Journal of Advanced Research in Applied Sciences and Engineering Technology

Journal homepage:
https://semarakilmu.com.my/journals/index.php/applied_sciences_eng_tech/index
ISSN: 2462-1943

Numerical Solution of the Porous Medium Equation with Source Terms Using the Four-Point Newton-EGKSOR Iterative Method Combined with Wave Variable Transformation

Graygorry Brayone Ekal[1,*], Jackel Vui Lung Chew[2], Khadizah Ghazali[1], Jumat Sulaiman[1]

[1] Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Kota Kinabalu, Sabah 88400, Malaysia
[2] Faculty of Computing and Informatics, Universiti Malaysia Sabah Labuan International Campus, 87000 Labuan F.T., Malaysia

ABSTRACT

The porous medium equation with source terms (PMES) is a nonlinear degenerate parabolic equation and serves as a model of many physical phenomena. To obtain its exact solution is a difficult task; hence it is necessary to find the approximate solution for the equation. In this paper, we proposed the 4-point Newton-Explicit Group Kaudd-SOR (4NEGKSOR) iterative method combined with the similarity transformation to solve the PMES numerically and obtain its approximate solution. The similarity transformation will be used to reduce the PMES into an ordinary differential equation, and we discretized the reduced form of the PMES using the finite difference scheme. Further, the processes for generating an approximation solution of the PMES proceeded via the 4NEGKSOR, and its formulation is derived. Moreover, the proposed method was tested with some numerical experiments to verify its effectiveness against existing iterative methods, i.e., the Newton-Gauss Seidal (NGS) and the Newton-Kaudd SOR (NKSOR). Based on the obtained results, the 4NEGKSOR iterative method proposed in this work is more efficient in getting the converged solution of the PMES compared to NGS and NKSOR iterative methods.

Keywords:
Porous medium equation; source term; explicit group; similarity transformation; Kaudd successive over-relaxation

1. Introduction

The nonlinear heat equation, usually called the porous medium equation (PME), is essential in developing a nonlinear analysis as it provides a number of interesting applications in modeling real-world problems. For instance, it is used to describe a groundwater flow, the flow of an ideal gas in a homogeneous porous medium, and heat propagation [1]. The complete form of the PME is referred to as the porous medium equation with source terms (PMES) [2]. The PMES is also useful in modeling realistic problems, such as representing population pressure in biological systems, modeling the unsteady heat transfer in the quiescent medium [3] describing a more realistic population dynamics modeling and wound healing process [4]. Hence, a solution to the PME, particularly the PMES, is needed in order to understand the nature of the related problems better. However, finding an

* *Corresponding author.*
E-mail address: graygorrybrayone@gmail.com

https://doi.org/10.37934/araset.60.2.3948

accurate and efficient solution is the main issue, as the PMES is cursed with nonlinearity [5]. Hence, an approximate solution to such equations is necessary to address those issues. Several methods, such as analytical dan numerical methods for solving the PMES using various approaches, have been suggested.

Estévez *et al.,* [6] investigated the functional separation of variables of the generalized porous medium equations with nonlinear source terms via the conditional symmetry method. On the basis of their results, they were able to deduce several exact solutions to the desired equation, including blow-up solutions, time-periodic solutions, and global solutions. Then, Pamuk, [7] utilized an analytical method, namely the Adomian decomposition method (ADM), to construct a series solution of the PMES that rapidly converges to the exact solution. In addition, Sari, [8] provided a numerical solution for the PMES by combining the compact finite difference method with a total variation diminishing third-order Runge-Kutta scheme. The proposed method is advantageous since it requires less storage space and solves PMES more precisely. Moreover, Biazar *et al.,* [9] adopted the homotopy perturbation technique (HPM) to solve the PMES and concluded that the HPM solution was comparable to the ADM solution. Next, Antar and Pamuk, [10] developed a novel technique for selecting a more accurate approximation of the trial function for the HPM to solve nonlinear problems, resulting in minimal computational load and time. Then, Saberi Nik *et al.,* [11] integrated the HPM with He's polynomial in order to solve the PMES. Chew *et al.,* [4] combine the quarter-sweep finite difference scheme with the modified sequential over-relaxation iterative method to approximate the PMES solution. The authors proved that their technique produced optimal solutions with fewer iterations and in less time.

Motivated by the work done by the previous researchers in solving the PMES problem and its applications, we propose the Newton-MKSOR iterative method for solving the PMES, we attempt to introduce a numerical method to solve the PMES using the Four-Point Newton-Explicit Group KSOR (4NEGKSOR) iterative method based on the wave variable transformation. The wave variable transformation is the general form of the similarity transformation that converts a partial differential equation into an ordinary differential equation [12]. Hence, we are able to reduce the PMES into a nonlinear second-order ordinary differential equation. Multiple researchers have studied the theory of travelling waves, particularly for PMES, in order to demonstrate the presence of travelling waves in such equations, as demonstrated in the reference terrain, [13, 14].

Additionally, other authors also presented other investigations of travelling waves analogous to PMES [11, 14-18]. Therefore, the PMES gives solutions based on travelling waves, allowing us to convert the Equation into a travelling wave equation. This reduction from two dependent variables to a single dependent variable will substantially cut computing costs. Then, we adopt the implicit finite difference method to discretize the transformed PMES in order to generate its approximation equation. This procedure returns a system of nonlinear equations. To address the nonlinearity, we employed the Newton method to linearize the nonlinear system into a simpler-to-solve linear system of equations. The proposed method was highly inspired by the method proposed by [20] to solve linear partial differential equations. Thereby, we extended their work to a nonlinear partial differential equation (PDE) that suffers from nonlinearity. The next section discussed the development of the approximate Equation for the PMES using the finite difference scheme. Then, we presented the formulation of the 4NEGKSOR iterative method to solve the PMES. The experiment's numerical findings will then be presented and analyzed. This paper concludes with a conclusion on our research.

2. Development of the Finite Difference Approximation Equation

The 1-dimensional porous medium equation with source terms is defined as [9]:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial t}\left(u^m \frac{\partial u}{\partial x}\right) + bu^r \tag{1}$$

such that a and b are constants, whereas m and r are known rational numbers. Usually, x will be treated as a spatial variable while t as a temporal variable. For convenience, we will only consider $x \in [0,1]$ and $t > 0$ in this study. To begin, we expand Eq. (1) into Eq. (2).

$$\frac{\partial u}{\partial t} = au^m \frac{\partial}{\partial t}\left(u^m \frac{\partial u}{\partial x}\right) + bu^r \tag{2}$$

Then, Eq. (2) will be simplified from $u(x,t)$ into $u(\xi)$ to utilising the wave variable transformation, which is dependent on only one variable. Then, Equation (2) will be reduced from $u(x,t)$ into $u(\xi)$ by using the wave variable transformation, which depends only on one variable. The traveling wave variable ξ is given as $\xi = x - ct$, where c is a constant representing the wave speed [12, 15]. Furthermore, we use the chain rule on ξ to get the appropriate changes for u_t, u_x, and u_{xx} with respect to ξ, namely, $u_t = -cu'$, $u_x = u'$ and $u_{xx} = u''$. Substituting these changes into Eq. (2) and thus, the equivalent traveling wave equation for PMES can be stated as:

$$-c\frac{du}{d\xi} = au^m \frac{d^2u}{d\xi^2} + amu^{m-1}\left(\frac{du}{d\xi}\right)^2 + bu^r, \tag{3}$$

where $-ct \le \xi \le 1 - ct$ with $t > 0$. Here, we are now dealing with an ODE instead of a PDE. Eq. (3) is discretized using the second-order central difference scheme given as [21, 22]:

$$u'(\xi_i) = \frac{u_{i+1} - u_{i-1}}{2h}, \tag{4}$$

$$u''(\xi_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \tag{5}$$

for $i = 1,2,3,\cdots,(M-1)$, where $h = ((1-ct) - (-ct))/M$ and M being the subinterval in ξ direction. Figure 1 illustrates the finite grid network described by Eq. (3).

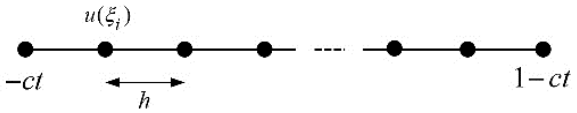

Fig. 1. Uniform node points for the solution domain for Eq. (1)

Substituting Eq. (4) and Eq. (5) into Eq. (3) will result in the Eq. (6), which is the finite difference approximation Eq. (2).

$$-c\left(\frac{u_{i+1} - u_{i-1}}{2h}\right) = au_i^m\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + amu_i^{m-1}\left(\frac{u_{i+1} - u_{i-1}}{2h}\right)^2 + bu_i^r, \tag{6}$$

Furthermore, we can rewrite Eq. (6) into Eq. (7):

$$\alpha(u_{i+1}-u_{i-1})=\beta u_i^m(u_{i+1}-2u_i+u_{i-1})+\gamma m u_i^{m-1}(u_{i+1}-u_{i-1})^2+bu_i^r\,, \tag{7}$$

where $\alpha=-c/2h$, $\beta=a/h^2$, and $\gamma=a/4h^2$. Next, to form a system of equations, we rearrange the Eq. (7) into a nonlinear function of,

$$f(u)=\alpha(u_{i+1}-u_{i-1})-\beta u_i^m(u_{i+1}-2u_i+u_{i-1})-\gamma m u_i^{m-1}(u_{i+1}-u_{i-1})^2-bu_i^r\,, \tag{8}$$

and applied the Newton method [23] to generate its corresponding linear system, which can be expressed as:

$$J\big(u^{(k)}\big)\delta u^{(k)}=-f\big(u^{(k)}\big)\,, \tag{9}$$

where $u=(u_1,u_2,\cdots,u_{M-1})^t$, $\delta u=(\delta u_1,\delta u_2,\cdots,\delta u_{M-1})^t$, $\boldsymbol{f}=(f_1,f_2,\cdots,f_{M-1})^t$, and k is the index of iteration. The J notation in Eq. (9) is referred to as the Jacobian matrix and is defined as:

$$J\big(u^{(k)}\big)=\begin{pmatrix}\frac{\partial f_1(u)}{\partial u_1} & \frac{\partial f_1(u)}{\partial u_2} & \cdots & \frac{\partial f_1(u)}{\partial u_{M-1}}\\ \frac{\partial f_2(u)}{\partial u_1} & \frac{\partial f_2(u)}{\partial u_2} & \cdots & \frac{\partial f_2(u)}{\partial u_{M-1}}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_{M-1}(u)}{\partial u_1} & \frac{\partial f_{M-1}(u)}{\partial u_2} & \cdots & \frac{\partial f_{M-1}(u)}{\partial u_{M-1}}\end{pmatrix}^{(k)}_{(M-1)\times(M-1)} \tag{10}$$

Finally, by solving the system of linear equation in Eq. (9), we will obtain the value of $\delta u^{(k)}$ that permits us to compute the approximate solutions $u^{(k)}$ using the formula:

$$u^{(k+1)}=\delta u^{(k)}+u^{(k)} \tag{11}$$

3. Implementation of the 4-Point Newton-KSOR Method

The Jacobian matrix in Eq. (9) is a large-scale tridiagonal matrix subjected to the value of M; hence the matrix is sparse. Therefore, we utilized the iterative approach to take full advantage of this sparsity as its implementation in computer programming becomes simpler. In this section, we described the development of the 4EGKSOR as our linear solver to Eq. (9). Recall that the formulation of the KSOR iterative method is given by [24]:

$$x_i^{(k+1)}=\frac{1}{1+\omega}\left[x_i^{(k)}+\frac{\omega}{a_{ii}}\left(b_i-\sum_{j=1}^{i-1}a_{ij}x_j^{(k+1)}-\sum_{j=i+1}^{n}a_{ij}x_j^{(k)}\right)\right] \tag{12}$$

where a_{ij} is the matrix entries of the matrix A in some systems of linear equations, say $A\boldsymbol{x}=b$, and $b=(b_1,b_2,\cdots,b_n)^t$. The ω notation is the relaxation parameter that takes values on $\mathbb{R}\setminus[-2,0]$. Then, to facilitate the iteration process, we employ the 4–Point Explicit Group iterative method that was proposed by Evans to solve a large sparse linear system [25]. The 4-point Explicit Group iterative method is constructed based on the linear system Eq. (8), and the solution domain will have several completed groups of four-points and ungroup points. In particular, the ungrouped points are handled

using a three-points iteration scheme [25]. Prior to that, the Jacobian matrix in Eq. 8 can be expressed as:

$$J\left(u^{(k)}\right)=\begin{pmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{M-2} & b_{M-2} & c_{M-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{M-1} & b_{M-1} \end{pmatrix}^{(k)}_{(M-1)\times(M-1)} \tag{13}$$

where $a_i=\partial f_i/\partial u_{i-1}$, $b_i=\partial f_i/\partial u_i$, and $c_i=\partial f_i/\partial u_{i+1}$ for $i=1,2,3,\cdots,(M-1)$. Now, the 4-point EG iterative method for the completed group is given by the following equation [26]:

$$\begin{pmatrix} \delta u_p \\ \delta u_{p+1} \\ \delta u_{p+2} \\ \delta u_{p+3} \end{pmatrix}^{(k+1)} = \begin{pmatrix} b_p & c_p & 0 & 0 \\ a_{p+1} & b_{p+1} & c_{p+1} & 0 \\ 0 & a_{p+2} & b_{p+2} & c_{p+2} \\ 0 & 0 & a_{p+3} & b_{p+3} \end{pmatrix}^{-1} \begin{pmatrix} s_p \\ s_{p+1} \\ s_{p+2} \\ s_{p+3} \end{pmatrix} \tag{14}$$

where $p=1,5,\cdots,(M-7)$, $s_p=-f_p-a_p\partial u_{p-1}^{(k)}$, $s_{p+1}=-f_{p+1}$, $s_{p+1}=-f_{p+1}$, $s_{p+2}=-f_{p+2}$, and $s_{p+3}=-f_{p+3}-a_{p+3}\delta u_{p+4}^{(k)}$. To solve the ungroup points, we consider the following iterative method:

$$\begin{pmatrix} \delta u_p \\ \delta u_{p+1} \\ \delta u_{p+2} \end{pmatrix}^{(k+1)} = \begin{pmatrix} b_p & c_p & 0 \\ a_{p+1} & b_{p+1} & c_{p+1} \\ 0 & a_{p+2} & b_{p+2} \end{pmatrix}^{-1} \begin{pmatrix} s_p \\ s_{p+1} \\ s_{p+2} \end{pmatrix} \tag{15}$$

where $p=(M-3)$, $s_p=-f_p-a_p\delta u_{p-1}^{(k)}$, $s_{p+1}=-f_{p+1}$, and $s_{p+2}=-f_{p+2}-c_{p+2}\delta u_{p+3}^{(k)}$. As a result, the 4-point Explicit Group KSOR iterative method may be derived from Eq. (12), and the formulation of the 4-point Explicit Group iterative method as:

$$\begin{pmatrix} \delta u_p \\ \delta u_{p+1} \\ \delta u_{p+2} \\ \delta u_{p+3} \end{pmatrix}^{(k+1)} = \frac{1}{1+\omega}\begin{pmatrix} \delta u_p \\ \delta u_{p+1} \\ \delta u_{p+2} \\ \delta u_{p+3} \end{pmatrix}^{(k)} + \frac{\omega}{1+\omega}\begin{pmatrix} b_p & c_p & 0 & 0 \\ a_{p+1} & b_{p+1} & c_{p+1} & 0 \\ 0 & a_{p+2} & b_{p+2} & c_{p+2} \\ 0 & 0 & a_{p+3} & b_{p+3} \end{pmatrix}^{-1} \begin{pmatrix} s_p \\ s_{p+1} \\ s_{p+2} \\ s_{p+3} \end{pmatrix} \tag{16}$$

for $p=1,5,\cdots,(M-7)$ and for $p=(M-3)$.

$$\begin{pmatrix} \delta u_p \\ \delta u_{p+1} \\ \delta u_{p+2} \end{pmatrix}^{(k+1)} = \frac{1}{1+\omega}\begin{pmatrix} \delta u_p \\ \delta u_{p+1} \\ \delta u_{p+2} \end{pmatrix}^{(k)} + \frac{\omega}{1+\omega}\begin{pmatrix} b_p & c_p & 0 \\ a_{p+1} & b_{p+1} & c_{p+1} \\ 0 & a_{p+2} & b_{p+2} \end{pmatrix}^{-1} \begin{pmatrix} s_p \\ s_{p+1} \\ s_{p+2} \end{pmatrix} \tag{17}$$

In addition, we presented the general algorithm for the 4-point Newton-Explicit Group KSOR iterative method to solve the PMES as follows.

Algorithm: 4-point Newton-Explicit Group KSOR iterative method

i. Set the value of $u^{(0)} = [-ct + (1 - ct)]/2$, $\varepsilon_u = 10^{-10}$, $\varepsilon_{\delta u} = 10^{-10}$.
ii. Set $\delta u^{(0)} = 0$.
iii. Calculate J and $-f$.
iv. Iterate the Eq. (16) and Eq. (17).
v. Check whether $\left\|\delta u^{(k+1)} - \delta u^{(k)}\right\| \leq \varepsilon_{\delta u}$. If true, then proceed to step vi. Else, repeat step ii.
vi. Calculate $u^{(k+1)} = \delta u^{(k)} + u^{(k)}$.
vii. Check whether $\left\|u^{(k+1)} - u^{(k)}\right\| \leq \varepsilon_u$. If true, then display approximate solutions, $\boldsymbol{u}$. Else, repeat step i until iv.

The optimum value of the relaxation parameter ω is estimated by running a computer programming with multiple values of ω until the smallest iteration is achieved.

4. Numerical Experiments

In this section, we analyze the performance of the 4-Point Newton-EG KSOR (4NEGKSOR) iterative methods by comparing it to the Newton-Gauss Seidel (NGS) and Newton-KSOR (NKSOR) iterative methods, which act as benchmarks for solving PMES. Four PMES problems are considered, and the number of iterations, the execution time, and the maximum absolute error required by each method to get numerical results are recorded. As for the tolerance error, $\varepsilon = 10^{-10}$ is utilized to determine the convergence of the solution at various matrix sizes, i.e., $M = 256, 512, 1024, 2048$ and 4096. The following are the problems of interest.

Example 1 [14]
Consider $m = -1$ and $r = 2$ in Eq. (1). Thus, the Equation becomes:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-1}\frac{\partial u}{\partial x}\right) + u^2 \tag{18}$$

which is a quasilinear fast diffusion equation with a quadratic reaction term. A solution for this equation which was provided by Polyanin and Zaitsev [14] was adopted for accuracy checking, namely:

$$u(x,t) = \left(\frac{(x+C_1)^2}{2t} + C_2 t - 2t\ln|t|\right)^{-1}, t \neq 0, \tag{19}$$

where c_1 and c_2 are arbitrary constants. In this implementation, c_1 and c_2 have been set to 0.35 and 1.35, respectively.

Example 2 [14]
Consider $m = 3$ and $r = 1$ in Eq. (1), which gives us an equation:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^3\frac{\partial u}{\partial x}\right) - bu \tag{20}$$

that describes the classical case of a gravity current in air. Polyanin and Zaitsev [14] gave a particular solution for this Equation which is

$$u(x,t) = \left(Ae^{\frac{2bmt}{m+2}} - \frac{bm^2(x+B)^2}{2a(m+2)}\right)^{\frac{1}{m}}, \tag{21}$$

where A and B are arbitrary constants. For this problem, we set A, a, B and b to be 1.35, 1, 0.35, and 1, respectively.

Example 3 [7]
Taking $m = 1$ and $r = 0$ in Eq. (1) then, it becomes:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + b \tag{22}$$

which is the heat conduction equation with source, and $u(x,t) = x + (a+b)t$ is the given exact solution. We set a and b to be 1 and –3, respectively.

Example 4 [4]
Let us take $m = 2$ and $r = 1$ in Eq. (1). Therefore Eq. (1) becomes:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right) - u \tag{23}$$

which represents a slow diffusion of particles on a fresh membrane. The solution for this problem provided by Chew *et al.* was utilized to verify the accuracy of the numerical solutions, which is

$$u(x,t) = \left(x - \frac{2}{\beta}\right)e^{\beta t} + \left(\frac{2}{\beta}\right)e^{2\beta t} \tag{24}$$

where β is an arbitrary constant that has been set to -1.

The numerical computations were executed on a laptop computer with an Intel(R) Core i7-6500U processor clocked at 2.60GHz and 8 GB of RAM, using C code generated by Visual Studio Code. The numerical results of this experiment are tabulated in Table 1, Table 2, and Table 3. Then the summarization of the finding is shown in Table 3.

The collected results in Tables 1, 2, and 3 show that the 4-Point Newton-EGKSOR iterative method required much less iteration and computation time than the N-GS iterative method. The reduction in the number of iterations and computation time (in seconds), referred to in Table 3, is about 99.13% - 99.91%, and 97.40% - 99.86%, respectively. Similarly, in Table 3, we can show that the 4-Point Newton-EGKSOR required fewer iterations and less calculation time against Newton-KSOR, which is about 51.18% - 77.06% and 16.00% - 80.25%, respectively. In terms of accuracy, all methods have good agreement. However, the 4-Point Newton-EGKSOR and Newton-KSOR iterative methods are slightly accurate compared to NGS.

Table 1
The iterations count (k), computational time (t), and maximum absolute error (ε) generated by NGS, NKSOR, and 4NEGKSOR at different grid sizes for Examples 1 and 2

M	Method	Example 1			Example 2		
		k	*t* (sec)	ε	*k*	*t* (sec)	ε
256	NGS	165519	0.75	1.8667E-04	289575	1.25	1.6586E-03
	NKSOR	2975 $\omega = -2.0240$	0.02	1.8709E-04	4346 $\omega = -2.0260$	0.03	1.6583E-03
	4NEGKSOR	1399 $\omega = -2.0462$	0.02	1.8709E-04	2079 $\omega = -2.0260$	0.02	1.6583E-03
512	NGS	547778	4.63	1.8766E-04	992710	8.33	1.6599E-03
	NKSOR	12039 $\omega = -2.0120$	0.16	1.8705E-04	8630 $\omega = -2.0130$	0.12	1.6584E-03
	4NEGKSOR	2762 $\omega = -2.0231$	0.03	1.8707E-04	4076 $\omega = -2.0260$	0.05	1.6584E-03
1024	NGS	1730412	28.93	1.9423E-04	3319955	55.51	1.6644E-03
	NKSOR	11667 $\omega = -2.0061$	0.30	1.8706E-04	32772 $\omega = -2.0065$	0.85	1.6584E-03
	4NEGKSOR	5420 $\omega = -2.0117$	0.16	1.8706E-04	8035 $\omega = -2.0260$	0.22	1.6584E-03
2048	NGS	5121184	171.14	2.2096E-04	10749133	360.14	1.6880E-03
	NKSOR	23221 $\omega = -2.0032$	1.15	1.8706E-04 1.8706E-04	34071 $\omega = -2.0033$	1.71	1.6584E-03
	4NEGKSOR	10709 $\omega = -2.0059$	0.55		15779 $\omega = -2.0260$	0.81	1.6584E-03
4096	NGS	13829405	925.35	3.3397E-04	33849665	2263.30	1.7731E-03
	NKSOR	45132 $\omega = -2.0016$	4.48	1.8706E-04 1.8706E-04	68403 $\omega = -2.0017$	7.46	1.6584E-03
	4NEGKSOR	21024 $\omega = -2.0030$	2.11		31028 $\omega = -2.0260$	3.11	1.6584E-03

Table 2
The iterations count (k), computational time (t), and maximum absolute error (ε) generated by NGS, NKSOR, and 4NEGKSOR at different grid sizes for Examples 3 and 4

M	Method	Example 3			Example 4		
		k	*t* (sec)	ε	*k*	*t* (sec)	ε
256	NGS	228186	0.97	6.0643E-07	189790	0.81	4.5050E-03
	NKSOR	3224 $\omega = -2.0255$	0.02	1.5739E-10	3433 $\omega = -2.0300$	0.03	4.5052E-03
	4NEGKSOR	1574 $\omega = -2.0518$	0.01	9.5551E-11	1649 $\omega = -2.0618$	0.02	4.5052E-03
512	NGS	779725	6.55	2.4306E-06	649967	5.54	4.5056E-03
	NKSOR	6424 $\omega = -2.0128$	0.09	1.7851E-10	6847 $\omega = -2.0150$	0.10	4.5052E-03
	4NEGKSOR	3118 $\omega = -2.0258$	0.06	1.1923E-10	3256 $\omega = -2.0306$	0.05	4.5052E-03
1024	NGS	2599500	43.52	9.9409E-06	2217716	38.24	4.5093E-03
	NKSOR	12782 $\omega = -2.0065$	0.33	2.3041E-10	25621 $\omega = -2.0038$	0.65	4.5052E-03
	4NEGKSOR	6186 $\omega = -2.0129$	0.16	1.3166E-10	6386 $\omega = -2.0154$	0.17	4.5052E-03
2048	NGS	8416693	281.10	4.4445E-05	7382980	246.89	4.5241E-03
	NKSOR	25327 $\omega = -2.0033$	1.28	3.0697E-10	27093 $\omega = -2.0038$	1.35	4.5052E-03
	4NEGKSOR	12222	0.60	1.9517E-10	12533	0.64	4.5052E-03

Table 2. Continued
The iterations count (k), computational time (t), and maximum absolute error (ε) generated by NGS, NKSOR, and 4NEGKSOR at different grid sizes for Examples 3 and 4

M	Method	Example 3			Example 4		
		k	*t* (sec)	ε	*k*	*t* (sec)	ε
		$\omega = -2.0065$			$\omega = -2.0077$		
4096	NGS	26872893	1802.07	1.5823E-04	23636169	1583.74	4.5851E-03
	NKSOR	50082	4.91	5.4396E-10	53123	5.21	4.5052E-03
		$\omega = -2.0017$			$\omega = -2.0020$		
	4NEGKSOR	24144	2.56	3.2026E-10	24584	2.93	4.5052E-03
		$\omega = -2.0033$			$\omega = -2.0039$		

Table 3
The reduction percentage by the 4NEGKSOR against NGS and NKSOR

Example	Method	*K* (%)	Seconds *t* (%)
1	NGS	99.15 - 99.85	97.99 - 99.77
	NKSOR	52.97 - 77.06	37.50 - 80.25
2	NGS	99.28 - 99.91	98.80 - 99.86
	NKSOR	52.16 - 75.48	52.58 - 73.76
3	NGS	99.31 - 99.91	98.97 - 99.86
	NKSOR	51.18 - 51.79	36.26 - 58.33
4	NGS	99.13 - 99.90	97.40 - 99.82
	NKSOR	51.97 - 75.08	16.00 - 73.64

4. Conclusions

In this study, the 4-Point Newton-Explicit Group KSOR (4N-EGKSOR) based on wave variable transformation is considered to solve porous medium equations with source terms (PMES). The integration of the wave transformation in the proposed method allows us to transform the original PMES into the traveling wave equation, which only depends on the spatial variable. Therefore, we do not have to calculate the approximate solution at each time step, hence minimizing the computational time. Then, the performance of the proposed method was examined by solving different PMES problems, and it was found, based on the numerical results, that the method is effective and accurate for handling such problems. Thus, the proposed method can provide a promising alternative way to approximate solutions for nonlinear parabolic partial differential equations that admit traveling wave solutions.

Acknowledgement
The authors are grateful for the fund received from Universiti Malaysia Sabah upon publication of this paper (GUG0624-1/2023).

References
[1] Chew, Jackel Vui Lung, Jumat Sulaiman, and Andang Sunarto. "Solving one-dimensional porous medium equation using unconditionally stable half-sweep finite difference and SOR method." *Mathematics and Statistics* 9, no. 2 (2021): 166-171. https://doi.org/10.13189/ms.2021.090211
[2] Vázquez, Juan Luis. *The porous medium equation: Mathematical theory*. Clarendon Press, 2006. https://doi.org/10.1093/acprof:oso/9780198569039.001.0001
[3] Fang, Zhong Bo, and Mei Wang. "Extinction properties of solutions for a fast diffusion equation with nonlocal source." *Boundary Value Problems* 2013 (2013): 1-15. https://doi.org/10.1186/1687-2770-2013-266
[4] Chew, Jackel Vui Lung, Jumat Sulaiman, Elayaraja Aruchunan, and Andang Sunarto. "Efficient iterative approximation for nonlinear porous medium equation with drainage model." In *Intelligent Systems Modeling and*

Simulation II: Machine Learning, Neural Networks, Efficient Numerical Algorithm and Statistical Methods, pp. 291-303. Cham: Springer International Publishing, 2022. https://doi.org/10.1007/978-3-031-04028-3_20

[5] Nasir, Sharifah Noha Zahirah Syed Abdul, Nurul Ain Ab Wahab, and Mohd Agos Salim Nasir. "Graphical user interface for solving non-linear equations for undergraduate students." *International Journal of Advanced Research in Future Ready Learning and Education* 30, no. 1 (2023): 25-34. https://doi.org/10.37934/frle.30.1.2534

[6] Estevez, P. G., Changzheng Qu, and Shunli Zhang. "Separation of variables of a generalized porous medium equation with nonlinear source." *Journal of mathematical analysis and applications* 275, no. 1 (2002): 44-59. https://doi.org/10.1016/S0022-247X(02)00214-7

[7] Pamuk, Nevin. "Series solution for porous medium equation with a source term by Adomian's decomposition method." *Applied mathematics and computation* 178, no. 2 (2006): 480-485. https://doi.org/10.1016/j.amc.2005.11.063

[8] Sari, Murat. "Solution of the porous media equation by a compact finite difference method." *Mathematical Problems in Engineering* 2009, no. 1 (2009): 912541. https://doi.org/10.1155/2009/912541

[9] Biazar, Jafar, Zainab Ayati, and Hamideh Ebrahimi. "Homotopy perturbation method for general form of porous medium equation." *Journal of Porous Media* 12, no. 11 (2009). https://doi.org/10.1615/JPorMedia.v12.i11.80

[10] Antar, Nalan, and Nevin Pamuk. "A new approach to the trial function for Homotopy Perturbation Method for nonlinear problems." *Elektronik Mesleki Gelişim Ve Araştırmalar Dergisi* 1, no. 2 (2013): 1-10.

[11] Saberi Nik, Hassan, M. S. Zahedi, Reza Buzhabadi, and Sohrab Effati. "Homotopy perturbation method and He's polynomials for solving the porous media equation." *Computational Mathematics and Modeling* 24 (2013): 279-292. https://doi.org/10.1007/s10598-013-9177-y

[12] Logan, J. David. *An introduction to nonlinear partial differential equations*. Vol. 89. John Wiley & Sons, 2008.

[13] Xu, T., and J. Yin. "Traveling waves in degenerate diffusion equations, Commun." *Math. Res* 39 (2023): 36-53. https://doi.org/10.4208/cmr.2021-0102

[14] Andrei, D. Polyanin, and Valentin F. Zaitsev. (2012). *Handbook of nonlinear partial differential equations*, Second Edition. Chapman & Hall/CRC Press, 2012.

[15] Ghazaryan, Anna R., Stéphane Lafortune, and Vahagn Manukian. *Introduction to Traveling Waves*. Chapman and Hall/CRC, 2022. https://doi.org/10.1201/9781003147619

[16] Bibi, Sadaf, and Syed Tauseef Mohyud-Din. "Traveling wave solutions of KdVs using sine–cosine method." *Journal of the Association of Arab Universities for Basic and Applied Sciences* 15 (2014): 90-93. https://doi.org/10.1016/j.jaubas.2013.03.006

[17] Ngamsaad, Waipot, and Kannika Khompurngson. "Self-similar solutions to a density-dependent reaction-diffusion model." *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics* 85, no. 6 (2012): 066120. https://doi.org/10.1103/PhysRevE.85.066120

[18] Mansour, Mahmoud Bakheet Ali. "Travelling wave solutions for doubly degenerate reaction–diffusion equations." *The ANZIAM Journal* 52, no. 1 (2010): 101-109. https://doi.org/10.1017/S144618111100054X

[19] Mansour, M. B. A. "Accurate computation of traveling wave solutions of some nonlinear diffusion equations." *Wave Motion* 44, no. 3 (2007): 222-230. https://doi.org/10.1016/j.wavemoti.2006.10.001

[20] Ali, N. A. M., R. Rahman, J. Sulaiman, and K. Ghazali. "SOR iterative method with wave variable transformation for solving advection-diffusion equations." In *AIP Conference Proceedings*, vol. 2013, no. 1. AIP Publishing, 2018. https://doi.org/10.1063/1.5054235

[21] Hoe, Hooi Mun, Wei KingTiong, Kim Gaik Tay, and Kang Leng Chiew. "Internal solitary waves of depression in rapidly varying topography." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 58, no. 2 (2019): 224-236.

[22] Ali, Nur Afza Mat, Rostang Rahman, Jumat Sulaiman, and Khadizah Ghazali. "Numerical solutions of unsteady advection-diffusion equations by using EG iteration with wave variable transformation." In *Journal of Physics: Conference Series*, vol. 1358, no. 1, p. 012049. IOP Publishing, 2019. https://doi.org/10.1088/1742-6596/1358/1/012049

[23] Kress, Rainer. *Numerical analysis*. Vol. 181. Springer Science & Business Media, 2012.

[24] Youssef, I. K. "On the successive overrelaxation method." *Journal of Mathematics and Statistics* 8, no. 2 (2012): 176-184. https://doi.org/10.3844/jmssp.2012.176.184

[25] Evans, D. J. "Group explicit iterative methods for solving large linear systems." *International Journal of Computer Mathematics* 17, no. 1 (1985): 81-108. https://doi.org/10.1080/00207168508803452

[26] Chew, J. V. L., J. Sulaiman, F. A. Muhiddin, and Kota Kinabalu. "Numerical evaluation on the solution of the porous medium equation by 4-Point Newton-Explicit Group." *International Journal of Mathematics and Computer Science* 16 (2021): 1487-1496.