x)\right) \tag{21}$$

We write the transformed initial condition Eq. (20) as

$$U_0(x) = e^{ix}. \tag{22}$$

Then, apply the multistep algorithm to have approximate solution for this example.

The graphs illustrated portray the comparison between the approximate solutions MsRDTM, RDTM and exact solution for $t \in [5.5, 6.0]$ and $x \in [-5, 5]$, which involves the real and imaginary part, as shown in Figure 2(a), Figure 2(b), Figure 2(c), and Figure 2(d) respectively. Based on Figure 2(a) and Figure 2(b), it shows that the graphs of the MsRDTM are similar with their exact solutions than the graph of RDTM as shown in Figure 2(c) and Figure 2(d). The MsRDTM solutions for this sort of NLSE are therefore proved to be near to the exact solutions.

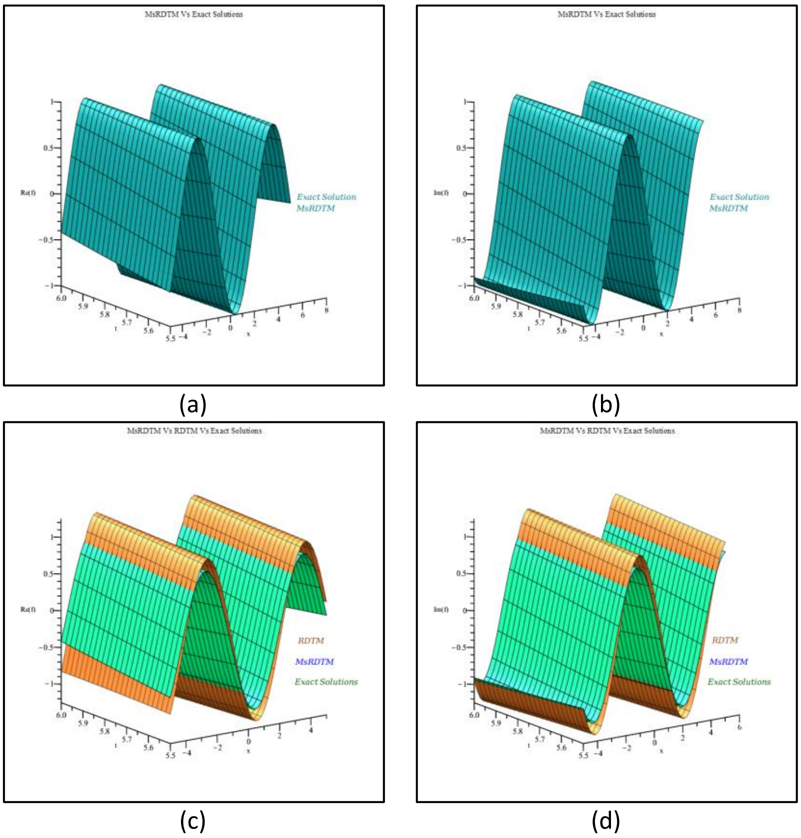


Fig. 2. The graphs shown in (a) and (b) are the comparison between exact solutions, and the MsRDTM, while (c) and (d) are the comparison the RDTM, the MsRDTM, and exact solutions which involve the real and imaginary part, respectively.

In Table 2, the performance error analysis is shown. The error analysis consists of absolute error and the error norms, L_2 and L_∞. Numerical results from MsRDTM are significantly more accurate than those of RDTM. MsRDTM give better results in solving the NLS equation compared to the RDTM.

Table 2
Error analysis of semi-analytic solution for RDTM and MsRDTM

T	Exact Solutions	Absolute Error RDTM	Absolute Error MsRDTM
0	0.5403023059 + 0.8414709848I	0	0
0.1	0.4975710479 + 0.8674232256I	$3.551056181 \times 10^{-10}$	$3.605551275 \times 10^{-10}$
0.2	0.4535961214 + 0.8912073601I	$2.996664813 \times 10^{-10}$	$5.099019514 \times 10^{-10}$
0.3	0.4084874409 + 0.9127639403I	$2.441311123 \times 10^{-10}$	$5.099019514\times 10^{-10}$
0.4	0.3623577545 + 0.9320390860I	$2.580697580 \times 10^{-9}$	$5.099019514 \times 10^{-10}$
0.5	0.3153223624 + 0.9489846194I	$1.204408569 \times 10^{-8}$	$5.385164807 \times 10^{-10}$
0.6	0.2674988286 + 0.9635581854I	$4.330415684 \times 10^{-8}$	$5.000000000 \times 10^{-10}$
0.7	0.2190066871 + 0.9757233578I	$1.276094040 \times 10^{-7}$	$5.656854249 \times 10^{-10}$
0.8	0.1699671429 + 0.9854497300I	$3.246369357 \times 10^{-7}$	$6.403124237 \times 10^{-10}$
0.9	0.1205027694 + 0.9927129910I	$7.404440897 \times 10^{-7}$	$7.071067812 \times 10^{-10}$
1.0	0.07073720167 + 0.9974949866I	$1.547822690 \times 10^{-6}$	$8.848163651 \times 10^{-10}$
	L_2	$1.751488817 \times 10^{-6}$	$7.277561405 \times 10^{-9}$
	L_∞	$1.547822690 \times 10^{-6}$	$8.848163651 \times 10^{-10}$

Example 3. We considered NLSE with trapping potential of the form [38]

$$iu_t + \frac{1}{2}u_{xx} - ucos^2(x) - |u|^2 u = 0 \tag{23}$$

with initial condition

$$u(x,0) = \sin(x). \tag{24}$$

$\sin(x)e^{(-\frac{3i}{2}t)}$ is this equation's exact solution. By applying the MsRDTM to Eq. (23) and using fundamental properties of MsRDTM, we have

$$U_{k+1,\mathrm{r}}(x) = \left(\frac{I}{k+1}\right)\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}U_{k,r}(x) - U_{k,r}(x)cos^2(x) - \sum_{l=0}^{k}\overline{U}_{k-\mathrm{l},r}(x)\sum_{m=0}^{l}U_{m,r}(x)U_{l-m,r}(x)\right) \tag{25}$$

We write the transformed initial condition Eq. (24) as

$$U_0(x) = \sin(x). \tag{26}$$

The multistep algorithm is then applied to get accurate approximate solution for this example. The comparison by graphical pictorials of approximate solutions MsRDTM, RDTM and exact solution for $t \in [0, 3.5]$ and $x \in [-3.5, 3.5]$, with piecewise solution of $u(x,t)$ for MsRDTM and $R = 7$ subintervals which involve the real and imaginary part, are shown in Figure 3(a), Figure 3(b), Figure 3(c) and Figure 3(d), respectively. Figure 3(a) and Figure 3(b) shows that the graphs of the MsRDTM are similar with their exact solutions than the graph of RDTM as shown in Figure 3(c) and Figure 3(d). The MsRDTM solutions for this form of NLSE are therefore proved to be significantly near to the exact solutions.

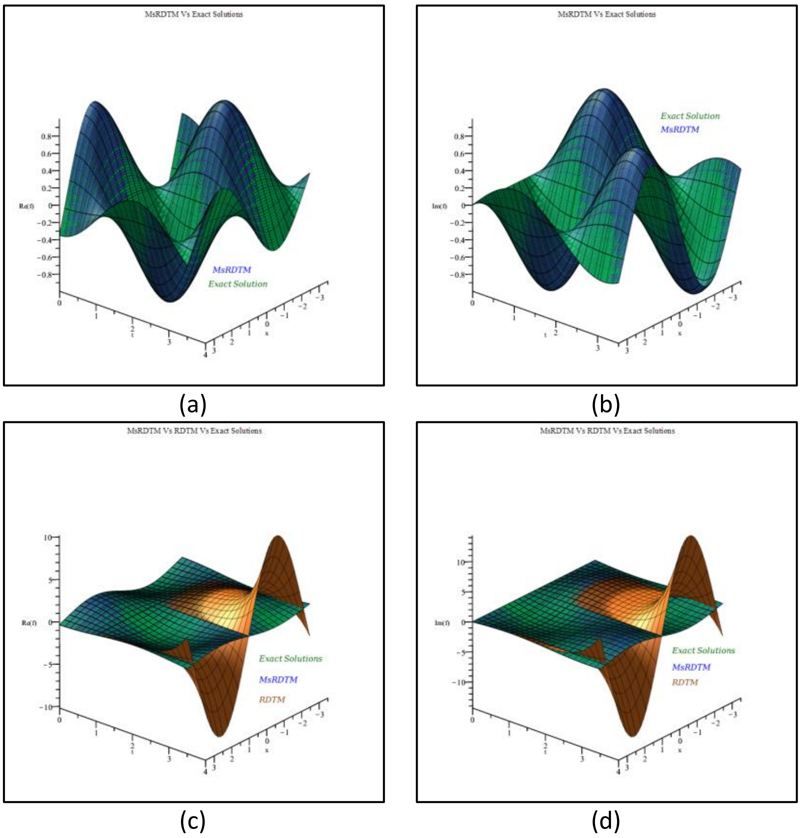


Fig. 3. The graphs shown in (a) and (b) are the exact solutions, and MsRDTM, while (c) and (d) are the exact solutions, the RDTM, and the MsRDTM which involve the real and imaginary part, respectively.

In Table 3, the performance error analysis is presented. Absolute error and error norms, L_2 and L_∞ have been used to elucidate the error analysis. Numerical results from MsRDTM are consequentially more accurate than those of RDTM. MsRDTM give better results in solving the NLS equation compared to the RDTM.

Table 3
Error analysis of semi-analytic solution for RDTM and MsRDTM

T	Exact Solutions	Absolute Error RDTM	Absolute Error MsRDTM
0	0.8414709848	0	0
0.1	0.8320221727 - 0.1257478525I	$2.660964897 \times 10^{-10}$	$2.714758068 \times 10^{-10}$
0.2	0.8038879363 - 0.2486716794I	$3.644744902 \times 10^{-8}$	$7.238607115 \times 10^{-10}$
0.3	0.7577001100 - 0.3660108763I	$6.230741499 \times 10^{-7}$	$1.077608651 \times 10^{-9}$
0.4	0.6944959727 - 0.4751302582I	$4.663560160 \times 10^{-6}$	$1.250936551 \times 10^{-9}$
0.5	0.6156949531 - 0.5735792387I	$2.221039714 \times 10^{-5}$	$1.551595830 \times 10^{-9}$
0.6	0.5230667522 - 0.6591468660I	$7.946428659 \times 10^{-5}$	$1.775084987 \times 10^{-9}$
0.7	0.4186915997 - 0.7299114759I	$2.333622902 \times 10^{-4}$	$1.957211586 \times 10^{-9}$
0.8	0.3049135365 - 0.7842838476I	$5.930457850 \times 10^{-4}$	$2.208761158 \times 10^{-9}$
0.9	0.1842877727 - 0.8210428948I	$1.349438602 \times 10^{-3}$	$2.428631626 \times 10^{-9}$
1.0	0.05952330275 - 0.8393630887I	$2.814081292 \times 10^{-3}$	$2.714758068 \times 10^{-9}$
	L_2	$3.186381856 \times 10^{-3}$	$5.555141621 \times 10^{-9}$
	L_∞	$2.814081292 \times 10^{-3}$	$2.714758068 \times 10^{-9}$

4. Conclusions

This work uses the MsRDTM approximation analytical method for dealing with NLSEs. The findings demonstrate the accuracy, effectiveness, and dependability of the procedure, as shown by the outcomes and the graphical representations. Moreover, MsRDTM is a valuable mathematical approach for dealing with nonlinear Schrodinger equations since it produces solutions with high accuracy, and it is significantly more accurate than RDTM. This paper's calculations were all performed using Maple 2021.

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