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Navigating Non-Classical Optimal Control Problem: A Hybrid Shooting Approach with Discretization Validation

Wan Noor Afifah Wan Ahmad[1], Suliadi Firdaus Sufahani[1,*], Mahmod Abd Hakim Mohamad[2], Mohd Saifullah Rusiman[1], Mohd Zulariffin Md Maarof[3], Muhamad Ali Imran Kamarudin[4], Naufal Ishartono[5], Lee Kah Howe[6], Asmala Ahmad[7]

1 Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology, Universiti Tun Hussein Onn Malaysia, Pagoh Educational Hub, 84600 Muar, Johor, Malaysia
2 Department of Mechanical Engineering, Center of Diploma, Universiti Tun Hussein Onn Malaysia, Pagoh Educational Hub, 84600 Muar, Johor, Malaysia
3 Department of Sciences and Mathematics, Center of Diploma, Universiti Tun Hussein Onn Malaysia, Pagoh Educational Hub, 84600 Muar, Johor, Malaysia
4 School of Business Management, College of Business, Universiti Utara Malaysia, 06010 Sintok, Kedah, Malaysia
5 Faculty of Education, Universitas Muhammadiyah Surakarta, Kabupaten Sukoharjo, Jawa Tengah 57169, Indonesia
6 Ecube Global Pte Ltd, 1 Claymore Drive, 08-03, Orchard Tower Rear Block APT, 229594 Singapore
7 Faculty of Information and Communication Technology, Universiti Teknikal Malaysia Melaka, 76100 Durian Tunggal, Melaka, Malaysia

ABSTRACT

This paper presents a solution to the challenges encountered in the non-classical Optimal Control (OC) problems, where the final state variable is unknown, leading to a non-zero final shadow value. The main objective is to maximize the performance index, but the presence of a piecewise royalty function in the performance index makes it non-differentiable at certain time frames. Therefore, we adopted a continuous approach using the hyperbolic tangent (tanh) function to overcome this difficulty. To compute the unknown final state value, we employed a hybrid shooting method, which combines the Newton and Brent methods implemented in the C++ programming language. Since the final shadow value is non-zero, a new equation is mathematically required to continue the investigation. Thusly, we established a new natural boundary condition based on the fundamental theory proposed by previous researchers. At the same time, the validation process involved discretization methods such as Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson approximations. The program was constructed in AMPL programming language with MINOS solver during the validation process. This study applied the proposed methods to fixed and three-stage piecewise royalty payments. The study expects the hybrid shooting method to produce a more accurate optimal result than the discretization method at the end of the investigation. This research highlights the significance of the fundamental theory in tackling real-world problems. In addition, the technique used here can serve as a stepping stone for future researchers exploring new mathematical approaches in real-world problem-solving. While at the same time, this ensures that the method remains up-to-date, making the academic field relevant for teaching and learning processes, especially in the domains of science and mathematics.

Keywords:
Discretization method; non-classical optimal control problem; piecewise royalty payment; shooting method

* *Corresponding author.*
E-mail address: suliadi@uthm.edu.my

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1. Introduction

Optimal Control (OC) problems play a crucial role in various scientific and engineering applications, enabling us to determine the optimal trajectory of a system to achieve desired objectives. In recent years, the study of the OC problem has been fascinating to many researchers. Xiaobing *et al.,* **Error! Reference source not found.** leverage efficient basis functions to efficiently handle the complexities arising in OC problems from fractional calculus. Meanwhile, Xie *et al.,* **Error! Reference source not found.** presented a novel hybrid improved neural network algorithm that combines L2 regularization and dropout regularization techniques. The proposed approach aims to enhance the performance and generalization capabilities of neural networks in solving mathematical problems in engineering. By integrating these regularization techniques, the algorithm mitigates overfitting and improves the robustness of the neural network model. The study demonstrates the effectiveness of the hybrid algorithm through experiments and simulations, showcasing its potential to achieve more accurate and reliable results in various engineering applications.

Huang *et al.,* **Error! Reference source not found.** introduced a novel approach for solving nonlinear time-fractional OC problems using the space-time Chebyshev spectral collocation method. Effati and Skandari **Error! Reference source not found.** proposed an OC approach for solving linear Volterra integral equations. The technique leverages the principles of OC theory to efficiently handle the linear Volterra integral equations, which are prevalent in various engineering and scientific applications. By formulating the problem as an OC problem, the study aims to find an OC function that satisfies the integral equation constraints while minimizing an objective function. Skandari and Tohidi **Error! Reference source not found.** combined linearization and discretization techniques to efficiently handle the complexities of solving a class of nonlinear OC problems.

Certain real-world OC scenarios present challenges that deviate from traditional OC problems. In particular, the non-classical OC problem arises when the final state variable is unknown, resulting in a non-zero final costate variable. These non-classical OC problems introduce complexities that demand innovative approaches for effective solutions. Within the realm of OC theory, the costate variable, alternatively referred to as the adjoint variable or shadow value, assumes a pivotal role **Error! Reference source not found.**,**Error! Reference source not found.**. It serves as a Lagrange multiplier, aiding in the resolution of OC problems that encompass differential equations and constraints. The terminology “costate” emerges from merging “cost,” which pertains to the objective function or performance index, and “state”, signifying the state variables characterizing the system.

This study draws inspiration from Spence’s work in 1981 **Error! Reference source not found.**, where an economic model incorporating variables such as demand, royalty payment structure, and discount factor was proposed. Building on this, Zinober and Kaivanto **Error! Reference source not found.** attempted to solve a similar economic model using matrix formulation for royalty payments. However, challenges emerged when dealing with changing levels of royalty payments, leading to computational difficulties arising from the non-differentiability of the model at certain timeframes. Further contributions in this realm include Cruz *et al.,* **Error! Reference source not found.**, addressing a non-classical OC problem featuring piecewise functions through nonlinear programming (NLP) methods like Runge-Kutta. More recently, Zinober and Sufahani **Error! Reference source not found.** introduced solutions for non-classical OC problems using hyperbolic tangent (tanh) modelling, a technique gaining traction in contemporary research. Al-Hawasy **Error! Reference source not found.** delved into continuous classical OC for nonlinear hyperbolic partial differential equations (PDEs), exploring the challenges posed by equality and inequality constraints. Al-Hawasy and Al-Rawdhanee **Error! Reference source not found.** presented a numerical approach merging the Galerkin finite element method with an implicit technique and the gradient projection method for solving classical

OC problems involving hyperbolic PDEs. Further expanding the toolbox, Al-Rawdanee and Al-Hawasy **Error! Reference source not found.** offered insights into employing mixed methods effectively for tackling classical OC problems governed by nonlinear hyperbolic equations. Al-Hawasy and Al-Rawdhanee **Error! Reference source not found.** focused on resolving OC problems associated with variable coefficients in nonlinear hyperbolic boundary value problems. Continuing this trajectory, Al-Hawasy and Al-Rawdhanee **Error! Reference source not found.** concentrated on solving nonlinear hyperbolic OC problems featuring state constraints.

The primary objective of this study is to maximize a performance index while considering system dynamics, constraints and boundary conditions. Unfortunately, this performance index involves royalty payment which is in a piecewise function that leads to non-differentiability at specific time frames. Overcoming this challenge requires sophisticated techniques that can handle such non-smooth conditions.

This paper proposed a novel hybrid shooting approach with discretization validation to effectively navigate the intricate landscapes of the non-classical OC problems. The main idea behind our method is to combine the advantages of the hybrid Newton-Brent shooting method to achieve accurate and optimal solutions for the non-classical OC scenario. A continuous approach was introduced based on the hyperbolic tangent (tanh) function to address the issue of non-differentiability in the performance index. This allows us to handle the piecewise components smoothly and enables the application of well-established optimization techniques.

Additionally, we incorporated the discretization method, which involves validating the continuous solution against discretized approximations using methods like Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson. This step ensures the accuracy and reliability of the obtained optimal trajectories.

The significance of this research lies in its ability to provide robust solutions to complex non-classical OC problems, which are prevalent in real-world applications. By presenting a hybrid approach that integrates continuous and discretized methods, we aim to demonstrate the versatility and efficiency of our proposed technique in solving challenging OC problems.

In the subsequent sections of this paper, we will provide a comprehensive explanation of the hybrid shooting approach, detailing its algorithm, implementation and validation process. Through numerical experiments on various non-classical OC scenarios, such as fixed royalty payment and three-stage piecewise royalty payment, we will demonstrate the effectiveness and superiority of our proposed method compared to traditional discretization approaches.

2. Hybrid Shooting Approach

The shooting method has found extensive application in solving OC problems. Zinober and Sufahani **Error! Reference source not found.** explored two distinct approaches to solving the non-classical OC problem: the shooting algorithm and the NLP technique. By comparing and analyzing the outcomes of both methods, the study seeks to find an effective and accurate solution to the non-classical OC problem encountered in economics. Abd Jalil and Roslan **Error! Reference source not found.** significantly contributed by combining the shooting method with automatic differentiation to tackle a singular heat transfer equation. They utilized the Taylor series expansion to compute the coefficients within the expansion and employed the shooting method to solve the boundary value problem (BVP) effectively.

In this research, the shooting method was adapted and enhanced in order to compute the optimal solution for the non-classical OC problem. The program was developed using the C++ programming language. The hybrid Newton-Brent shooting method is an advanced numerical approach that integrates elements of both the Newton and Brent methods within the shooting algorithm. Similar to the traditional shooting method, the hybrid shooting method begins with an initial guess for the unknown final state value and iteratively refines these values to meet the desired optimality conditions. The Brent method is employed to efficiently update the initial guess, leading to improved accuracy in the boundary value solution. The obtained values are then passed to the Newton iteration to compute the scalar function. During this phase, an Ordinary Differential Equation (ODE) solver is utilized to transform the BVP into an Initial Value Problem (IVP) by guessing an initial shadow value for one of the boundary conditions. Following the IVP solution, the final value of the state trajectory is compared to the desired optimality condition. If the final state value satisfies the optimality condition, the shooting process is successful, and the performance index will be computed. However, suppose the final state value does not meet the optimality condition. In that case, the initial guess for the final state value is adjusted, and the iterative process continues until the optimality condition is satisfied.

As per Press *et al.*, **Error! Reference source not found.**, the Brent method falls under the category of minimization techniques. Consequently, when solving a maximization problem, the optimal performance index is multiplied by a negative one at the end of the computation. This adjustment allows the minimization algorithm, such as the Brent method, to effectively find the maximum value of the performance index. Figure 1 provides a concise overview of the working process of the hybrid shooting method, illustrating the flow of operations in the numerical approach.

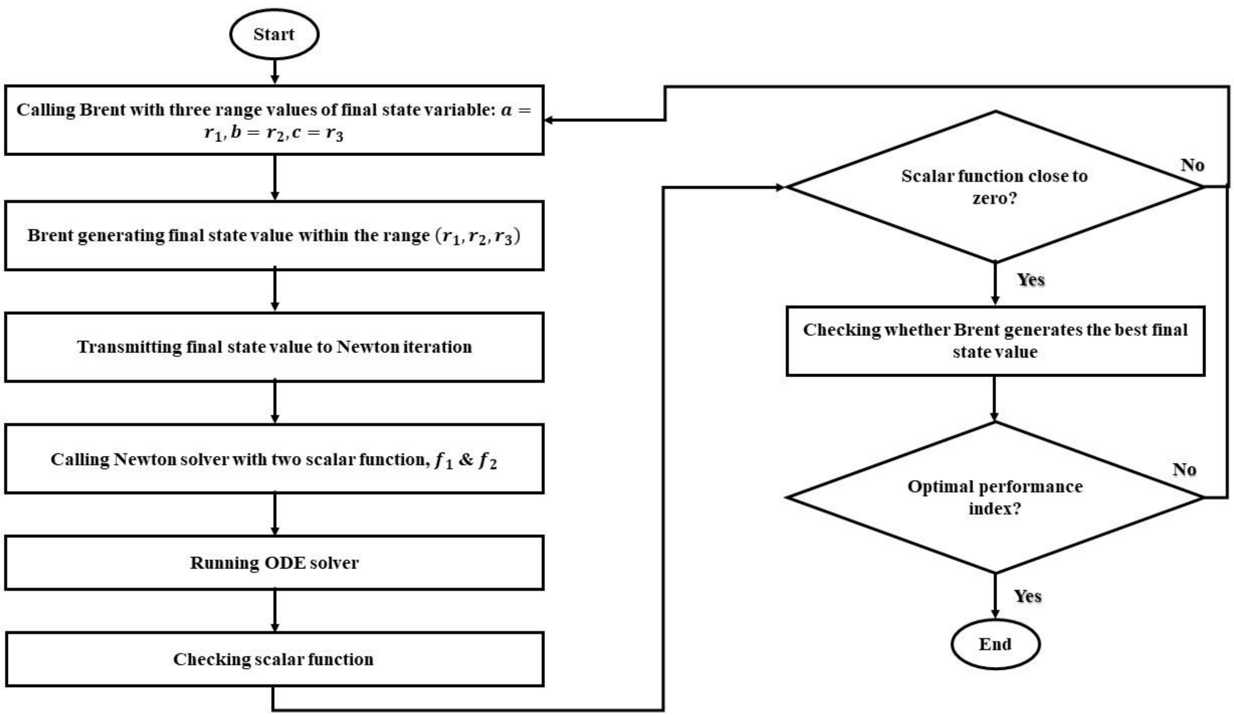


Fig. 1. Flowchart of hybrid shooting approach for calculation process

3. Validation Process Via Discretization Method

The validation process via the discretization method involved approximating the non-classical OC problem using discrete steps to derive an approximate solution. This numerical integration technique transformed the continuous OC problem into a sequence of discrete time points.

The commonly used Euler approximation was applied, where the state and control variables were updated at each time step based on the derivatives of the governing differential equations.

Additionally, the widely used Runge-Kutta method, known for its higher accuracy **Error! Reference source not found.** was also employed, considering multiple intermediate steps within each time interval. Moreover, this validation process explored other sophisticated nonlinear NLP methods, such as Trapezoidal and Hermite-Simpson **Error! Reference source not found.**. The AMPL programming language was utilized to construct the program for this discretization process **Error! Reference source not found.**. An appropriate time step size was selected to achieve the right balance between accuracy and computational efficiency. Smaller time steps generally provided more accurate results but required greater computation time, while larger time steps reduced computational costs but might sacrifice accuracy **Error! Reference source not found.**. Step size refers to the magnitude of the interval or increment used in numerical methods or algorithms, especially in the context of discretization. Cruz *et al.,* **Error! Reference source not found.** applied a 40 step size, while Zinober and Sufahani **Error! Reference source not found.** used a larger time step size equivalent to 450. For this paper, the step size was set to 50.

After discretizing the OC problem, the MINOS solver was employed to find the state and control variables at each time step. These values were then used to calculate the performance index over the entire time interval. By comparing the results with the hybrid shooting approach, the validation process ensured the consistency and accuracy of the discretization method. Upon successful validation, the discretization method can be regarded as a reliable tool for solving non-classical OC and other related problems.

4. Non-Classical Optimal Control Problem: Case Study for Royalty Payment Problem

Spence **Error! Reference source not found.** focuses on the concept of learning curves and their relevance to competition within the field of economics. The study delves into how firms' performance improves over time through experience, resulting in cost reductions as they acquire greater knowledge and expertise in their production processes. The researcher thoroughly examines the influence of learning curves on various market aspects, including dynamics, competition, and firm behaviour. By shedding light on the impact of learning effects, the study provides valuable insights into how they can shape market structure and pricing strategies and ultimately affect firms' profitability. In this case, let us consider the following economic model where we wish to maximize the performance index J **Error! Reference source not found.**.

$$J\left[u(t)\right]=\int_{t_i}^{t_f} g\left(t,y(t),u(t)\right)dt \tag{1}$$

The integrand function, denoted as g, depends on time t, the state variable $y(t)$ and the control variable $u(t)$. This integrand function encompasses several components, including demand, royalty payment function and discount rate.

$$g=\left(au^{1-\alpha}-\left(\rho+m_0+c_0e^{-\lambda y}\right)u\right)e^{-rt} \tag{2}$$

The performance index described by Eq. (1) is contingent upon the integrand expressed in Eq. (2) . This integrand is governed by various factors, including a representing the demand, α denoting the price elasticity of demand, ρ indicating the royalty function, m_0 representing the asymptote of the learning curve, c_0 reflecting the unit cost component affected by learning, λ defining the learning rate, r accounting for the discount factor, $u(t)$ representing the control variable, and $y(t)$ representing the state variable.

Zinober and Kaivanto **Error! Reference source not found.** explored the optimization of production decisions while considering continuous royalty payment obligations that vary across different segments or stages. Their research aims to identify efficient production strategies that accommodate these specific royalty payment structures. They attempted to solve the economic model Eq. (1) and Eq. (2) using matrix formulation. In this scenario, the focus was on considering the ODE.

$$y'(t) = u(t) \tag{3}$$

However, they encountered challenges in solving the problem when the level of stage payment increased, leading to non-differentiability at certain time frames. This difficulty served as motivation for the development of a new approach to address and solve the model.

Zinober and Sufahani **Error! Reference source not found.** investigated a distinctive OC problem within the realm of economics. The problem is characterized by an unknown and unconstrained state value at the final time, while the Lagrangian integrand in the functional is a piecewise constant function dependent on this unknown value. As discussed in Section 1, dealing with an unknown final state results in a non-zero final shadow value. This necessitates the introduction of a new equation to proceed with the investigation mathematically. In addressing this issue, the works of Cruz *et al.,* **Error! Reference source not found.** and Malinowska and Torres **Error! Reference source not found.** presented a novel boundary condition for handling the unknown final state. Building upon their contributions, this study will follow their approach to effectively overcome the challenge posed by the non-zero final shadow value.

Example 1. Let us examine the fixed royalty payment that remains constant throughout the contract's entire duration.

$$\rho\left(y(t)\right) = 10\% \tag{4}$$

In the first example, the royalty function ρ equivalent to a constant value of 10% is applied in Eq. (2) to maximize the performance index Eq. (1). Figure 2 depicts the fixed amount of royalty payment made from the initial time to the contract's end time.

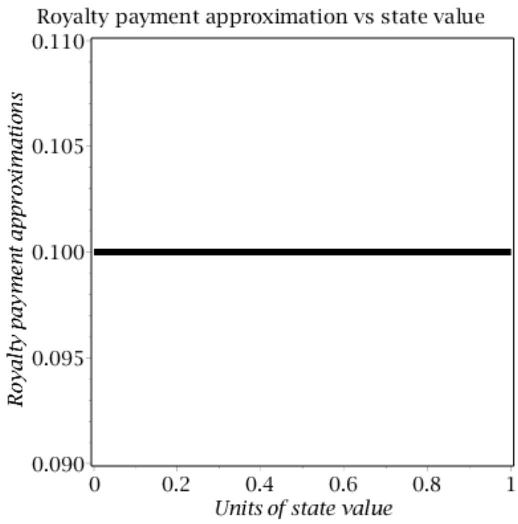


Fig. 2. Fixed royalty payment

With the given royalty payment value Eq. (4), the performance index that we aim to maximize is as follows:

$$J\left[u(t)\right]=\int_{0}^{10}\left(au^{1-\alpha}-\left(0.1+m_0+c_0e^{-\lambda y}\right)u\right)e^{-rt}dt \tag{5}$$

Considering Eq. (1), Eq.(2) and Eq.(4), Eq. (5) emerges as the performance index targeted for maximization, with the initial time t_i being set to zero while the final time t_f is ten. Notably, for 10% fixed royalty payments, the computed optimal final shadow value is equal to zero. The optimal results of the final state value, initial shadow value and performance index for both the hybrid shooting approach and the discretization method are presented in Table 1.

Table 1
Results of the shooting and discretization methods for fixed royalty payment

Methods	Final state value	Initial shadow value	Performance index
Hybrid Newton-Brent shooting	0.794328	0.055411	0.954043
Euler	0.788524	0.055631	0.958304
Runge-Kutta	0.793436	0.055643	0.958404
Trapezoidal	0.783348	0.055643	0.958404
Hermite-Simpson	0.807736	0.055675	0.958699

According to the data presented in Table 1, the hybrid Newton-Brent shooting method yields optimal solutions similar up to two decimal places for the final state value when compared with the Runge-Kutta method. Overall, the optimal final state values are comparable up to one decimal place for all methods except for the Hermite-Simpson approximation. Meanwhile, the optimal shadow values at the initial time are comparable up to three decimal places for all approaches. Additionally, the optimal performance index is comparable up to two decimal places for both the hybrid shooting and discretization methods. The results show that the performance is maximized up to 95% to meet the optimality condition of optimizing the state, costate and control variables. This observation suggests that a fixed amount of royalty payment can be categorized as a classical OC problem since the final shadow value is equal to zero.

Figure 3 illustrates the plots for the optimal state, shadow value, control, and performance index obtained for $t_i = 0$ and $t_f = 10$. In conclusion, all the methods yield nearly similar values during the iteration process.

However, the rigidity of a fixed royalty payment presents various disadvantages. Firstly, it does not adapt to changing market conditions or business performance, potentially resulting in overpayment or underpayment based on actual sales or revenue. Secondly, fixed payments may not incentivize licensees or partners to invest in marketing, sales efforts, or product improvements, as the payment remains constant, leading to a lack of motivation for maximizing sales or enhancing product quality.

Additionally, fixed royalty payments can adversely affect profit margins for licensees or franchisees, especially during periods of low sales, as the fixed cost may become burdensome during business downturns. Over time, if the fixed payment rate becomes unfavourable for the licensee, it may create tension and dissatisfaction in long-term partnerships, causing the licensee to perceive the initial agreement terms as no longer fair or competitive.

From the licensor's perspective, a fixed royalty payment might not adequately compensate for potential risks or fluctuations in the market. This could lead to missed opportunities for additional revenue if the licensed product or service experiences significant success. Furthermore, fixed royalty payments might not align with the licensee's actual performance or promotional efforts, creating a disconnect between the licensee's actions and the reward they receive.

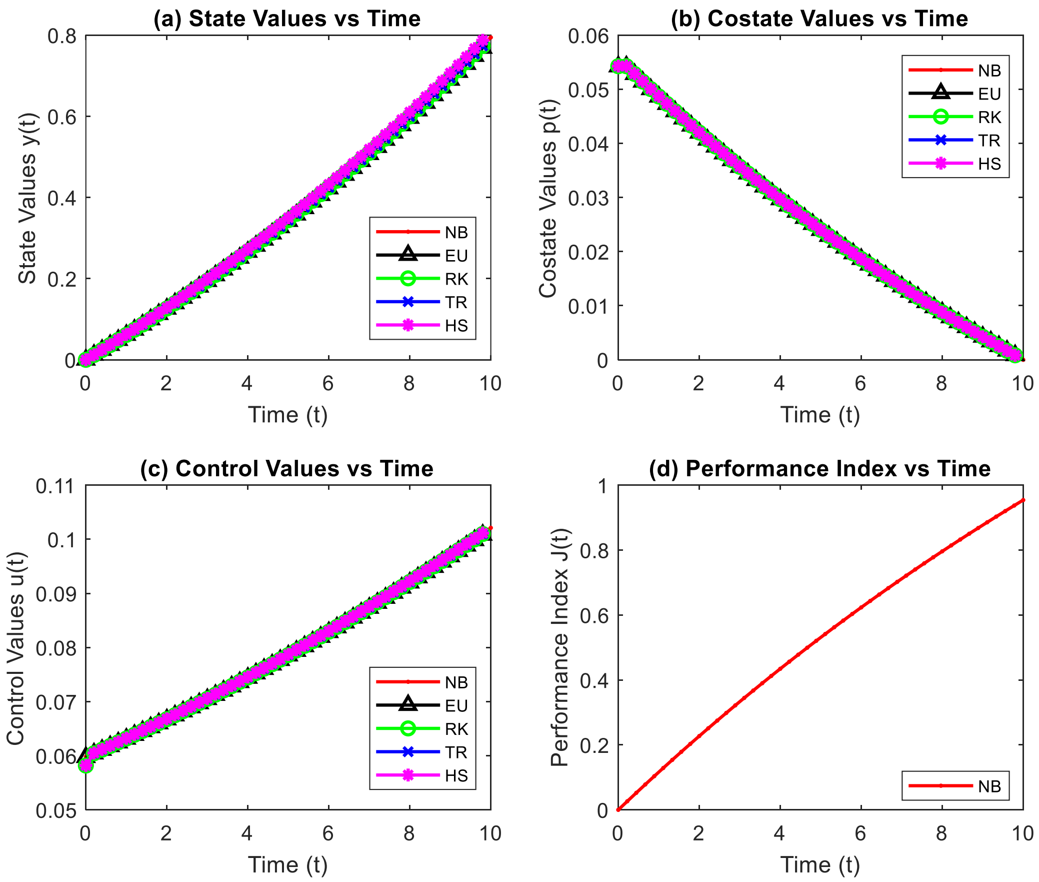


Fig. 3. Plot for the state, shadow value, control and performance index for fixed royalty payment (NB=hybrid Newton-Brent shooting; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson)

Despite the initial ease of setting up a fixed royalty payment structure, it could lead to legal or contractual disputes in the future if the involved parties disagree on the appropriateness of the fixed amount. As a result, considering more dynamic royalty structures that address these shortcomings and align incentives between licensors and licensees becomes crucial for fostering successful and mutually beneficial business partnerships.

To mitigate these drawbacks, we explore a more dynamic approach by testing the method with percentage-based royalties linked to sales or revenue. This adjustment aims to better align both parties' incentives and accommodate market conditions fluctuations. The royalty payment function can be structured in various stages. For instance, let us consider Example 2, a three-stage piecewise royalty function, to illustrate the application of this dynamic approach.

Example 2. Let us examine the following three-stage piecewise function representing a decreasing trend of royalty payment.

$$\rho\left(y(t)\right)=\begin{cases}50\% & \text{for } 0\le y(t)\le 0.2z\\ 40\% & \text{for } 0.2z< y(t)\le 0.8z\\ 10\% & \text{for } 0.8z< y(t)\le z\end{cases} \tag{6}$$

Figure 4 presents the three-stage piecewise function Eq. (6) for the royalty payment, illustrating its discontinuous form.

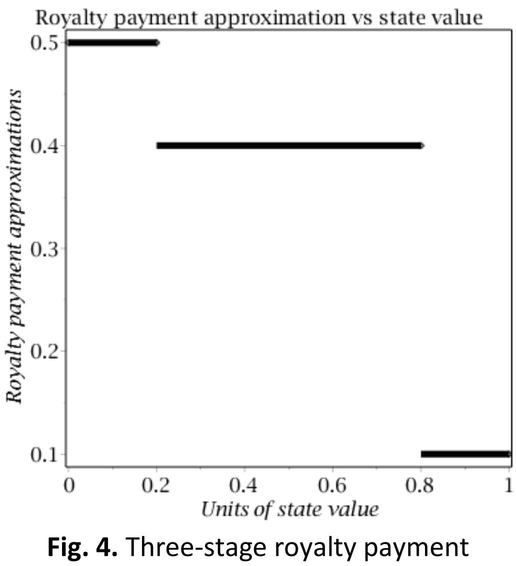


Fig. 4. Three-stage royalty payment

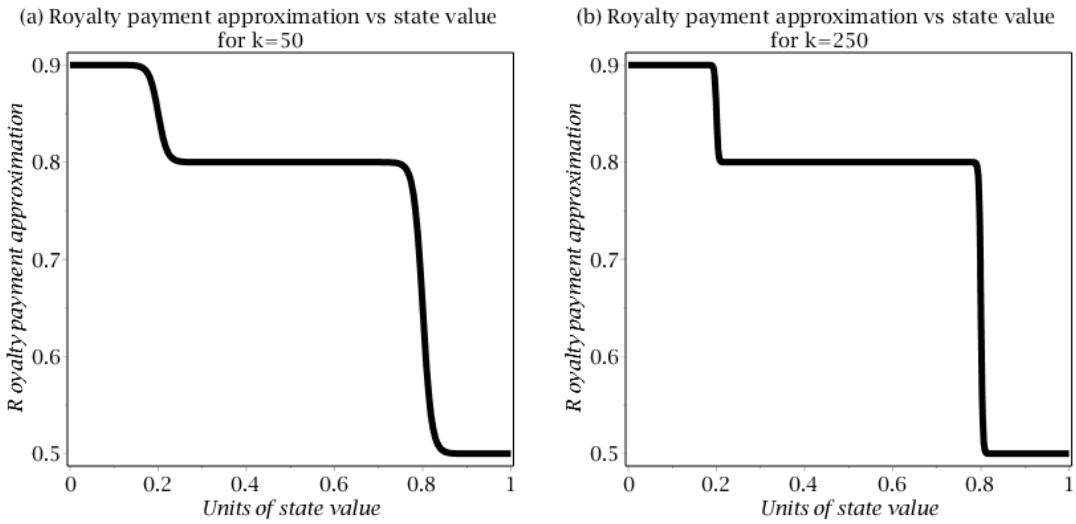
As mentioned in previous research, difficulties arise when dealing with increasing levels of stage payment, making the royalty payment in conditions Eq. (6) non-differentiable at certain time frames **Error! Reference source not found.**. Later, Zinober and Sufahani **Error! Reference source not found.** investigated a continuous approximation of the piecewise constant integrand function to address this complexity. Consequently, the royalty function $\rho(y)$ can be transformed into the hyperbolic tangent (tanh) function.

$$\rho(y) = 0.7 - 0.05\tanh\left(k\left(y - 0.2z\right)\right) - 0.15\tanh\left(k\left(y - 0.8z\right)\right) \tag{7}$$

Skandari **Error! Reference source not found.** primary focus lies on the space of hyperbolic tangent (tanh) functions, and the author successfully demonstrated its universal approximator property. The findings contributed to the field of stabilizer control design for nonlinear dynamical systems by utilizing these functions effectively. Skandari, Ghaznavi and Abedian **Error! Reference source not found.** contributed to a novel methodology for designing stabilizer control in nonlinear systems by employing hyperbolic modelling. The authors successfully demonstrated that the hyperbolic model can be formulated as a state-space model, proving its efficacy in stabilizing intricate nonlinear control systems. One significant advantage of their proposed approach is that it does not depend on fuzzy concepts or expert experience, making it more accessible and easier to implement. Additionally, the model parameters required for the approach can be readily obtained, further enhancing its practicality and effectiveness in designing stabilizer control for nonlinear systems.

The hyperbolic model can be designed as a state-space model, allowing for easy implementation and analysis **Error! Reference source not found.**. Furthermore, the hyperbolic model can be utilized for complex nonlinear control systems, making it applicable to a wide range of systems **Error! Reference source not found.**. The proposed approach does not rely on fuzzy concepts or expert experience, making it applicable to complex nonlinear control systems without the need for linguistic information or fuzzy rule-based systems **Error! Reference source not found.**. Moreover, the model parameters can be easily obtained by solving an NLP problem, simplifying the parameter identification process **Error! Reference source not found.**. According to previous findings, Skandari **Error! Reference source not found.**, the stabilizer control designed based on the hyperbolic model is bounded and can effectively stabilize the original nonlinear system. In other words, the hyperbolic model can approach the original system at any desired accuracy near the equilibrium system **Error! Reference source not found.**. Overall, the use of hyperbolic modelling provides a flexible and efficient approach for stabilizer control in nonlinear systems **Error! Reference source not found.**,**Error! Reference source not found.**.

Taking into account the benefits of hyperbolic modelling, we proceeded to solve Example 2 by approximating the three-stage piecewise royalty payment Eq. (6) with the continuous hyperbolic tangent (tanh) function Eq. (7). Two k values were selected for this study, namely 50 and 250. The smoothing value k plays a crucial role in determining the step size during iteration. Figure 5 reveals that an increase in the value of k results in a smoother plot for the royalty payment.



(a) (b)

Fig. 5. Royalty payment function in continuous hyperbolic tangent (tanh) approximation for smoothing values k equal to (a) 50 and (b) 250

Let us now proceed to maximize the performance index considering Eq. (1), Eq. (2) and Eq. (7) as follows.

$$\text{Maximize } J\left[u(t)\right] = \int_0^{10} \left(a u^{1-\alpha} - \begin{pmatrix} 0.7 - 0.05\tanh\left(k\left(y-0.2z\right)\right) \\ -0.15\tanh\left(k\left(y-0.8z\right)\right) + m_0 + c_0 e^{-\lambda y} \end{pmatrix} u \right) e^{-rt} dt \tag{8}$$

subject to smoothing value k is equal to 50. The optimal final shadow value equals -0.119813 when the smoothing value k equals 50.

Based on Table 2, the optimal final state values obtained from all methods are comparable up to one decimal place. Additionally, the optimal initial shadow value and performance index generated by the hybrid Newton-Brent shooting method are similar up to two decimal places. Overall, the performance is maximized up to 71%, satisfying the optimality condition for optimizing the state, shadow value and control variables.

Table 2
Results of the shooting and discretization methods with $k = 50$

Methods	Final state value	Initial shadow value	Performance index
Hybrid Newton-Brent shooting	0.430334	0.119221	0.710674
Euler	0.426435	0.119783	0.713131
Runge-Kutta	0.428299	0.119410	0.713225
Trapezoidal	0.424676	0.119741	0.713829
Hermite-Simpson	0.435671	0.117956	0.713312

Figure 6 illustrates that the optimal plot for the shadow value follows the pattern of the royalty payment function. When the smoothing value k is set to 250 for maximizing the performance index Eq. (8), the optimal final shadow value is approximately -0.119701, which is quite close to the value obtained with the smoothing value k equal to 50.

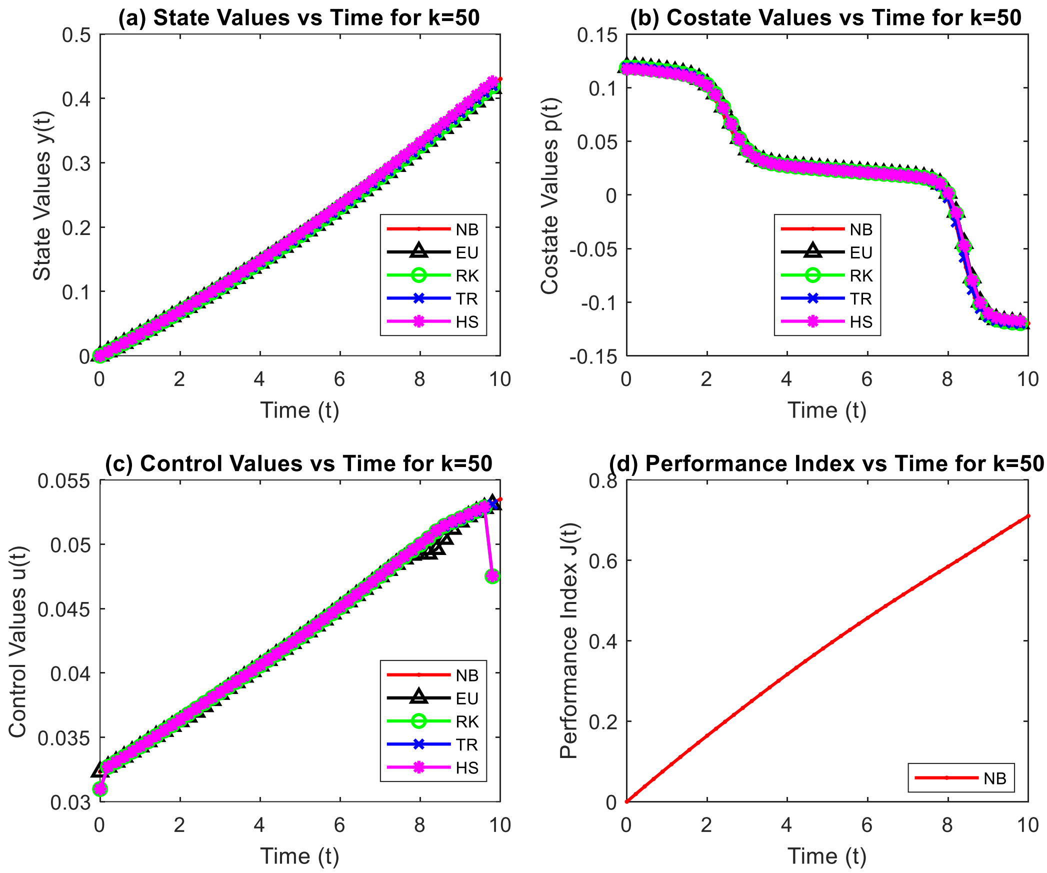


Fig. 6. Plot of the optimal state, shadow value, control and performance index for three-stage royalty payment when the smoothing value k equals 50 (NB=hybrid Newton-Brent shooting; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson)

Based on the findings in Table 3, the optimal values of the final state, initial shadow value, and performance index exhibit noteworthy distinctions for the hybrid Newton-Brent shooting method when compared with the discretization techniques. The hybrid shooting method yields a similar final state value, only up to one decimal place, compared to the results from the discretization technique. Similarly, the optimal initial shadow value obtained through the hybrid shooting method is comparable up to one decimal place when contrasted with the discretization results. However, at the final time, the optimal performance index shows consistent results with a similarity of up to two decimal places for all approaches. Overall, the performance is maximized up to 71% while fulfilling the optimality condition for optimizing the components of the state, shadow value, and control in the context of continuous royalty payment mode.

Table 3
Results of the shooting and discretization methods with $k = 250$

Methods	Final state value	Initial shadow value	Performance index
Hybrid Newton-Brent shooting	0.430452	0.119115	0.710601
Euler	0.427219	0.115111	0.71312
Runge-Kutta	0.429520	0.112206	0.713221

Trapezoidal	0.426555	0.109614	0.713774
Hermite-Simpson	0.436957	0.110323	0.713313

Based on Figure 6 and Figure 7, all the methods exhibit similar plots, with the exception of the discretization plot for the shadow value and control variable. This discrepancy could potentially be attributed to discretization errors that might have occurred during the process, as discussed in previous works **Error! Reference source not found.**,**Error! Reference source not found.**. To conclude, the optimal performance is maximized at the same rate for both chosen smoothing values—however, the higher smoothing value results in a smoother plot, displaying more transparent values during iteration. Due to the non-zero final shadow value, the royalty payment, which occurs in stages, adheres to the non-classical OC formulation.

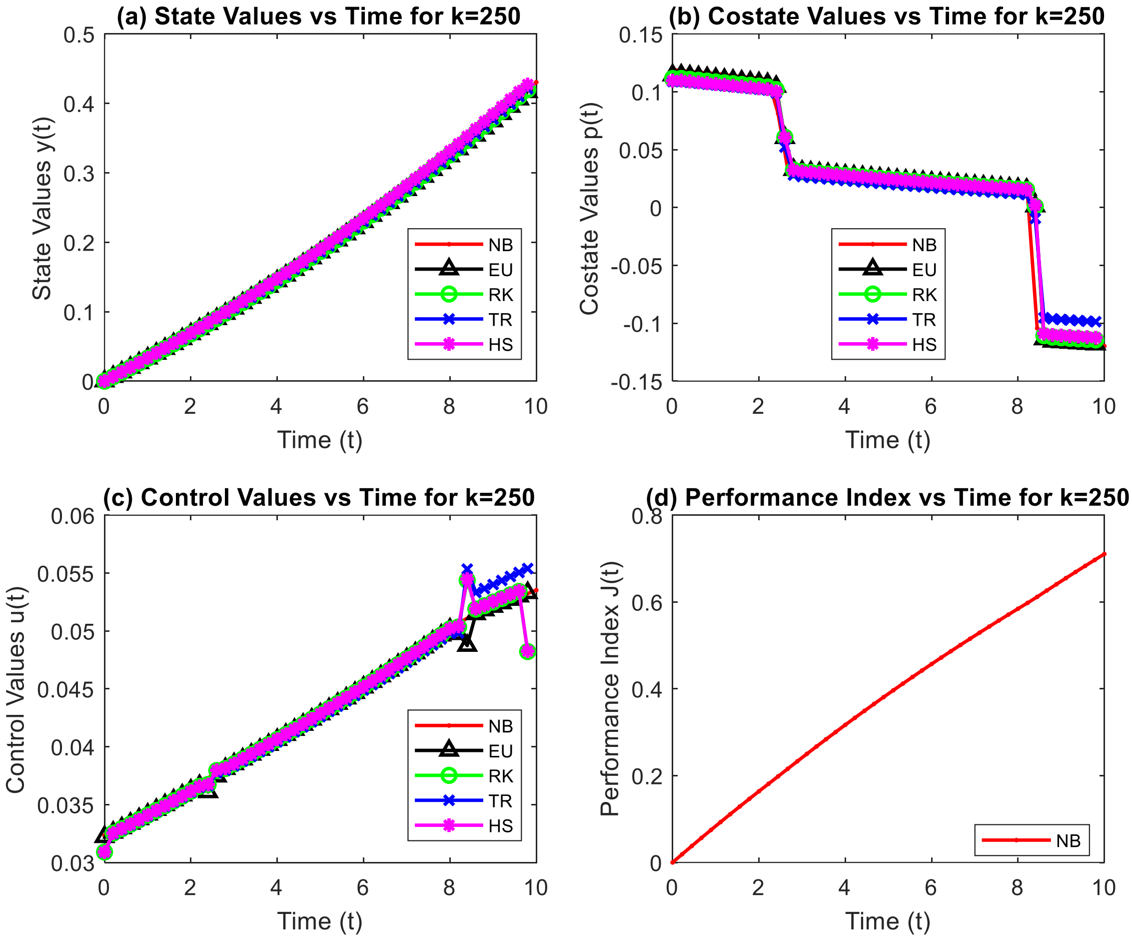


Fig. 7. Plot of the optimal state, shadow value, control and performance index for three-stage royalty payment when the smoothing value k equals 250 (NB=hybrid Newton-Brent shooting; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson)

5. Discussion and Future Research Directions

Through a thorough comparative analysis, it is evident that in Example 1, the discretization method yields an optimal solution similar up to four decimal places for the initial shadow value in comparison to the shooting result. Moreover, when considering the optimal performance index, the results from the discretization approach display a similarity of up to three decimal places compared to the shooting method's outcomes. Notably, the plots depicting the optimal state, costate, control, and performance index while incorporating fixed royalty payments consistently align for both the shooting and discretization methods.

In the case of Example 2, where the royalty payment follows a piecewise function and the program is developed with two distinct smoothing values, notable observations arise. For the first smoothing value set at 50, the discretization results generate an optimal solution that consistently matches up to two decimal places, with the exception of the Hermite-Simpson approximation. Additionally, the performance index's optimal solution for the discretization approach is consistent up to three decimal places. Nevertheless, the shooting method appears to exhibit a higher level of consistency throughout the iteration process when compared to the discretization computation. This distinction is illustrated in Figure 6, wherein the control variable plot highlights potential discretization errors that might occur during the iterative stages.

For a smoothing value equal to 250, the discretization method similarly provides a consistent optimal solution up to two decimal places, except for the Hermite-Simpson approximation for the final state value. Correspondingly, the optimal performance index maintains a consistency of up to three decimal places when compared with the shooting method's results. However, the shooting method showcases enhanced consistency during the iteration process, as evidenced by Figure 7. Notably, the shooting plot exhibits smoother behaviour than the discretization method, particularly evident in the costate and control variable plots.

Promising directions for future research will build the foundation that has been established. Other successful research such as Goh [30] and Syed Abdul Nasir, Ab Wahab and Nasir [31] can be a guideline to further the research towards advance application. Consideration is given to the proposal of investigations that extend, refine, or complement the work. Some potential avenues that could be suggested include:

i. Further refinement and optimization of the hybrid shooting approach by replacing Newton methods such as the Galerkin method.
ii. Exploration of the applicability of the method to different problem domains or industry sectors.
iii. Investigation into the impact of varying model parameters on the performance of the hybrid approach.
iv. Comparison of the hybrid approach with other cutting-edge optimization techniques.
v. Integration of real-world data to enhance the accuracy and applicability of the method.
vi. Application of the methodology to more complex or multi-dimensional non-classical optimal control problems.

6. Conclusion

In conclusion, this paper has presented the effective hybrid Newton-Brent shooting approach with discretization validation to address the challenges posed by the non-classical OC problem. The non-classical OC problem, characterized by an unknown final state variable and a non-differentiable performance index, has been shown to demand innovative techniques for accurate and efficient solutions. The hybrid shooting approach has proven valuable in handling the non-classical OC problem with an unknown final state. By iteratively adjusting initial conditions and leveraging boundary value solvers, this method efficiently computes the unknown final state variable, overcoming one of the key challenges in the non-classical OC problem. We introduced a continuous approach based on the hyperbolic tangent (tanh) function to handle non-differentiability in the performance index. This approach smoothly runs piecewise components, allowing for the application of optimization techniques, and ultimately contributes to improved accuracy in obtaining optimal solutions. Moreover, incorporating discretization methods such as Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson approximations further enhances the results' reliability. By validating the continuous solution against discretized approximations, we ensure the accuracy and robustness of our proposed approach. Our approach has demonstrated its effectiveness and superiority over traditional discretization methods through extensive numerical experiments on various OC scenarios, such as fixed royalty payment and three-stage piecewise royalty function. The results have showcased the efficiency and accuracy of the hybrid shooting approach with discretization validation in navigating complex OC problems. This research highlights the significance of fundamental theory in solving real-world control problems and emphasizes the importance of innovative mathematical approaches. The proposed method served as a valuable contribution to non-classical OC, enabling researchers to tackle challenging OC problems in practical applications. Furthermore, the versatility of our approach opens up new possibilities for future researchers to explore and develop advanced mathematical techniques for OC problem-solving. As the academic field continuously evolves, the hybrid shooting approach with discretization validation ensures that the methods employed remain current and relevant for teaching and learning, particularly in the domain of science and mathematics.

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