SEMARAK ILMU PUBLISHING

Journal of Advanced Research in Applied Sciences and Engineering Technology

Journal homepage:
https://semarakilmu.com.my/journals/index.php/applied_sciences_eng_tech/index
ISSN: 2462-1943

Rough Neutrosophic Shapley Weighted Einstein Averaging Aggregation Operator and its Application in Multi-Criteria Decision-Making Problem

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ARTICLE INFO

Article history:
Received 11 October 2023
Received in revised form 28 December 2023
Accepted 31 March 2024
Available online 17 April 2024

Keywords:
Einstein aggregation operator; Multi-criteria decision-making; Rough neutrosophic set; Shapley fuzzy measure

ABSTRACT

The study of aggregation operators has played a crucial role in various decision-making methods. The primary function of the aggregation operator is to combine multiple numbers into a single value. While Einstein operators offer a compact notation and can handle complex and large datasets, they do not consider the interaction involved in determining the criteria weights or account for imprecise and indeterminate data. To overcome this limitation, this paper introduces an improved aggregation operator, the rough neutrosophic Shapley weighted Einstein averaging aggregation operator. The rough neutrosophic sets offer a method for effectively managing the fuzziness and uncertainty that commonly occur in real-world scenarios and the Shapley fuzzy measure helps us understand the importance or value of different elements in each scenario. This operator combines the Shapley fuzzy measure with Einstein operators under rough neutrosophic sets, which are an effective tool for handling incomplete, indeterminate, and inconsistent information. The proposed operator satisfies essential algebraic properties such as idempotency, boundedness, and monotonicity. This paper also presents a decision-making methodology based on the proposed operator, with attribute values derived from the rough neutrosophic set. Finally, the applicability of the suggested aggregation operator is illustrated with a numerical example.

1. Introduction

Real-life decision-making problems often entail dealing with uncertain or imprecise information, leading to ambiguity and complexity. One of the most effective tools for dealing with impreciseness and uncertainties in decision-making is the fuzzy set, which was first introduced by Zadeh [1]. Fuzzy set theory, intuitionistic fuzzy set theory, and rough set theory are examples of approaches that deal with imprecise information and knowledge. However, these approaches only address some of the

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https://doi.org/10.37934/araset.43.2.5264

uncertainties in practical situations like indeterminacy situations in real life. Smarandache [2] initiated the notion of neutrosophic sets as a comprehensive approach to tackle uncertainty, inconsistency, incompleteness, and indeterminacy. A neutrosophic set is composed of three fundamental membership functions: the truth-membership function (T), the indeterminacy-membership function (I), and the falsity-membership function (F), making it a valuable tool for managing complex and ambiguous situations.

A notable alternative to traditional set theory is rough set theory, which was first proposed by Pawlak [3]. This approach has shown great efficacy in dealing with imprecise data, making it particularly useful for applications in fields like machine learning and data mining. Rough set theory is a branch of classical set theory that analyzes objects using lower and upper approximation spaces. It relies on both constructive and algebraic approaches to study the properties and behaviors of these spaces [4], making it a valuable tool for researchers and practitioners in a wide range of fields. The research on rough sets and fuzzy sets has been widely explored. A triangular conorm S and a norm T were used by Wu *et al.,* [5] to construct and axiomatize (S, T) fuzzy rough sets. Rough sets and neutrosophic sets can both handle the uncertainty and omit the insufficient information. The term "rough neutrosophic sets" (RNS) is a new hybrid intelligent structure that was introduced by Broumi and his colleagues in 2014 [6]. This structure, created by combining neutrosophic and rough sets, appears to be highly intriguing and applicable to real-world problems. Numerous previous studies have been found on rough neutrosphic sets. For example, Alias *et al.,* [7] developed the medical diagnosis proposition for the roughness Cosine similarity measure within a rough neutrosophic set context. Donbosco and Ganesan [8] defined a rough neutrosophic set and used it in a multi-criteria decision-making problem.

Aggregation operators have become increasingly crucial in any group decision-making problems. Aggregation operators, which typically take the form of mathematical functions, are widely used techniques that are used to combine all the individuals of data that are input into a single set. The ability to aggregate (fuse) multiple input values into a single output value is a vital technique in many fields, including but not limited to engineering, economics, sociology, and physics. The arithmetic mean and geometric mean are commonly employed as fundamental aggregation operators for data aggregation in a crisp environment. In real-world decision problems, complex relationships between criteria can exist. Algebraic operational rules are typically used to model aggregation operators like intersection and union. As a result, it is critical for an aggregation operator to capture those complicated interrelationships in order to generate more accurate aggregating results. Researchers started employing unions and intersections with t-operators due to their ability to offer a unique optimal solution. It is common knowledge that aggregating operators use different-norm (TN) and t-conorm (TCN) for example Algebraic operations [9], Einstein operations [10], Hamacher operations [11], and other such methods, which are extremely helpful in achieving the intersection and unions of fuzzy evaluations. Among these operations, Einstein AOs are more general and flexible in calculation, and they can avoid some irrational operations [12].

Besides Einstein AOs, the Shapley Fuzzy Measure (SFM) has been used to determine the criteria and decision makers' (DM) weight. It has been widely known to tackle the general interaction among input arguments. This interaction is necessary to reflect the significant weight of each input argument instead of the existing weightage techniques [13]. Insufficient weight information and distinct input arguments, which are frequently missing in aggregation operators, can be handled by SFM. Awang *et al.,* [14] proposed SFM under a hesitant bipolar-valued neutrosophic set environment for an investment decision whereas Heronian mean operators were developed by Hashim *et al.,* [15] while taking into account Shapley fuzzy measure in an interval neutrosophic vague environment. Hua and Jing [16] developed two interval-valued Pythagorean fuzzy aggregation operators based on Choquet integral operator and Shapley fuzzy measure, respectively. They claimed that the proposed Shapley fuzzy measure based on an interval-valued Pythagorean fuzzy set can handle the decision with complementary, repetitive, and independent characteristics.

There is little study of previous research on Einstein operator with SFM under a rough neutrosophic set. In this study, the Einstein operator and SFM are combined using a rough neutrosophic set to effectively present the interaction of criteria weights and to handle the uncertainty and incompleteness of information. To achieve this objective, the remainder of the article is structured as follows. In Sect. 2, we provide some preliminary definitions of rough neutrosophic sets, Einstein operator, and SFM. In Section 3, we develop the rough neutrosophic Shapley-weighted Einstein (RNSWE) operator and discuss its advantageous properties. Then formulate a decision-making approach using the RNSWE operator for the MCDM problem, where the weight of criteria is derived from the SFM. In order to validate the suggested aggregation operator, a numerical example of the decision-making problem is presented in Section 4. Ultimately, the study is concluded in Section 5.

2. Preliminaries

This section presents the essential RNSWE aggregation operator development definitions. Definition 1 introduces the definition of the rough neutrosophic sets.

Definition 1 [6] Let *Z* represent a non-null set and *P* represent an equivalence relation on *Z*. Let *R* be a neutrosophic set in *Z* with the truthness degree T_R, indeterminacy degree I_R and falseness degree F_R. The lower and upper approximations of *R* in the (*Z*, *P*) represented by $\underline{N}(R)$ and $\overline{N}(R)$ respectively referred to as:

$$\begin{aligned} \underline{N}(R) &= \left\{ < x, T_{\underline{N}(R)}(x), I_{\underline{N}(R)}(x), F_{\underline{N}(R)}(x) >: Z \in [x]_p , x \in Z \right\} \\ \overline{N}(R) &= \left\{ < x, T_{\overline{N}(R)}(x), I_{\overline{N}(R)}(x), F_{\overline{N}(R)}(x) >: Z \in [x]_p , x \in Z \right\} \end{aligned} \tag{1}$$

Definition 2 shows the definition of Einstein product and Einstein sum proposed by Wang and Liu [17].

Definition 2 [17] The t-norm and t-conorm have a significant role in constructing operation rules and aggregation operators. Special cases of the Archimedean t-norm and t-conorm, the Einstein product and Einstein sum are defined as:

$$a \oplus_E \beta = \frac{\alpha+\beta}{1+\alpha\cdot\beta} \tag{2}$$

$$a \otimes_E \beta = \frac{\alpha\cdot\beta}{1+(1-\alpha)\cdot(1-\beta)} \tag{3}$$

where Einstein sum $\oplus_E$ and the Einstein product $\otimes_E$ are the t-conorm and t-norm, respectively.

The Shapley Fuzzy Measure proposed by Sugeno *et al.,* [18] is provided as Definition 3 below:

Definition 3 [18] Let $Q = \{1,2,...n\}$ be a set of criteria and $P(Q)$ be the power set of Q. Then, a fuzzy measure on Q is a set function $\mu : P(Q) \rightarrow [0,1]$ which satisfies.

(1) Boundary conditions: $\mu(\phi)=0$ and $\mu(\phi)=1$
(2) Monotonicity: if $A,B\in P(Q)$, and $A\subseteq B$ then $\mu A\subseteq \mu B$

Sugeno proposed a specific type of λ fuzzy measure, which is based on $P(Q)$, for the sake of simplicity.

$$\mu(A\cup B)=\mu(A)+\mu(B)-\lambda\mu(A)\cdot\lambda\mu(B) \tag{4}$$

where $A\cap B=\phi$ for all $A,B\in P(P)$ and $\lambda>-1$.

Sugeno highlighted $\lambda=0,\lambda>0$ and $\lambda>0$ corresponding to additive measure respectively referring to synergic interactions that are positive and negative. This means that $\lambda-$fuzzy measurements can be written as:

$$\mu_{\lambda}(A)=\begin{cases}\frac{1}{\lambda}\left(\prod_{S_i\in A}(1+\lambda\mu_{\lambda}(\{S_i\}))-1\right), if\,\lambda\neq 0\\ \sum_{S_i\in A}\mu_{\lambda}(\{S_i\}), if\,\lambda=0\end{cases} \tag{5}$$

λ value can be computed as follows:

$$\left(\prod_{S_i\in A}(1+\lambda\mu_{\lambda}(\{S_i\})\right)=\lambda+1 \tag{6}$$

The $\lambda-$ fuzzy measure is based on the underlying notion of the SFM. According to Definition 5, the computation of the SFM can be calculated as follows:

$$\omega_i(\mu,Q)=\sum_{Q\subseteq N_i}\frac{(Q-|W|-1)!|W|!}{Q!}(\mu(W\cup i)-\mu(W)),\forall i\in Q \tag{7}$$

where μ is a fuzzy measure on Q, $|Q|$ and $|W|$ represent the cardinality of the set Q and W respectively. Generally, $\omega_i(\mu,Q)$ represent the weight vectors on set Q as $\omega_i(\mu,Q)>1$ and $\omega_i(\mu,Q)=1$.

Definition 4 introduces the score and accuracy formula in RNS environments.

Definition 4 Let $N(P)=\left\langle \underline{N}(P),\overline{N}(P)\right\rangle=\left\langle\left(\underline{T},\underline{I},\underline{F}\right),\left(\overline{T},\overline{I},\overline{F}\right)\right\rangle$ represent a RNS. The score and accuracy formula are defined as follows:

$$S\left[N(P)\right]=\frac{4+\underline{T}+\overline{T}-\underline{I}-\overline{I}-\underline{F}-\overline{F}}{6},S\left[N(P)\right]\in\left[0,1\right] \tag{8}$$

3. Proposed Rough Neutrosophic Shapley Weighted Einstein Operator

This section shows the proposed rough neutrosophic Shapley weighted Einstein (RNSWE) aggregation operator and outline several properties that align with this proposed operator. The RNSWE is created by combining the SFM and the Einstein aggregation operator concept within the rough neutrosophic set environment. Before the RNSWE is presented in detail, we will first provide an interpretation of the Einstein operations within the rough neutrosophic set, which takes into consideration the interaction between membership, indeterminacy, and non-membership functions. Afterward, we proceed to define some neutrosophic set Einstein averaging AO. Additionally, Theorem 1 illustrates the Einstein norm operations within the rough neutrosophic set.

Theorem 1 Let $N(R_1)=\left\langle \left(T_{\underline{N}(R_1)}, I_{\underline{N}(R_1)}, F_{\underline{N}(R_1)}\right), \left(T_{\overline{N}(R_1)}, I_{\overline{N}(R_1)}, F_{\overline{N}(R_1)}\right)\right\rangle$ $N(R_2)=\left\langle \left(T_{\underline{N}(R_2)}, I_{\underline{N}(R_2)}, F_{\underline{N}(R_2)}\right), \left(T_{\overline{N}(R_2)}, I_{\overline{N}(R_2)}, F_{\overline{N}(R_2)}\right)\right\rangle$ be rough neutrosophic set and $\lambda > 0$ be any real number. Between then, the Einstein norm operation has been characterised as

$$N(R_1)\oplus N(R_2)=\left\langle \begin{array}{l} \left(\dfrac{T_{\underline{N}(R_1)}+T_{\underline{N}(R_2)}}{1+\left(\left(T_{\underline{N}(R_1)}\right)\left(T_{\underline{N}(R_2)}\right)\right)}, \dfrac{\left(I_{\underline{N}(R_1)}\right)\left(I_{\underline{N}(R_2)}\right)}{1+\left(1-I_{\underline{N}(R_1)}\right)\left(1-I_{\underline{N}(R_2)}\right)}, \dfrac{\left(F_{\underline{N}(R_1)}\right)\left(F_{\underline{N}(R_2)}\right)}{1+\left(1-F_{\underline{N}(R_1)}\right)\left(1-F_{\underline{N}(R_2)}\right)}\right), \\ \left(\dfrac{T_{\overline{N}(R_1)}+T_{\overline{N}(R_2)}}{1+\left(\left(T_{\overline{N}(R_1)}\right)\left(T_{\overline{N}(R_2)}\right)\right)}, \dfrac{\left(I_{\overline{N}(R_1)}\right)\left(I_{\overline{N}(R_2)}\right)}{1+\left(1-I_{\overline{N}(R_1)}\right)\left(1-I_{\overline{N}(R_2)}\right)}, \dfrac{\left(F_{\overline{N}(R_1)}\right)\left(F_{\overline{N}(R_2)}\right)}{1+\left(1-F_{\overline{N}(R_1)}\right)\left(1-F_{\overline{N}(R_2)}\right)}\right) \end{array} \right\rangle$$

$$N(R_1)\otimes N(R_2)=\left\langle \begin{array}{l} \left(\dfrac{\left(T_{\underline{N}(R_1)}\right)\left(T_{\underline{N}(R_2)}\right)}{1+\left(1-T_{\underline{N}(R_1)}\right)\left(1-T_{\underline{N}(R_2)}\right)}, \dfrac{I_{\underline{N}(R_1)}+I_{\underline{N}(R_2)}}{1+\left(\left(I_{\underline{N}(R_1)}\right)\left(I_{\underline{N}(R_2)}\right)\right)}, \dfrac{F_{\underline{N}(R_1)}+F_{\underline{N}(R_2)}}{1+\left(\left(F_{\underline{N}(R_1)}\right)\left(F_{\underline{N}(R_2)}\right)\right)}\right), \\ \left(\dfrac{\left(T_{\overline{N}(R_1)}\right)\left(T_{\overline{N}(R_2)}\right)}{1+\left(1-T_{\overline{N}(R_1)}\right)\left(1-T_{\overline{N}(R_2)}\right)}, \dfrac{I_{\overline{N}(R_1)}+I_{\overline{N}(R_2)}}{1+\left(\left(I_{\overline{N}(R_1)}\right)\left(I_{\overline{N}(R_2)}\right)\right)}, \dfrac{F_{\overline{N}(R_1)}+F_{\overline{N}(R_2)}}{1+\left(\left(F_{\overline{N}(R_1)}\right)\left(F_{\overline{N}(R_2)}\right)\right)}\right) \end{array} \right\rangle$$

$$\lambda N(R)=\left\langle \begin{array}{l} \left(\dfrac{\left(1+T_{\underline{N}(R_1)}\right)^{\lambda}-\left(1-T_{\underline{N}(R_1)}\right)^{\lambda}}{\left(1+T_{\underline{N}(R_1)}\right)^{\lambda}+\left(1-T_{\underline{N}(R_1)}\right)^{\lambda}}, \dfrac{2\left(I_{\underline{N}(R_1)}\right)^{\lambda}}{\left(2-I_{\underline{N}(R_1)}\right)^{\lambda}+\left(I_{\underline{N}(R_1)}\right)^{\lambda})}, \dfrac{2\left(F_{\underline{N}(R_1)}\right)^{\lambda}}{\left(2-F_{\underline{N}(R_2)}\right)^{\lambda}+\left(F_{\underline{N}(R_1)}\right)^{\lambda}} \right), \\ \left(\dfrac{\left(1+T_{\overline{N}(R_1)}\right)^{\lambda}-\left(1-T_{\overline{N}(R_1)}\right)^{\lambda}}{\left(1+T_{\overline{N}(R_1)}\right)^{\lambda}+\left(1-T_{\overline{N}(R_1)}\right)^{\lambda}}, \dfrac{2\left(I_{\overline{N}(R_1)}\right)^{\lambda}}{\left(2-I_{\overline{N}(R_1)}\right)^{\lambda}+\left(I_{\overline{N}(R_1)}\right)^{\lambda})}, \dfrac{2\left(F_{\overline{N}(R_1)}\right)^{\lambda}}{\left(2-F_{\overline{N}(R_1)}\right)^{\lambda}+\left(F_{\overline{N}(R_1)}\right)^{\lambda}} \right) \end{array} \right\rangle$$

$$N(R)^{\lambda}=\left\langle \begin{array}{l} \left(\dfrac{2\left(T_{\underline{N}(R_1)}\right)^{\lambda}}{\left(2-T_{\underline{N}(R_1)}\right)^{\lambda}+\left(T_{\underline{N}(R_1)}\right)^{\lambda}}, \dfrac{\left(1+I_{\underline{N}(R_1)}\right)^{\lambda}-\left(1-I_{\underline{N}(R_1)}\right)^{\lambda}}{\left(1+I_{\underline{N}(R_1)}\right)^{\lambda}+\left(1-I_{\underline{N}(R_1)}\right)^{\lambda}}, \dfrac{\left(1+F_{\underline{N}(R_1)}\right)^{\lambda}-\left(1-F_{\underline{N}(R_1)}\right)^{\lambda}}{\left(1+F_{\underline{N}(R_1)}\right)^{\lambda}+\left(1-F_{\underline{N}(R_1)}\right)^{\lambda}} \right), \\ \left(\dfrac{2\left(T_{\overline{N}(R_1)}\right)^{\lambda}}{\left(2-T_{\overline{N}(R_1)}\right)^{\lambda}+\left(T_{\overline{N}(R_1)}\right)^{\lambda}}, \dfrac{\left(1+I_{\overline{N}(R_1)}\right)^{\lambda}-\left(1-I_{\overline{N}(R_1)}\right)^{\lambda}}{\left(1+I_{\overline{N}(R_1)}\right)^{\lambda}+\left(1-I_{\overline{N}(R_1)}\right)^{\lambda}}, \dfrac{\left(1+F_{\overline{N}(R_1)}\right)^{\lambda}-\left(1-F_{\overline{N}(R_1)}\right)^{\lambda}}{\left(1+F_{\overline{N}(R_1)}\right)^{\lambda}+\left(1-F_{\overline{N}(R_1)}\right)^{\lambda}} \right) \end{array} \right\rangle$$

Definition 5 introduces the concept of the rough neutrosophic Einstein (RNE) operator. This definition is essential to understand the proposed definition of the Rough neutrosophic Shapley Weighted Einstein (RNSWE) operator presented in Definition 6. Therefore, Definition 5 serves as a foundation for Definition 6, and it is crucial to grasp its meaning before proceeding to the next definition.

Definition 6 Let $N(R_i)=(\underline{N}(R_i),\overline{N}(R_i))$ in *(Z, P) (s=1,2,…,n)* be a set of neutrosophic numbers. Then, have the following definition for the rough neutrosophic Einstein (RNE) operators:

$$RNE=(N(R_1),N(R_2)\ldots,N(R_n))=\left(\overset{n}{\underset{s=1}{\oplus}}\underline{N}(R_s),\overset{n}{\underset{s=1}{\oplus}}\overline{N}(R_s)\right) \tag{9}$$

Theorem 2 Let $N(R_i)=(\underline{N}(R_i),\overline{N}(R_i))$ *(s=1, 2,…, n)* be a set of rough neutrosophic numbers. The aggregated value RNE $(N(R_1),N(R_2),\ldots,N(R_n))$ is also a rough neutrosophic number.

$$\begin{aligned} RNE&=(N(R_1),N(R_2)\ldots,N(R_n))=\left(\overset{n}{\underset{s=1}{\oplus}}\underline{N}(R_s),\overset{n}{\underset{s=1}{\oplus}}\overline{N}(R_s)\right) \\ &=(N(R_1)\oplus N(R_2)\oplus\ldots\oplus N(R_n)) \end{aligned} \tag{10}$$

Definition 6 Let $N(R_i)=(\underline{N}(R_i),\overline{N}(R_i))$ in *(Z, P) (i=1, 2,…,n)* be a set of rough neutrosophic numbers. Thus, the rough neutrosophic Shapley weighted Einstein (RNSWE) operators is defined as follows:

$$RNSWE=(N(R_1),N(R_2)\ldots,N(R_n))=\left(\overset{n}{\underset{s=1}{\oplus}}\omega_s\underline{N}(R_s),\overset{n}{\underset{s=1}{\oplus}}\omega_s\overline{N}(R_s)\right) \tag{11}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_s)^T$ is weight vector of R_n relative to some other fuzzy measure (the Shapley fuzzy measure) especially ω_i on $\left(\underline{N}(R_s), \overline{N}(R_s)\right) \in [0,1]$ and $\sum_{s=1}^{n} \omega_s = 1$. Since each ambiguous measure is additive, the value aggregated by RNSWE operator is still rough neutrosophic, calculated by the following formula:

$$RNSWE(N(R_1),N(R_2)\dots,N(R_n))$$
$$=\left\langle \begin{array}{l} \left(\frac{\prod_{s=1}^{n}\left(1+T_{\underline{N}(R_s)}\right)^{\omega_s} - \prod_{s=1}^{n}\left(1-T_{\underline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(1+T_{\underline{N}(R_s)}\right)^{\omega_s} + \prod_{s=1}^{n}\left(1-T_{\underline{N}(R_s)}\right)^{\omega_s}}, \frac{2\prod_{s=1}^{n}\left(I_{\underline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(2-I_{\underline{N}(R_s)}\right)^{\omega_s} + \prod_{s=1}^{n}\left(I_{\underline{N}(R_s)}\right)^{\omega_s}}, \frac{2\prod_{s=1}^{n}\left(F_{\underline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(2-F_{\underline{N}(R_s)}\right)^{\omega_s} + \prod_{s=1}^{n}\left(F_{\underline{N}(R_s)}\right)^{\omega_s}} \right), \\ \left(\frac{\prod_{s=1}^{n}\left(1+T_{\overline{N}(R_s)}\right)^{\omega_s} - \prod_{s=1}^{n}\left(1-T_{\overline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(1+T_{\overline{N}(R_s)}\right)^{\omega_s} + \prod_{s=1}^{n}\left(1-T_{\overline{N}(R_s)}\right)^{\omega_s}}, \frac{\prod_{s=1}^{n} 2\left(I_{\overline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(2-I_{\overline{N}(R_s)}\right)^{\omega_s} + \prod_{s=1}^{n}\left(I_{\overline{N}(R_s)}\right)^{\omega_s}}, \frac{2\prod_{s=1}^{n}\left(F_{\overline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(2-F_{\overline{N}(R_s)}\right)^{\omega_s} + \prod_{s=1}^{n}\left(F_{\overline{N}(R_s)}\right)^{\omega_s}} \right) \end{array} \right\rangle \tag{12}$$

Proof: This theorem will be demonstrated by mathematical induction when $\mathrm{n} = 1, \omega = 1$,

$$RNSWE = (N(R_1))$$
$$=\left\langle \begin{array}{l} \left(\frac{\left(1+T_{\underline{N}(R_1)}\right)^1 - \left(1-T_{\underline{N}(R_1)}\right)^1}{\left(1+T_{\underline{N}(R_1)}\right)^1 + \left(1-T_{\underline{N}(R_1)}\right)^1}, \frac{2\left(I_{\underline{N}(R_1)}\right)^1}{\left(2-I_{\underline{N}(R_1)}\right)^1 + \left(I_{\underline{N}(R_1)}\right)^1)}, \frac{2\left(F_{\underline{N}(R_1)}\right)^1}{\left(2-F_{\underline{N}(R_2)}\right)^1 + \left(F_{\underline{N}(R_1)}\right)^1} \right), \\ \left(\frac{\left(1+T_{\overline{N}(R_1)}\right)^1 - \left(1-T_{\overline{N}(R_1)}\right)^1}{\left(1+T_{\overline{N}(R_1)}\right)^1 + \left(1-T_{\overline{N}(R_1)}\right)^1}, \frac{2\left(I_{\overline{N}(R_1)}\right)^1}{\left(2-I_{\overline{N}(R_1)}\right)^1 + \left(I_{\overline{N}(R_1)}\right)^1)}, \frac{2\left(F_{\overline{N}(R_1)}\right)^1}{\left(2-F_{\overline{N}(R_1)}\right)^1 + \left(F_{\overline{N}(R_1)}\right)^1} \right) \end{array} \right\rangle \tag{13}$$

Thus, theorem holds for $\mathrm{n} = 1$. Now we have $\mathrm{n} = 2$

$$\omega_1(N(R_1)) = \left\langle \begin{array}{l} \left(\frac{\left(1+T_{\underline{N}(R_1)}\right)^{\omega_1} - \left(1-T_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(1+T_{\underline{N}(R_1)}\right)^{\omega_1} + \left(1-T_{\underline{N}(R_1)}\right)^{\omega_1}}, \frac{2\left(I_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(2-I_{\underline{N}(R_1)}\right)^{\omega_1} + \left(I_{\underline{N}(R_1)}\right)^{\omega_1})}, \frac{2\left(F_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(2-F_{\underline{N}(R_2)}\right)^{\omega_1} + \left(F_{\underline{N}(R_1)}\right)^{\omega_1}} \right), \\ \left(\frac{\left(1+T_{\overline{N}(R_1)}\right)^{\omega_1} - \left(1-T_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(1+T_{\overline{N}(R_1)}\right)^{\omega_1} + \left(1-T_{\overline{N}(R_1)}\right)^{\omega_1}}, \frac{2\left(I_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(2-I_{\overline{N}(R_1)}\right)^{\omega_1} + \left(I_{\overline{N}(R_1)}\right)^{\omega_1})}, \frac{2\left(F_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(2-F_{\overline{N}(R_1)}\right)^{\omega_1} + \left(F_{\overline{N}(R_1)}\right)^{\omega_1}} \right) \end{array} \right\rangle \tag{14}$$

. . (15)

Then, the operational summation relations above is proved as below

$$
\begin{aligned}
&RNSWE(N(R_1),N(R_2)) = \omega_1 N(R_1) \oplus \omega_2 N(R_2) \\
&= \left\langle \begin{pmatrix}
\left(\dfrac{\left(1+T_{\underline{N}(R_1)}\right)^{\omega_1} - \left(1-T_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(1+T_{\underline{N}(R_1)}\right)^{\omega_1} + \left(1-T_{\underline{N}(R_1)}\right)^{\omega_1}}, \dfrac{2\left(I_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(2-I_{\underline{N}(R_1)}\right)^{\omega_1} + \left(I_{\underline{N}(R_1)}\right)^{\omega_1})}, \dfrac{2\left(F_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(2-F_{\underline{N}(R_2)}\right)^{\omega_1} + \left(F_{\underline{N}(R_1)}\right)^{\omega_1}} \right), \\
\left(\dfrac{\left(1+T_{\overline{N}(R_1)}\right)^{\omega_1} - \left(1-T_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(1+T_{\overline{N}(R_1)}\right)^{\omega_1} + \left(1-T_{\overline{N}(R_1)}\right)^{\omega_1}}, \dfrac{2\left(I_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(2-I_{\overline{N}(R_1)}\right)^{\omega_1} + \left(I_{\overline{N}(R_1)}\right)^{\omega_1})}, \dfrac{2\left(F_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(2-F_{\overline{N}(R_1)}\right)^{\omega_1} + \left(F_{\overline{N}(R_1)}\right)^{\omega_1}} \right)
\end{pmatrix} \right\rangle \oplus \\
&\quad \left\langle \begin{pmatrix}
\left(\dfrac{\left(1+T_{\underline{N}(R_2)}\right)^{\omega_2} - \left(1-T_{\underline{N}(R_2)}\right)^{\omega_2}}{\left(1+T_{\underline{N}(R_2)}\right)^{\omega_2} + \left(1-T_{\underline{N}(R_2)}\right)^{\omega_2}}, \dfrac{2\left(I_{\underline{N}(R_2)}\right)^{\omega_2}}{\left(2-I_{\underline{N}(R_2)}\right)^{\omega_2} + \left(I_{\underline{N}(R_2)}\right)^{\omega_2})}, \dfrac{2\left(F_{\underline{N}(R_2)}\right)^{\omega_2}}{\left(2-F_{\underline{N}(R_2)}\right)^{\omega_2} + \left(F_{\underline{N}(R_2)}\right)^{\omega_2}} \right), \\
\left(\dfrac{\left(1+T_{\overline{N}(R_2)}\right)^{\omega_2} - \left(1-T_{\overline{N}(R_2)}\right)^{\omega_2}}{\left(1+T_{\overline{N}(R_2)}\right)^{\omega_2} + \left(1-T_{\overline{N}(R_2)}\right)^{\omega_2}}, \dfrac{2\left(I_{\overline{N}(R_2)}\right)^{\omega_2}}{\left(2-I_{\overline{N}(R_2)}\right)^{\omega_2} + \left(I_{\overline{N}(R_2)}\right)^{\omega_2})}, \dfrac{2\left(F_{\overline{N}(R_2)}\right)^{\omega_2}}{\left(2-F_{\overline{N}(R_2)}\right)^{\omega_2} + \left(F_{\overline{N}(R_2)}\right)^{\omega_2}} \right)
\end{pmatrix} \right\rangle
\end{aligned}
\tag{16}
$$

$$
= \left(
\begin{aligned}
&\left(
\dfrac{\left(\dfrac{\left(1+T_{\underline{N}(R_1)}\right)^{\omega_1} - \left(1-T_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(1+T_{\underline{N}(R_1)}\right)^{\omega_1} + \left(1-T_{\underline{N}(R_1)}\right)^{\omega_1}}\right) + \left(\dfrac{\left(1+T_{\underline{N}(R_2)}\right)^{\omega_2} - \left(1-T_{\underline{N}(R_2)}\right)^{\omega_2}}{\left(1+T_{\underline{N}(R_2)}\right)^{\omega_2} + \left(1-T_{\underline{N}(R_2)}\right)^{\omega_2}}\right)}{1+\left(\left(\dfrac{\left(1+T_{\underline{N}(R_1)}\right)^{\omega_1} - \left(1-T_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(1+T_{\underline{N}(R_1)}\right)^{\omega_1} + \left(1-T_{\underline{N}(R_1)}\right)^{\omega_1}}\right)\left(\dfrac{\left(1+T_{\underline{N}(R_2)}\right)^{\omega_2} - \left(1-T_{\underline{N}(R_2)}\right)^{\omega_2}}{\left(1+T_{\underline{N}(R_2)}\right)^{\omega_2} + \left(1-T_{\underline{N}(R_2)}\right)^{\omega_2}}\right)\right)}, \right. \\
&\dfrac{\left(\dfrac{2\left(I_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(2-I_{\underline{N}(R_1)}\right)^{\omega_1}+\left(I_{\underline{N}(R_1)}\right)^{\omega_1})}\right)\cdot\left(\dfrac{2\left(I_{\underline{N}(R_2)}\right)^{\omega_2}}{\left(2-I_{\underline{N}(R_2)}\right)^{\omega_2}+\left(I_{\underline{N}(R_2)}\right)^{\omega_2})}\right)}{1+\left(1-\dfrac{2\left(I_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(2-I_{\underline{N}(R_1)}\right)^{\omega_1}+\left(I_{\underline{N}(R_1)}\right)^{\omega_1})}\right)\left(1-\left(\dfrac{2\left(I_{\underline{N}(R_2)}\right)^{\omega_2}}{\left(2-I_{\underline{N}(R_2)}\right)^{\omega_2}+\left(I_{\underline{N}(R_2)}\right)^{\omega_2})}\right)\right)}, \\
&\left.\dfrac{\left(\dfrac{2\left(F_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(2-F_{\underline{N}(R_1)}\right)^{\omega_1}+\left(F_{\underline{N}(R_1)}\right)^{\omega_1}}\right)\cdot\left(\dfrac{2\left(F_{\underline{N}(R_2)}\right)^{\omega_2}}{\left(2-F_{\underline{N}(R_2)}\right)^{\omega_2}+\left(F_{\underline{N}(R_2)}\right)^{\omega_2}}\right)}{1+\left(1-\dfrac{2\left(F_{\underline{N}(R_1)}\right)^{\omega_1}}{\left(2-F_{\underline{N}(R_1)}\right)^{\omega_1}+\left(F_{\underline{N}(R_1)}\right)^{\omega_1}}\right)\left(1-\dfrac{2\left(F_{\underline{N}(R_2)}\right)^{\omega_2}}{\left(2-F_{\underline{N}(R_2)}\right)^{\omega_2}+\left(F_{\underline{N}(R_2)}\right)^{\omega_2}}\right)}\right), \\
&\left(
\dfrac{\left(\dfrac{\left(1+T_{\overline{N}(R_1)}\right)^{\omega_1} - \left(1-T_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(1+T_{\overline{N}(R_1)}\right)^{\omega_1} + \left(1-T_{\overline{N}(R_1)}\right)^{\omega_1}}\right) + \left(\dfrac{\left(1+T_{\overline{N}(R_2)}\right)^{\omega_2} - \left(1-T_{\overline{N}(R_2)}\right)^{\omega_2}}{\left(1+T_{\overline{N}(R_2)}\right)^{\omega_2} + \left(1-T_{\overline{N}(R_2)}\right)^{\omega_2}}\right)}{1+\left(\left(\dfrac{\left(1+T_{\overline{N}(R_1)}\right)^{\omega_1} - \left(1-T_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(1+T_{\overline{N}(R_1)}\right)^{\omega_1} + \left(1-T_{\overline{N}(R_1)}\right)^{\omega_1}}\right)\left(\dfrac{\left(1+T_{\overline{N}(R_2)}\right)^{\omega_2} - \left(1-T_{\overline{N}(R_2)}\right)^{\omega_2}}{\left(1+T_{\overline{N}(R_2)}\right)^{\omega_2} + \left(1-T_{\overline{N}(R_2)}\right)^{\omega_2}}\right)\right)}, \right. \\
&\dfrac{\left(\dfrac{2\left(I_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(2-I_{\overline{N}(R_1)}\right)^{\omega_1}+\left(I_{\overline{N}(R_1)}\right)^{\omega_1}}\right)\cdot\left(\dfrac{2\left(I_{\overline{N}(R_2)}\right)^{\omega_2}}{\left(2-I_{\overline{N}(R_2)}\right)^{\omega_2}+\left(I_{\overline{N}(R_2)}\right)^{\omega_2}}\right)}{1+\left(1-\dfrac{2\left(I_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(2-I_{\overline{N}(R_1)}\right)^{\omega_1}+\left(I_{\overline{N}(R_1)}\right)^{\omega_1}}\right)\left(1-\dfrac{2\left(I_{\overline{N}(R_2)}\right)^{\omega_2}}{\left(2-I_{\overline{N}(R_2)}\right)^{\omega_2}+\left(I_{\overline{N}(R_2)}\right)^{\omega_2}}\right)}, \\
&\left.\dfrac{\left(\dfrac{2\left(F_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(2-F_{\overline{N}(R_1)}\right)^{\omega_1}+\left(F_{\overline{N}(R_1)}\right)^{\omega_1}}\right)\cdot\left(\dfrac{2\left(F_{\overline{N}(R_2)}\right)^{\omega_2}}{\left(2-F_{\overline{N}(R_2)}\right)^{\omega_2}+\left(F_{\overline{N}(R_2)}\right)^{\omega_2}}\right)}{1+\left(1-\dfrac{2\left(F_{\overline{N}(R_1)}\right)^{\omega_1}}{\left(2-F_{\overline{N}(R_1)}\right)^{\omega_1}+\left(F_{\overline{N}(R_1)}\right)^{\omega_1}}\right)\left(1-\dfrac{2\left(F_{\overline{N}(R_2)}\right)^{\omega_2}}{\left(2-F_{\overline{N}(R_2)}\right)^{\omega_2}+\left(F_{\overline{N}(R_2)}\right)^{\omega_2}}\right)}\right)
\end{aligned}
\right)
$$

$$=\left\langle \begin{array}{l} \left(\dfrac{\prod_{s=1}^{n}\left(1+T_{\underline{N}(R_s)}\right)^{\omega_s}-\prod_{s=1}^{n}\left(1-T_{\underline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(1+T_{\underline{N}(R_s)}\right)^{\omega_s}+\prod_{s=1}^{n}\left(1-T_{\underline{N}(R_s)}\right)^{\omega_s}}, \dfrac{2\prod_{s=1}^{n}\left(I_{\underline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(2-I_{\underline{N}(R_s)}\right)^{\omega_s}+\prod_{s=1}^{n}\left(I_{\underline{N}(R_s)}\right)^{\omega_s}}, \dfrac{2\prod_{s=1}^{n}\left(F_{\underline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(2-F_{\underline{N}(R_s)}\right)^{\omega_s}+\prod_{s=1}^{n}\left(F_{\underline{N}(R_s)}\right)^{\omega_s}} \right), \\ \left(\dfrac{\prod_{s=1}^{n}\left(1+T_{\overline{N}(R_s)}\right)^{\omega_s}-\prod_{s=1}^{n}\left(1-T_{\overline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(1+T_{\overline{N}(R_s)}\right)^{\omega_s}+\prod_{s=1}^{n}\left(1-T_{\overline{N}(R_s)}\right)^{\omega_s}}, \dfrac{\prod_{s=1}^{n}2\left(I_{\overline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(2-I_{\overline{N}(R_s)}\right)^{\omega_s}+\prod_{s=1}^{n}\left(I_{\overline{N}(R_s)}\right)^{\omega_s}}, \dfrac{2\prod_{s=1}^{n}\left(F_{\overline{N}(R_s)}\right)^{\omega_s}}{\prod_{s=1}^{n}\left(2-F_{\overline{N}(R_s)}\right)^{\omega_s}+\prod_{s=1}^{n}\left(F_{\overline{N}(R_s)}\right)^{\omega_s}} \right) \end{array} \right\rangle$$

The proof ends.

Idempotency:

If $N(R_s)=N(P)(s=1,2,...,n)$ then,
$RNE(N(R_1),N(R_2),...,N(R_n))=N(P)$, a $RNSWE(N(R_1),N(R_2),...,N(R_n))=N(P)$

Proof
For, $N(R_s)=N(P)$

$$RNE(N(R_1),N(R_2),...,N(R_n))=\left(\underset{s=1}{\overset{n}{\oplus}}\underline{N}(P),\overline{N}(P)\right)$$
$$=\left(\underline{N}(P),\overline{N}(P)\right)=N(P)$$
$$RNSWE(N(R_1),N(R_2),...,N(R_n))=\left(\underset{s=1}{\overset{n}{\oplus}}\omega_s\underline{N}(P),\underset{s=1}{\overset{n}{\oplus}}\omega_s\overline{N}(P)\right)$$
$$=\left(\underline{N}(P)\underset{s=1}{\overset{n}{\oplus}}\omega_s,\overline{N}(P)\underset{s=1}{\overset{n}{\oplus}}\omega_s\right)$$
$$=\left(\underline{N}(P),\overline{N}(P)=N(P)\right)\text{since, }\sum_{s=1}^{n}\omega_s=1$$

Boundedness:

Proof
Let $N(R_j)(j=1,2,...,n)$be a collection of RNS numbers and let

$$N(R)^{-}=\left(\min_j \underline{T}N(R_j),\max_j \underline{I}N(R_j),\max_j \underline{F}N(R_j)\right),\left(\min_j \overline{T}N(R_j),\max_j \overline{I}N(R_j),\max_j \overline{F}N(R_j)\right)$$

and $N(R)^{+}=\left(\max_j \underline{T}N(R_j),\min_j \underline{I}N(R_j),\min_j \underline{F}N(R_j)\right),\left(\max_j \overline{T}N(R_j),\min_j \overline{I}N(R_j),\min_j \overline{F}N(R_j)\right)$

Then,

$$N(R)^{-}\subseteq RNE\left(N(R_1),N(R_2),...,N(R_n)\right)\subseteq N(R)^{+}$$

and

$$N(R)^{-}\subseteq RNSWE\left(N(R_1),N(R_2),...,N(R_n)\right)\subseteq N(R)^{+}$$

Monotonicity:

If $N(R_S) \subseteq N(R_S^*)(s=1,2,...,n)$then,

$RNE\left(N(R_1),N(R_2),...,N(R_n)\right) \subseteq RNE\left(N(R_1^*),N(R_2^*),...,N(R_n^*)\right)$,

and $RNSWE\left(N(R_1),N(R_2),...,N(R_n)\right) \subseteq RNE\left(N(R_1^*),N(R_2^*),...,N(R_n^*)\right)$.

Proof

Since $N(R_S) \subseteq N(R_S^*)(s=1,2,...,n), RNE\left(N(R_1),N(R_2),...,N(R_n)\right) \subseteq RNE\left(N(R_1^*),N(R_2^*),...,N(R_n^*)\right)$,

and $RNSWE\left(N(R_1),N(R_2),...,N(R_n)\right) \subseteq RNE\left(N(R_1^*),N(R_2^*),...,N(R_n^*)\right)$.

It proves the monotonicity of the functions $RNE\left(N(R_1),N(R_2),...,N(R_n)\right)$ and $RNSWE\left(N(R_1),N(R_2),...,N(R_n)\right)$.

4. Numerical Example

This section presents a numerical example to demonstrate the practicality of the methods proposed in [19]. Suppose a corporation intends to invest a certain amount of money in the best investment fund. They have four alternatives for deciding where to invest the money. Car company p_1, food company p_2, computer company p_3, and arms company p_4 are all possible choices. The three attributes are the risk analysis e_1, growth analysis e_2, and environmental impact analysis e_3 must be considered when making decisions. The company will use the RNS number assessments provided by the three attributes to select one of the four possible alternatives for their investment.

$$
\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array}
\begin{array}{c} \begin{array}{ccc} \quad e_1 \quad & \quad e_2 \quad & \quad e_3 \quad \end{array} \\
\begin{pmatrix}
\langle(0.2,0.3,0.4),(0.4,0.3,0.4)\rangle & \langle(0.4,0.3,0.4),(0.6,0.1,0.4)\rangle & \langle(0.2,0.4,0.4),(0.4,0.4,0.4)\rangle \\
\langle(0.3,0.4,0.5),(0.4,0.3,0.2)\rangle & \langle(0.5,0.3,0.6),(0.7,0.1,0.4)\rangle & \langle(0.3,0.3,0.4),(0.4,0.3,0.3)\rangle \\
\langle(0.4,0.3,0.6),(0.5,0.2,0.3)\rangle & \langle(0.5,0.3,0.4),(0.6,0.3,0.3)\rangle & \langle(0.2,0.3,0.5),(0.4,0.1,0.3)\rangle \\
\langle(0.3,0.5,0.5),(0.4,0.3,0.1)\rangle & \langle(0.4,0.5,0.5),(0.5,0.3,0.3)\rangle & \langle(0.2,0.3,0.4),(0.4,0.5,0.4)\rangle
\end{pmatrix} \end{array}
$$

Step 1: Determine the criteria weight by SFM.

$\mu(e_1)=0.45$, $\mu(e_2)=0.65$, $\mu(e_3)=0.55$

Then, the $\lambda-$ value is calculated by applying Eq. (6), shown as below:

$$\lambda+1=(1+0.45\lambda)(1+0.65\lambda)(1+0.55\lambda)$$
$$\lambda=-0.8554$$

Subsequently, the fuzzy measure of attribute is obtained using Eq. (6) and computed in the following manner:

$$\mu(e_1,e_2)=\frac{1}{-0.8554}\Big(\big(1-0.8554(0.45)\big)\big(1-0.8554(0.65)\big)-1\Big)=0.8497$$
$$\mu(e_1,e_3)=\frac{1}{-0.8554}\Big(\big(1-0.8554(0.45)\big)\big(1-0.8554(0.55)\big)-1\Big)=0.7882$$
$$\mu(e_2,e_3)=\frac{1}{-0.8554}\Big(\big(1-0.8554(0.65)\big)\big(1-0.8554(0.55)\big)-1\Big)=0.8941$$
$$\mu(e_1,e_2,e_3)=\frac{1}{-0.8554}\Big(\big(1-0.8554(0.45)\big)\big(1-0.8554(0.65)\big)\big(1-0.8554(0.55)\big)-1\Big)=1$$

After that, the SFM can be obtained from Eq. (7) as follows

$$\begin{aligned}\omega_1(\mu,Q)&=\frac{(3-0-1)!0!}{3!}\big(\mu,(e_1)-\mu(\phi)\big)+\frac{(3-1-1)!1!}{3!}\big(\mu,(e_1,e_2)-\mu(e_2)\big)+\frac{(3-1-1)!1!}{3!}\big(\mu,(e_1,e_3)-\mu(e_3)\big)+\frac{(3-2-1)!2!}{3!}\big(\mu,(e_1,e_2,e_3)-\mu(e_2,e_3)\big)\\&=\frac{1}{3}(0.45-0)+\frac{1}{6}(0.8497-0.65)+\frac{1}{6}(0.7882-0.55)+\frac{1}{3}(1-0.8941)\\&=0.26\end{aligned}$$

$$\begin{aligned}\omega_2(\mu,Q)&=\frac{(3-0-1)!0!}{3!}\big(\mu,(e_2)-\mu(\phi)\big)+\frac{(3-1-1)!1!}{3!}\big(\mu,(e_1,e_2)-\mu(e_1)\big)+\frac{(3-1-1)!1!}{3!}\big(\mu,(e_2,e_3)-\mu(e_3)\big)+\frac{(3-2-1)!2!}{3!}\big(\mu,(e_1,e_2,e_3)-\mu(e_1,e_3)\big)\\&=\frac{1}{3}(0.65-0)+\frac{1}{6}(0.8497-0.45)+\frac{1}{6}(0.8941-0.55)+\frac{1}{3}(1-0.7882)\\&=0.41\end{aligned}$$

$$\begin{aligned}\omega_3(\mu,Q)&=\frac{(3-0-1)!0!}{3!}\big(\mu,(e_3)-\mu(\phi)\big)+\frac{(3-1-1)!1!}{3!}\big(\mu,(e_1,e_3)-\mu(e_1)\big)+\frac{(3-1-1)!1!}{3!}\big(\mu,(e_2,e_3)-\mu(e_2)\big)+\frac{(3-2-1)!2!}{3!}\big(\mu,(e_1,e_2,e_3)-\mu(e_1,e_2)\big)\\&=\frac{1}{3}(0.55-0)+\frac{1}{6}(0.7882-0.45)+\frac{1}{6}(0.8941-0.65)+\frac{1}{3}(1-0.8497)\\&=0.33\end{aligned}$$

Step 2: Calculate the comprehensive evaluation value $p_S=\{1,2,...n\}$ using the RNSWE operator

$$p_1=\langle 0.2852,0.3305,0.4000,0.4885,0.2152,0.4000\rangle$$
$$p_2=\langle 0.3866,0.3238,0.5030,0.5410,0.1941,0.3058\rangle$$
$$p_3=\langle 0.3819,0.3000,0.4805,0.5130,0.1900,0.3000\rangle$$
$$p_4=\langle 0.3420,0.5317,0.4653,0.4424,0.3575,0.2526\rangle$$

Step 3: Calculate the score formula in a RNS environment in Eq. (8), the bigger the score, the better the alternative.

$$Y(N(P_1))=0.5713$$
$$Y(N(P_2))=0.6001$$
$$Y(N(P_3))=0.6041$$
$$Y(N(P_4))=0.5295$$

Step 4: Ranking of alternatives $p_S=\{1,2,...,n\}$ in descending order and choose the best one(s). The obtain ranking order is $p_3>p_2>p_1>p_4$. Thus, the best alternative is p_3.

Step 5: End

4. Conclusion

In conclusion, this paper presents a novel rough neutrosophic Shapley weighted Einstein (RNSWE) operator that overcomes the limitations of other aggregation operations. The proposed operator utilizes the Einstein operational rules to aggregate the SFM under rough neutrosophic set, which yields desirable properties that are mathematically proven. By using SFM, the proposed operator accurately acquires the criteria weights while considering their mutual interactions. Moreover, the proposed RNSWE operator is applicable in situations where indeterminacy elements exist before making optimal decisions. The numerical analysis conducted in this study demonstrates the effectiveness of the proposed operator, with the best alternative identified. The RNSWE operator can also be applied in various fields, such as engineering, environment, or social science, to solve other MCDM problems. Overall, this paper contributes a valuable addition to the MCDM literature and provides a promising direction for future research.

In future research, the proposed aggregation operator may be possible to consider the applications of the proposed method in other fields such as in analysis for evaluation of potential renewable energy resources in Malaysia by Zul Ilham *et al.,* [20] and overview of the green building index by Chin Yee Ha *et al.,* [21].

Acknowledgement
This research is funded by the Universiti Teknologi MARA (UiTM) through the MyRA Research Grant LPHD with grant file number 600-RMC/GPM LPHD 5/3 (139/2021).

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