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The Edge Metric Dimension of Comb Product Cycles Versus Stars and Fans

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ABSTRACT

The edge metric dimension finding of a graph resulting from the comb product on a cycle graph to the complete graph and the fact about the dominant vertex in the complete graph arise a problem about the edge metric dimension of a graph resulting from the comb product on a cycle graph to the other graphs that have dominant vertex such as star and fan graph. This study aims to determine the edge metric dimension of the graph resulting from the comb product on a cycle graph to the star and fan graph, respectively. The results show that the upper and lower bound of the edge metric dimension of both graphs are equivalent so that an exact edge metric dimension is obtained. The edge metric dimension of the graph comb product of the cycle graph C_n to the star graph $S_{1,m}$ is $n(m-1)$. The edge metric dimension of the graph comb product of the cycle graph C_n to the fan graph $F_{1,m}$ is $n(m-1)$.

1. Introduction

A graph is a pair of sets: in the first set, its members are called vertices and in the second set, the members are pair of vertices. In some papers, pair of vertices is called edge so the second set is called the set of edges. If the graph is denoted G, then the graph can be written as $G = (V, E)$ in which $V = V(G)$ is called a set of vertices and $E = E(G)$ is called a set of edges. Therefore, a graph consists of two sets, namely vertex and edge set in book [13]. Graph theory is a concept in mathematics that can be used to represent complex problems in a simple form so that they are easy to understand and analyse. In this case, a graph is used to explained discrete objects and the relationships between them in a system. In graph theory, the object is represented by a vertex and the relationship between two objects is represented by an edge.

Suppose that graph G has n vertices and k edges in which $x, y \in V(G)$ and $e = xy \in E(G)$. Thus, we said x is adjacent to y, also vice versa, and the edge e is incident with x and y. Also, that graph has n order and k size. Some graphs have certain characteristics in graph theory so that they are called special graphs in book [10].

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A graph is said to be a simple graph if it has not any loop or multiple edges. If $\forall x, y \in V(G), xy \in E(G)$ and G is a simple graph, then graph G is called a complete graph. The complete graph that has n order is denoted by K_n . The set of neighbors of x is denoted $N(x)$ and the degree of x is denoted $deg(x)$. If the graph is simple, then $deg(x) = |N(x)|$. A simple graph of order n which has two vertices of degree one and the other of degree two is called a path graph, denoted by P_n. If every two points on graph G lie in a path of G, then graph G is called a connected graph from previous study [7]. A connected and simple graph of order n that every vertex has degree two is called a cycle graph, denoted by C_n. The star graph denoted by $S_{1,n}$ in this paper is a simple graph obtained from an empty graph that has n vertices in which all vertex of the empty graph is adjacent to an adding vertex, a common vertex. The empty graph is a graph with none of the edge as said by previous study [6]. Thus, graph $S_{1,n}$ has a n-degree vertex and n one-degree vertices. Mufti *et al.,* [20] defined fan graph $F_{1,n}$ as the simple graph obtained from a path graph P_n by connecting every vertex to an adding vertex, a common vertex. Thus, graph $F_{1,n}$ has a n-degree vertex, 2 two-degree vertices, and other three-degree vertices. A windmill graph is a graph obtained from some of complete graphs that are attached by a common vertex. A n-sunlet graph is obtained by attaching the pendant edges to every vertex in a cycle graph**.** Meanwhile, a prism graph is obtained from the cartesian product of a 2-order path graph and a cycle graph.

A vertex $w \in V(G)$ is called a dominant vertex in graph G if $d(w, v) = 1$ for every $v \in V(G)\backslash\{w\}$ as said from previous study [8]. Thus, from the special graph explanation, the complete graph, star graph $S_{1,n}$ and fan graph $F_{1,n}$ have a dominant vertex. For certain purposes, we can choose a vertex on G in which the selected vertex is called a fixed point of G.

Suppose G is a simple graph and $u, v \in V(G)$. Diestel [17] explains that the distance between u and v, written $d(u, v)$) is

$$d(u,v) = \begin{cases} 0;\ \text{if } u = v \\ k;\ \text{if } u, v \text{ is connected by a path that has length k} \\ \infty;\ \text{if } u, v \text{ is not connected by any path} \end{cases}$$

Now, suppose $u, v, w \in V(G)$ and $e = uw \in E(G)$, the distance between the edge e and the vertex v is $s(e, v) = min\,\{d(u, v), d(w, v)\}$ as stated by previous study [2,4,5].

The concept of metric in graph theory is used in many studies. One of them is the study of the metric dimension of a graph. Metric dimension has many applications, in engineering there are navigation system, navigation of robots and telecommunication. Over time, the study of this topic is growing and one of the results is the edge metric dimension of a graph. The concept of edge metric dimension was introduced by Kelenc *et al.,* [1] because the metric basis of a graph only distinguishes its vertices and it is not sufficient to distinguish the edges of the graph. Kelenc *et al.,* [2] proved that if $\beta(G)$ represents the metric dimension and $\beta_e(G)$ represents the edge metric dimension of a graph G, then only one happens: $\beta(G) > \beta_e(G), \beta(G) < \beta_e(G)$, or $\beta(G) = \beta_e(G)$. In this paper, the metric dimension of a graph is called the vertex metric dimension of a graph.

Although relatively new, research on the edge metric dimension of the graph is quite massive. For example, Knor *et al.,* [11] was explained about remarks on the vertex and the edge metric dimension of 2-connected graphs and Knor *et al.,* [12] was explained about graphs with the edge metric dimension smaller than the metric dimension. Nasir *et al.,* [18] explained the edge metric dimension of n-sunlet family and prism graphs. Meanwhile, Zubrilina [15] provides a characterization of n order graphs that has edge metric dimension $n - 1$ and proposed an open problem about what graphs of n order that has edge metric dimension $n - 2$. In addition, Singh *et al.,* [16] determined the vertex metric dimension and edge metric dimension of windmill graphs. The edge metric dimension of the graph resulting from the comb product of C_m to the K_n (french cycle windmill) is obtained by Singh *et al.,* [16]. The result is the main rationale for this paper. In this paper, we determine the edge metric dimension of the graph from the graph resulting from the comb product of a cycle graph to the star and fan graph, respectively.

1.1 Comb Product of Two Graphs

In graph theory, many graph operations attract the attention of researchers as stated by previous study [9]. Graph operations are the way to obtain a new graph, such as join (∪), sums $(+)$, comb products (⊳), and others. The focus of paper is the comb product, following is the definition of that operation.

Suppose two graphs, G and $H,$ were connected and $u \in V(H)$. The graph resulting from the comb product of G to the H *(*denoted by $G \rhd H$*)* is the graph obtained by taking all the vertices and edges of the graph $G,$ also for H as much as $|V(G)|$ called: $H_i, i = 1,2, \dots, |V(G)|$ and attach a fixed point u_i of each graph H_i to the vertex i-th of the graph G as stated by previous study [19].

For example: Suppose that $V(C_5) = \{u_1, u_2, \dots, u_5\}$ and $V(S_{1,5}) = \{v_0, v_1, \dots, v_5\}$. It means, $E(C_5) = \{u_1u_2, u_2u_3, \dots, u_4u_5, u_5u_1\}$ and $E(S_{1,5}) = \{v_0v_j | 1 \le j \le 5\}$. In this case, $|V(C_5)| = 5$ so that graph $S_{1,5}$ is multiplied by 5, call $S_{1,5}^i$ with vertices $v_j^i, i = 1,2 \dots,5$ and $j = 0,1,2 \dots,5$. The fixed point of $S_{1,5}^i$ is v_0^i for $i = 1,2 \dots,5$. Therefore, $V(C_5 \rhd S_{1,5}) = \{u_1 = v_0^1, u_2 = v_0^2, u_3 = v_0^3, \dots, u_5 = v_0^5\} \cup \{v_j^i | i = 1,2, \dots,5 \text{ and } j = 1,2, \dots,5\}$ and $E(C_5 \rhd S_{1,5}) = \{u_1u_2, u_2u_3, \dots, u_4u_5, u_5u_1\} \cup \{u_iv_j^i | i = 1,2, \dots,5 \ and \ j = 1,2, \dots,5\}$. The graph figure of $C_5 \rhd S_{1,5}$ is in Figure 1.

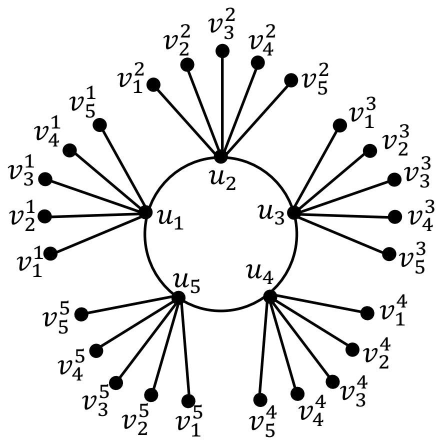

Fig. 1. Graph $C_5 \rhd S_{1,5}$

Now, suppose that a fan graph has $V(F_{1,4}) = \{w_0, w_1, w_2, \dots, w_4\}$ and $E(F_{1,4}) = \{w_jw_{j+1} | 1 \le j \le 3\} \cup \{w_0w_j | 1 \le j \le 4\}$. For graph $C_5 \rhd F_{1,4}$, based on the comb product definition, then fan graph $F_{1,4}$ is multiplied by 5, call $F_{1,4}^i$ with vertices $w_j^i, i = 1,2 \dots,5$ and $j = 0,1,2 \dots,4$. The fixed point of graph $F_{1,4}^i$ is w_0^i for $i = 1,2 \dots,5$. Therefore, the vertex set is $V(C_5 \rhd F_{1,4}) = \{u_1 = w_0^1, u_2 = w_0^2, u_3 = w_0^3, \dots, u_5 = w_0^5\} \cup \{w_j^i | i = 1,2, \dots,5 \ and \ j = 1,2, \dots,4\}$ and edge set is $E(C_5 \rhd F_{1,4}) = \{u_1u_2, u_2u_3, \dots, u_4u_5, u_5u_1\} \cup \{w_j^iw_{j+1}^i | 1 \le i \le 5, 1 \le j \le 4\} \cup \{u_iw_j^i | i = 1,2, \dots,5 \text{ and } j = 1,2, \dots,4\}$. The graph figure of $C_5 \rhd F_{1,4}$ is in Figure 2.

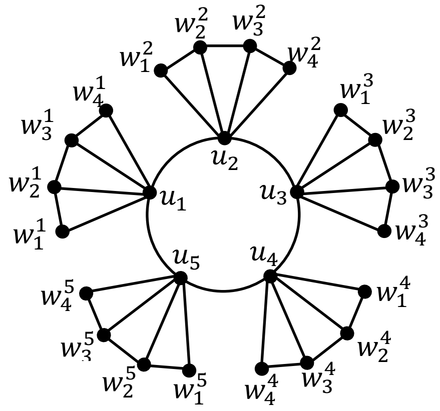

Fig. 2. Graph $C_5 \rhd F_{1,4}$

1.2 The Edge Metric Dimension of a Graph

In this paper, we mention the vertex and edge metric dimension of a graph. For the vertex metric dimension, several terms are used, such as the vertices distance, the resolving set, and the basis of the graph. Suppose a k-tuple $S = (s_1, s_2, s_3, \dots, s_k), s_i \in V(G)$ and a vertex $v \in V(G)$. The representation of vertex v to the S that denoted by $r(v|S)$ is k-tuple:

$$r(v|S) = \big(d(v, s_1), d(v, s_2), d(v, s_3), \dots, d(v, s_k)\big).$$

If every two vertices called u, v in graph G satisfied $r(u|S) \neq r(v|S)$, then set $\bar{S} = \{s_i | 1 \leq i \leq k\}$ is called the resolving set of G. The resolving set with minimum cardinality is called a basis of G. The cardinality of the basis is called the vertex metric dimension of G, denoted by $\beta(G)$ in this paper.

Now, suppose e is an edge on a connected graph G and a p-tuple $T = (t_1, t_2, t_3, \dots, t_p), t_i \in V(G)$. The representation of edge e to the T that denoted $r(e|T)$ *is* p-tuple:

$$r(e|T) = \Big(s(e, t_1), s(e, t_2), s(e, t_3), \dots, s\big(e, t_p\big)\Big)$$

$\bar{T} = \{t_i | 1 \leq i \leq p\}$ is called the edge resolving set if for any $e_i, e_j \in E(G)$, $r(e_i|T) \neq r(e_j|T)$. The edge resolving set with minimum cardinality is an edge basis and the cardinality is the edge metric dimension of the graph as stated by previous study [14]. In this paper, the edge metric dimension of graph G is denoted by $\beta_e(G)$. In previous study [3] used term p-vector as the metric code (edge representation in this paper).

For example: Suppose $V(C_4 \rhd S_{1,3}) = \{u_1, u_2, u_3, u_4\} \cup \{v_j^i | 1 \leq i \leq 4, 1 \leq j \leq 3\}$ and $E(C_4 \rhd S_{1,3}) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\} \cup \{u_iv_j^i | 1 \leq i \leq 4, 1 \leq j \leq 3\}$.

Choose $T = (v_1^1, v_2^1, v_1^2, v_2^2, v_1^3, v_2^3, v_1^4, v_2^4), v_j^i \in V(C_4 \rhd S_{1,3})$.

The representation of edge $e_i = u_iu_{(i+1)modn}$ to the T is

$$\begin{aligned} r(e_i|T) &= \big(s(e_i, v_1^1), s(e_i, v_2^1), s(e_i, v_1^2), s(e_i, v_2^2), \dots, s(e_i, v_1^4), s(e_i, v_2^4)\big) \\ &= (2,2, \underbrace{1,1}_{i-th}, \underbrace{1,1}_{(i+1)-th}, 2,2) \end{aligned}$$

The representation of edge $e_j^i = u_i v_j^i$ to the T is

$$r(e_j^i\,|T) = \begin{cases} (2,2,\underbrace{0,1}_{i-th},2,2,3,3) \text{ for } j = 1 \text{ on } v_j^i \\ (2,2,\underbrace{1,0}_{i-th},2,2,3,3) \text{ for } j = 2 \text{ on } v_j^i \\ (2,2,\underbrace{1,1}_{i-th},2,2,3,3) \text{ for } j = 3 \text{ on } v_j^i \end{cases}$$

Thus, we obtained that every edge has the different representation to the T, then set $\bar{T} = \{v_j^i | 1 \leq i \leq 4, 1 \leq j \leq 2\}$ is the edge resolving set of graphs $C_4 \rhd S_{1,3}$.

2. Methodology

2.1 Research Type, Subject and Object

This study was literature research by studying and examining references from a graph theory book, five e-books, and nineteen journal articles that are relevant to the research topic. The subject studied was the edge metric dimension of comb product graphs. The objects studied were the graphs resulting from the comb product of C_n to the star and fan graph, respectively.

2.2 Research Stages

The problem identification was the initial stage to obtain a problem in graph theory regarding the development of metric dimension, namely the edge metric dimension of a graph. Then, literature research was carried out. We collected, studied, and examined various references that are relevant to the research topic so that it assisted researchers in understanding the topic. In addition, researchers obtained a lot of information that can be used in the research process. Then, we did an observation of careful review regarding the edge metric dimension of the graph that has been studied previously and what the development can be done. Researchers found that Singh *et al.,* determine the edge metric dimension of $C_n \rhd K_m$. We knew that graph K_m has a unique characteristic about the existence of a dominant vertex. Therefore, this study took the specific topic about the edge metric dimension of graph $C_n \rhd G$ in which G is a graph that has a dominant vertex, such as the star and fan graph. After the specific topic is determined, then the upper and lower bound of the edge metric dimension of the studied graphs are determined. Last, we obtained the edge metric dimension of the graph.

3. Main Results

The main results are presented in Theorem 3.1 and Theorem 3.2 about the edge metric dimension of the graph resulting from the comb product of C_n to the star graph $S_{1,m}$ and fan graph $F_{1,m}$. Before presenting them, we present our finding definition and propositions to support the main results.

First, we propose the "equivalent edges on a graph". This term has contributed to determine the edge metric dimension of a graph.

Definition 3.1 Suppose G is a connected graph and $e_1, e_2 \in E(G)$. Edges e_1 and e_2 are equivalent if they satisfy:

i. $e_1 = va$ and $e_2 = vb$ for certain $a, b, v \in V(G)$.
ii. $s(e_1, u) = s(e_2, u)$, for any $u \in V(G) \backslash \{a, b\}$.

Following is the proposition that explained the relation between equivalent edges on a graph and the edge resolving set.

3.1 Proposition 3.1

Suppose that $T \subseteq V(G)$ and the edges, $e_1 = ua$ and $e_2 = ub$, are equivalent in graph G in which $u, a, b \in V(G)$. If T is the edge resolving set, then $a \in T$ or $b \in T$.

<u>Proof:</u>

Suppose that $T \subseteq V(G)$ and there are two equivalent edges of graph $G, e_1 = ua$ and $e_2 = ub$.

Because of e_1 and e_2 are equivalent edges that have a common vertex u, then $s(e_1, x) = s(e_2, x)$ for any $x \in V(G)\backslash\{a, b\}$.

By using contraposition, we will show that if $a, b \notin T$, then T is not edge resolving set.

Suppose that $T = \{v_1, v_2, \dots, v_k\} \subseteq V(G)$ and $a, b \notin T$. Construct $T' = (v_1, v_2, \dots, v_k), v_i \in T$, then

$$\begin{aligned} r(e_1|T') &= \big(s(e_1, v_1), s(e_1, v_2), \dots, s(e_1, v_k)\big) \\ &= \big(s(e_2, v_1), s(e_2, v_2), \dots, s(e_2, v_k)\big) \text{ ; because of } e_1 \text{and } e_2 \text{ are equivalent} \\ &= r(e_2|T') \end{aligned}$$

Therefore, $r(e_1|T') = r(e_2|T')$. There is the same edge representation to T'. Thus, T is not an edge resolving set. Then, it is proved that if T is the edge resolving set, then $a \in T$ or $b \in T$. ■

Proposition 3.1 showed that the identification of equivalent edges on a graph is useful to determine the edge resolving set of the graph. Now, we present the distance of vertex-to-vertex and side-to-vertex of our studied graphs. The results showed on Proposition 3.2, 3.3, 3.4, and 3.5.

3.2 Proposition 3.2

Suppose that $a, b \in V\big(S_{1,m}\big)$. If a, b are not adjacent, then $d(a, b) = 2$.

<u>Proof:</u>

Suppose that $a, b \in V(S_{1,m})$ and a, b are not adjacent. Thus, there exists a vertex $u \in V(S_{1,m})$ as the dominant vertex in which $d(u, v) = 1$, for any $v \in V\big(S_{1,m}\big)\backslash\{u\}$. Therefore, $d(a, b) = d(a, u) + d(u, b) = 1 + 1 = 2$. Therefore, $d(a, b) = 2$. ■

3.3 Proposition 3.3

Suppose that $m, n \in V\big(F_{1,m}\big)$. If m, n are not adjacent, then $d(m, n) = 2$.

<u>Proof:</u>

Suppose that $m, n \in V(F_{1,m})$. If m, n are not adjacent, then there exists a vertex $w \in V(F_{1,m})$ as the dominant vertex in which $d(w, x) = 1$, for any $x \in V\big(F_{1,m}\big)\backslash\{w\}$ so that $d(m, n) = d(m, w) + d(w, n) = 1 + 1 = 2$. Thus, $d(m, n) = 2$. ■

From Proposition 3.2 and 3.3, we obtained that for every two non-adjacent vertices in graph $S_{1,m}$ or $F_{1,m}$, the distance is 2.

3.4 Proposition 3.4

Suppose that $a, b, v \in V(S_{1,m})$. If $e = ab$, then $s(e, v) \leq 1$.
Proof:
Suppose that $a, b, v \in V(S_{1,m})$ and $e = ab$. Therefore, we divided this case in two cases.
Case 1: If $a = v$ or $b = v$, then
For $a = v$, $s(e, v) = min\{d(a, v), d(b, v)\} = min\{d(v, v), d(b, v)\} = min\,\{0, d(b, v)\} = 0$
For $b = v$, $s(e, v) = min\{d(a, v), d(b, v)\} = min\{d(a, v), d(v, v)\} = min\{d(a, v), 0\} = 0$
Thus, $s(e, v) = 0$.
Case 2: If $a \neq b \neq v$, then
If a is dominant vertex, $d(a, v) = 1$ and $d(b, v) \geq 1$, then

$$s(e, v) = min\{d(a, v), d(b, v)\} = min\{1, d(b, v)\} = 1$$

If b is dominant vertex, $d(b, v) = 1$ and $d(a, v) \geq 1$, then

$$s(e, v) = min\{d(a, v), d(b, v)\} = min\{d(a, v), 1\} = 1$$

Then, $s(e, v) = 1$.
Based on Case 1 and 2, we obtained $s(e, v) \leq 1$. ■

3.5 Proposition 3.5

Suppose that $m, n, u \in V(F_{1,m})$. If $e = mn$, then $s(e, u) \leq 2$.
Proof:
Suppose that $m, n, u \in V(F_{1,m})$. If $e = mn$, based on the definition about the distance between an edge and a vertex, we obtained three cases.
Case 1: If $m = u$ or $n = u$, then
For $m = u$,

$$s(e, u) = min\{d(m, u), d(u, n)\} = min\{d(u, u), d(u, n)\} = min\,\{0, d(u, n)\} = 0$$

For $n = u$,

$$s(e, u) = min\{d(m, u), d(u, n)\} = min\{d(m, u), d(u, u)\} = min\{d(m, u), 0\} = 0$$

Therefore, if $m = u$ or $n = u$, then $s(e, u) = 0$.
Case 2: If $m \neq n \neq u$, however, m or n is adjacent to the u, then
If m is adjacent to u, $d(m, u) = 1$ and $d(n, u) \geq 1$, then

$$s(e, u) = min\{d(m, u), d(u, n)\} = min\{1, d(u, n)\} = 1$$

If n is adjacent to u, $d(n, u) = 1$ and $d(m, u) \geq 1$,then

$$s(e, u) = min\{d(m, u), d(u, n)\} = min\{d(m, u), 1\} = 1$$

Thus, if $m \neq n \neq u$, however, m or n is adjacent to the u, then $s(e, u) = 1$.
Case 3: If $m \neq n \neq u$, however, m and n are not adjacent to the u, then based on Proposition 3.3, $s(e, u) = min\{d(m, u), d(u, n)\} = min\{2,2\} = 2$.
Based on Case 1, 2, and 3, then $s(e, w) \leq 2$. ■
The finding definition and propositions have been presented. Next, we present the main results, namely the edge metric dimension of the graph resulting from the comb product of C_n to the star graph $S_{1,m}$ and fan graph $F_{1,m}$. Theorem 3.1 will provide the edge metric dimension of the graph resulting from the comb product of C_n to the star graph $S_{1,m}$, while Theorem 3.2 about the graph resulting from the comb product of C_n to the fan graph $F_{1,m}$.
Theorem 3.1 $\beta_e(C_n \rhd S_{1,m}) = n(m-1)$

Proof:

Suppose that $V(C_n)=\{u_i|1\le i\le n\}$ and $E(C_n)=\{e_i|1\le i\le n;\ e_i=u_iu_{(i+1)(mod\ n)}\}$ also $V(S_{1,m})=\{v_0,v_1,v_2,\dots,v_m\}$ and $E(S_{1,m})=\{v_0v_j|1\le j\le m\}$.

On graph $C_n \rhd S_{1,m}$, the fixed point of graph $S_{1,m}$ is dominant vertex, assume it is v_0. Based on the comb product definition, following are the vertex and edge set of graphs $C_n \rhd S_{1,m}$.

$$V(C_n \rhd S_{1,m})=V(C_n)\cup\{v_j^i|1\le i\le n,1\le j\le m\}$$
$$=\{u_i|1\le i\le n\}\cup\{v_j^i|1\le i\le n,1\le j\le m\}$$
$$E(C_n \rhd S_{1,m})=E(C_n)\cup\{u_iv_j^i|\ 1\le i\le n,1\le j\le m\}$$
$$E(C_n \rhd S_{1,m})=\{\ e_i=u_iu_{(i+1)modn}|1\le i\le n;\}\cup\{e_j^i=u_iv_j^i|\ 1\le i\le n,1\le j\le m\}$$

We show the graph $C_n \rhd S_{1,m}$ in Figure 3.

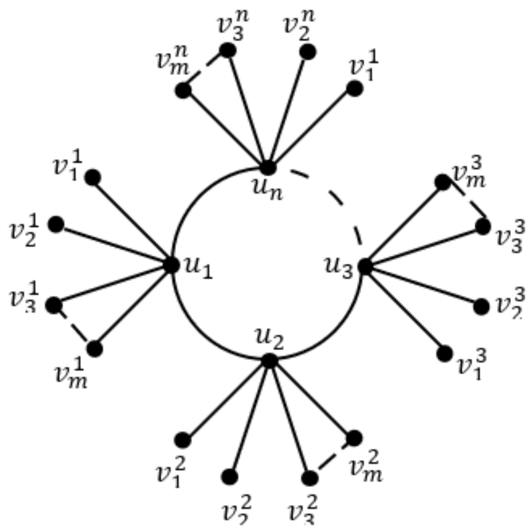

Fig. 3. Graph $C_n \rhd S_{1,m}$

Next, we will look for the upper and lower bound edge metric dimension of the graph $C_n \rhd S_{1,m}$.

Upper bound of the edge metric dimension:

Choose $T=(v_1^1,v_2^1,\dots,v_{m-1}^1,v_1^2,v_2^2,\dots,v_{m-1}^2,\dots,v_1^n,v_2^n,\dots,v_{m-1}^n)$, $v_j^i\in V(C_n \rhd S_{1,m})$. Two cases appeared, such as n is odd or even.

Case 1: If n is odd, then

$$r(e_i|T)=\left(\left\lceil\frac{n}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil,\dots,\left\lceil\frac{n}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil-1,\left\lceil\frac{n}{2}\right\rceil-1,\dots,\left\lceil\frac{n}{2}\right\rceil-1,\dots,3,3,\dots,3,2,2,\dots,2,\underbrace{1,1,\dots,1}_{i-th},\underbrace{1,1,\dots,1}_{(i+1)-th},2,2,\dots,2,3,3,\dots,3,\dots,\left\lceil\frac{n}{2}\right\rceil-1,\left\lceil\frac{n}{2}\right\rceil-1,\dots,\left\lceil\frac{n}{2}\right\rceil-1\right)$$

$$r(e_j^i|T)=\begin{cases}\left(\left\lceil\frac{n}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil,\dots,\left\lceil\frac{n}{2}\right\rceil,\dots,3,3,\dots,3,2,2,\dots,2,\underbrace{1,1,\dots,\underbrace{0}_{j-th}\dots,1}_{i-th},2,2,\dots,2,3,3,\dots,3,\dots,\left\lceil\frac{n}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil,\dots,\left\lceil\frac{n}{2}\right\rceil\right); \boldsymbol{j\ne m}\\ \left(\left\lceil\frac{n}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil,\dots,\left\lceil\frac{n}{2}\right\rceil,\dots,3,3,\dots,3,2,2,\dots,2,\underbrace{1,1,\dots,1}_{i-th},,2,2,\dots,2,3,3,\dots,3,\dots,,\left\lceil\frac{n}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil,\dots,\left\lceil\frac{n}{2}\right\rceil\right); \boldsymbol{j=m}\end{cases}$$

Case 2: If n is even, then

$$r(e_i|T) = \left(\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil, \dots, 3,3, \dots, 3,2,2, \dots, 2, \underbrace{1,1, \dots, 1}_{i-th}, \underbrace{1,1, \dots, 1}_{(i+1)-th}, 2,2, \dots, 2,3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil\right)$$

$$r(e_j^i|T) = \begin{cases} \left(\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil, \dots, 3,3, \dots, 3,2,2, \dots, 2, \underbrace{1,1,, \dots, \underbrace{0}_{j-th} \dots, 1}_{i-th}, 2,2, \dots, 2,3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil - 1, \dots, \left\lceil\frac{n}{2}\right\rceil - 1\right); \boldsymbol{j \neq m} \\ \left(\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil, \dots, 3,3, \dots, 3,2,2, \dots, 2, \underbrace{1,1, \dots, 1}_{i-th}, ,2,2, \dots, 2,3,3, \dots, 3, \dots, , \left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil - 1, \dots, \left\lceil\frac{n}{2}\right\rceil - 1\right); \boldsymbol{j = m} \end{cases}$$

Based on Case 1 and 2, we obtain the different representation for every edge to the T in graph $C_n \rhd S_{1,m}$. Therefore, the set with member vertex of T is the edge resolving set. Thus, $\beta_e(C_n \rhd S_{1,m}) \leq n(m-1)$.

Lower bound of the edge metric dimension:

We will show that none edge resolving set of graph $C_n \rhd S_{1,m}$ with cardinality less than $n(m-1)$.

Assume there exist edge resolving set $T' \subseteq V(C_n \rhd S_{1,m})$, $|T'| < n(m-1)$.

Also, edges $e_j^i = u_i v_j^i \in E(C_n \rhd S_{1,m})$ for certain i and $1 \leq j \leq m$ were equivalent edges.

Therefore, the number of equivalent edges is m for certain $i \in (1,2, \dots, n)$. By the fact about T' is edge resolving set, using Proposition 4.3., then for any $i \in (1,2, \dots, n)$, at most there is one j so that $v_j^i \notin T'$.

We know that $v_0^i = u_i$, we obtain,

$$\begin{aligned} |T'| &\geq |V(C_n)|\left(\left|V\left(S_{1,m}\right)\right| - 1\right) - \left|\left\{v_j^i | 1 \leq i \leq n, \text{a certain } j\right\}\right| \\ &= n((m+1)-1) - n \\ &= nm - n \\ &= n(m-1) \end{aligned}$$

Now we obtain $|T'| \geq n(m-1)$. This is contradicted with our assumption, $|T'| < n(m-1)$. Thus, none edge resolving set of graphs $C_n \rhd S_{1,m}$ with cardinality less than $n(m-1)$. Therefore, $\beta_e(C_n \rhd S_{1,m}) \geq n(m-1)$.

Based on the upper and lower bound of the edge metric dimension, we obtain $\beta_e\left(C_n \rhd \left(S_{1,m}\right)\right) = n(m-1)$. ■

Theorem 3.2 $\beta_e(C_n \rhd F_{1,m}) = n(m-1)$

Proof:

Assume that $V(C_n) = \{x_i | 1 \leq i \leq n\}$ and $E(C_n) = \{e_i | 1 \leq i \leq n;\ e_i = x_i x_{(i+1)(mod\ n)}\}$ also $V\left(F_{1,m}\right) = \{w_0, w_1, w_2, \dots, w_m\}$ and $E\left(F_{1,m}\right) = \left\{f_j = w_j w_{j+1} |\ 1 \leq j \leq m-1\right\} \cup \left\{w_0 w_j | 1 \leq j \leq m\right\}$.

The fixed point of $F_{1,m}$ is the dominant vertex, said w_0. Following are the vertex and edge set of graphs $C_n \rhd F_{1,m}$ according to the comb product definition.

$$\begin{aligned} V\left(C_n \rhd F_{1,m}\right) &= V(C_n) \cup \left\{w_j^i | 1 \leq i \leq n, 1 \leq j \leq m\right\} \\ &= \{x_i | 1 \leq i \leq n\} \cup \left\{w_j^i | 1 \leq i \leq n, 1 \leq j \leq m\right\} \end{aligned}$$

$$\begin{aligned} E\left(C_n \rhd F_{1,m}\right) = &\left\{ e_i = x_i x_{(i+1)mod n} | 1 \leq i \leq n; \right\} \cup \left\{f_j^i = w_j^i w_{j+1}^i |\ 1 \leq j \leq m-1\right\} \\ &\cup \left\{e_j^i = x_i w_j^i |\ 1 \leq i \leq n, 1 \leq j \leq m\right\} \end{aligned}$$

Graph $C_n \rhd F_{1,m}$ in Figure 4.

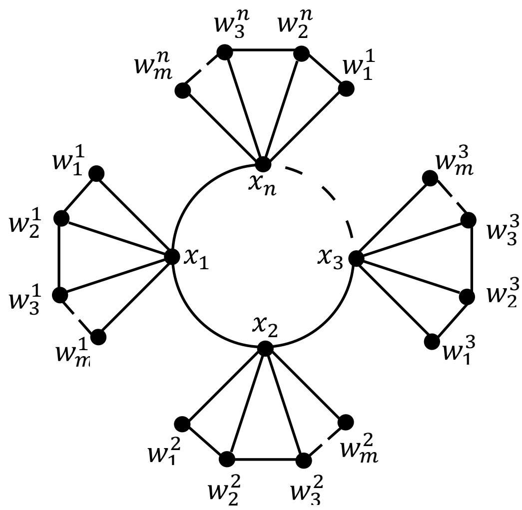

Fig. 4. Graph $C_n \rhd F_{1,m}$

Now, we will find the upper and lower bound of the edge metric dimension of graph $C_n \rhd F_{1,m}$.

Upper bound of the edge metric dimension:

Now, choose $T = (w_1^1, w_2^1, \dots, w_{m-1}^1, w_1^2, w_2^2, \dots, w_{m-1}^2, \dots, w_1^n, w_2^n, \dots, w_{m-1}^n)$, $w_j^i \in V(C_n \rhd F_{1,m})$. Following two cases are given, when n is odd or even.

Case 1: If n is odd, then we obtained

$$r(e_i|T) = \left(\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil - 1, \dots, \left\lceil\frac{n}{2}\right\rceil - 1, \dots, 3,3, \dots, 3,2,2, \dots, 2, \underbrace{1,1, \dots, 1}_{i-th}, \underbrace{1,1, \dots, 1}_{(i+1)-th}, 2,2, \dots, 2,3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil - 1, \dots, \left\lceil\frac{n}{2}\right\rceil - 1\right)$$

$$r(e_j^i|T) = \begin{cases} \left(\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil, \dots, 3,3, \dots, 3,2,2, \dots, 2, \underbrace{1,1,, \dots, \underbrace{0}_{j-th} \dots, 1}_{i-th}, 2,2, \dots, 2,3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil\right); \boldsymbol{j \neq m} \\ \left(\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil, \dots, 3,3, \dots, 3,2,2, \dots, 2, \underbrace{1,1, \dots, 1}_{i-th}, , 2,2, \dots, 2,3,3, \dots, 3, \dots, , \left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil\right); \boldsymbol{j = m} \end{cases}$$

$$r(f_j^i|T) = \begin{cases} \left(\left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \dots, 3,3, \dots, 3, \underbrace{0,0,1,2 \dots, 2}_{i-th}, 3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1\right); \boldsymbol{j=1} \\ \left(\left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \dots, 3,3, \dots, 3, \underbrace{2,2, \dots 2,1, \underbrace{0}_{j-th}, \underbrace{0}_{(j+1)-th}, 1,2, \dots, 2}_{i-th}, 3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1\right); \boldsymbol{2 \le j \le m-3} \\ \left(\left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \dots, 3,3, \dots, 3, \underbrace{2,2, \dots, 2,1,0,0}_{i-th}, 3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1\right); \boldsymbol{j = m-2} \\ \left(\left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \dots, 3,3, \dots, 3, \underbrace{2,2, \dots, 2,1,0}_{i-th}, 3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1\right); \boldsymbol{j = m-1} \end{cases}$$

Case 2: If n is even, then we obtained

$$r(e_i|T) = \left(\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil, \dots, 3,3, \dots, 3,2,2, \dots, 2, \underbrace{1,1, \dots, 1}_{i-th}, \underbrace{1,1, \dots, 1}_{(i+1)-th}, 2,2, \dots, 2,3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil\right)$$

$$r(e_j^i|T) = \begin{cases} \left(\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil, \dots, 3,3, \dots, 3,2,2, \dots, 2, \underbrace{1,1,, \dots, \underbrace{0}_{j-th} \dots, 1}_{i-th}, 2,2, \dots, 2,3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil - 1, \dots, \left\lceil\frac{n}{2}\right\rceil - 1\right) \\ \left(\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil, \dots, 3,3, \dots, 3,2,2, \dots, 2, \underbrace{1,1, \dots, 1}_{i-th},, 2,2, \dots, 2,3,3, \dots, 3, \dots,, \left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil - 1, \dots, \left\lceil\frac{n}{2}\right\rceil - 1\right); \boldsymbol{j = m} \end{cases}$$

$$r(f_j^i|T) = \begin{cases} \left(\left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \dots, 3,3, \dots, 3, \underbrace{0,0,1,2 \dots, 2}_{i-th}, 3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil\right); \boldsymbol{j = 1} \\ \left(\left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \dots, 3,3, \dots, 3, \underbrace{2,2, \dots 2,1, \underbrace{0}_{j-th}, \underbrace{0}_{(j+1)-th}, 1,2, \dots, 2}_{i-th}, 3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil\right); \boldsymbol{2 \le j \le m-3} \\ \left(\left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \dots, 3,3, \dots, 3, \underbrace{2,2, \dots, 2,1,0,0}_{i-th}, 3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil\right); \boldsymbol{j = m-2} \\ \left(\left\lceil\frac{n}{2}\right\rceil+1, \left\lceil\frac{n}{2}\right\rceil+1, \dots, \left\lceil\frac{n}{2}\right\rceil+1, \dots, 3,3, \dots, 3, \underbrace{2,2, \dots, 2,1,0}_{i-th}, 3,3, \dots, 3, \dots, \left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil, \dots, \left\lceil\frac{n}{2}\right\rceil\right); \boldsymbol{j = m-1} \end{cases}$$

From both cases, it can be concluded that every edge in graph $C_n \trianglerighteq F_{1,m}$ has the different representation to the T. Thus, the set with member vertices of T is the edge resolving set. We get, $\beta_e(C_n \trianglerighteq F_{1,m}) \le n(m-1)$.

Lower bound of the edge metric dimension:

Now, we prove that none edge resolving set of graphs $C_n \trianglerighteq F_{1,m}$ that has cardinality less than $n(m-1)$.

Suppose there exist edge resolving set $T' \subseteq V(C_n \trianglerighteq F_{1,m})$, $|T'| < n(m-1)$.

We know that edges $e_j^i = x_i w_j^i \in E(C_n \trianglerighteq F_{1,m})$ for certain i and $1 \le j \le m$ were equivalent so that the number of equivalent edges is m for certain $i \in (1,2, \dots, n)$. T' is edge resolving set, using Proposition 4.3., then for any $i \in (1,2, \dots, n)$, at most there is one j so that $w_j^i \notin T'$. Then, because of $w_0^i = x_i$ so we obtain,

$$\begin{aligned} |T'| &\ge |V(C_n)|\left(\left|V(F_{1,m})\right| - 1\right) - \left|\{v_j^i | 1 \le i \le n, \text{a certain } j\}\right| \\ &= n((m+1)-1) - n \\ &= nm - n \\ &= n(m-1) \end{aligned}$$

Thus, $|T'| \ge n(m-1)$. There is a contradiction because of the assumption $|T'| < n(m-1)$. Therefore, none edge resolving set of graphs $C_n \trianglerighteq F_{1,m}$ with cardinality less than $n(m-1)$. We obtained, $\beta_e(C_n \trianglerighteq F_{1,m}) \ge n(m-1)$.

Based on the upper and lower bound of the edge metric dimension, $\beta_e\left(C_n \trianglerighteq \left(F_{1,m}\right)\right) = n(m-1)$. ■

4. Conclusions

The edge metric dimension of the graph resulting from the comb product of the cycle C_n respectively to the star graph $S_{1,m}$ and fan graph $F_{1,m}$ is $n(m-1)$. The edge metric dimension of the graph resulting from the comb product of cycle respectively to the star graph or fan graph is the product of the cardinality of vertex set cycle and the cardinality of vertex set star or fan graph that was reduced by 2.

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