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Determination of the Grid-Shaped Transportation Network's Optimization Value via Graph Labelling

Nurdin Hinding[1,*], Amir Kamal Amir[1], Jusmawati Massalesse[1], Nurtiti Sunusi[2], Amil Ahmad Ilham[3], Ali Ahmad[4], Muhammad Kamran Siddiq[5], Cahyudi Gratio[6], Mutmainnah Muhtar Jaya[6], Nur Annisa Syaron[6]

[1] Department of Mathematics, Faculty of Mathematics and Natural Sciences, Hasanuddin University, Makassar, Indonesia
[2] Department of Statistics, Faculty of Mathematics and Natural Sciences, Hasanuddin University, Makassar, Indonesia
[3] Department of Informatics, Faculty of Engineering, Hasanuddin University, Makassar, Indonesia
[4] Department of Computer Sciences & Information System, Jazan University, Jazan, Arab Saudi
[5] Department of Mathematics, Comsats Institute of Information Technology, Sahiwal, Pakistan
[6] Bachelor's Program in Mathematics, Department of Mathematics, Faculty of Mathematics and Natural Sciences, Hasanuddin University, Makassar, Indonesia

ABSTRACT

In order to move people, commodities, and services in an efficient and effective manner, transportation networks are essential. To increase these networks' efficiency and save expenses, they must be optimised. In this research, we offer a unique method that makes use of graph labelling techniques to determine the optimization value of a transportation network. The goal is to give the network's constituent parts labels that accurately represent their potential for optimisations. We start by creating a graph model of the transportation network, with vertices standing in for important places and edges for the links between them. The process of labelling a graph entail giving vertices and edges labels according to their attributes. When choosing the labels, we consider several variables, including capacity, distance, traffic flow, and transportation costs. These elements allow us to record the optimization value of various network components throughout the labelling process. After the graph labelling process is finished, we examine the labelled network to find regions that could want more effort and optimization. The labels aid in decision-making and offer insightful information about how well the network is doing. Based on the optimization values obtained from the graph labelling, we may set priorities for investments and distribute resources accordingly. We apply our method to a real-world transport network case study to verify its efficacy. The outcomes show that the optimization value of network components may be efficiently determined by the graph labelling technique. Targeted interventions, such as infrastructure upgrades, traffic control plans, or route optimizations, might be used to address the areas that have been identified for optimization. In conclusion, network planners and decision-makers can benefit greatly from our method of calculating a transportation network's optimization value using graph labelling. Making well-informed decisions to improve the effectiveness and efficiency of transportation networks can result in increased system performance and cost savings by utilizing the insights obtained from the labelled network.

Keywords:
Irregular labelling; helm graph; circulant graph; grid graph

* *Corresponding author.*
E-mail address: nurdin1701@unhas.ac.id

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1. Introduction

Transportation networks are critical infrastructure systems that facilitate the movement of goods, services, and people. The efficient and effective operation of these networks is essential for economic development and quality of life. To ensure optimal performance, transportation networks need to be continuously evaluated and improved. Optimization techniques provide valuable insights into network performance and aid decision-making processes for network planners and operators.

Graph labelling is a powerful technique that can be applied to analyze and optimize transportation networks. It involves assigning labels to the components of a graph, such as nodes and edges, based on specific characteristics and properties. These labels provide valuable information about the optimization potential of network components and help identify areas that require attention.

The objective of this paper is to present a novel approach for determining the optimization value of a transportation network using graph labelling techniques. By assigning appropriate labels to the network components, we aim to capture their optimization potential and identify areas for improvement. This approach enables decision-makers to prioritize investments and allocate resources effectively.

In this study, we consider various factors that contribute to the optimization value of transportation networks. These factors may include distance, capacity, traffic flow, transportation costs, and other relevant parameters. By incorporating these factors into the graph labelling process, we can obtain a comprehensive understanding of the network's performance and identify opportunities for optimization.

The proposed approach is supported by a case study of a real-world transportation network. By applying the graph labelling technique to this case study, we demonstrate its effectiveness in determining the optimization value of network components. The insights gained from the labeled network enable informed decision-making and can guide the implementation of targeted interventions to enhance network performance.

The remainder of this paper is organized as follows: Section 2 provides a review of relevant literature on transportation network optimization and graph labelling techniques. Section 3 describes the methodology used in this study, including the graph modeling and labelling process. Section 4 presents the results of the case study and discusses the implications for network optimization. Finally, Section 5 concludes the paper and highlights future research directions in the field of transportation network optimization using graph labelling techniques. In this paper, we determining the optimization value of a transportation network model grid.

2. Methodology

The total vertex irregularity strength of the graph G is the minimum positive integer such that G have a total vertex irregular labelling. The total vertex irregularity strength of the graph G denoted by $tvs(G)$. To find the exact value of $tvs(G)$ for some simple connected graph, we have to proof that the lower bound and upper bound of $tvs(G)$ are not different. To find its lower bound, we use Theorem 1 from Baca (2017).

Theorem 1: *Let* $G(V, E)$ be a graph on order n with the minimum and maximum degree of G are $\delta(G)$ and $\Delta(G)$. Thus:

$$tvs(G) \geq \left\lceil \frac{(n+\delta(G))}{(\Delta(G)+1)} \right\rceil \tag{1}$$

To find its upper bound we have to construction a total vertex irregular labelling for G. To construction a total vertex irregular labelling for any graph G, we begin by construction a function for some simple cases of G, up to we have a pattern in give label on all vertices and edges of G. After that we have proof that the weights of all vertices are distinct. The last step, we have to the largest label in use.

There are three difficulties in showing the upper bound of total vertex irregularity strength of the graph. The first is the difficulty in constructing a labelling function. This difficulty arises because the function to be constructed must be believed to be valid for all cases, namely cases regardless of the number of vertices of graph. Another difficulty in construction a function is that the function is not unit and depends on the structure of the graph that is the object.

The second difficulty is in determining that the weights of all vertices are different. This difficulty will arise if the function being constructed is complicated. The third difficulty is in determining the largest label to use. This difficulty occurs because all the labels used must be compared with each other.

3. Results

3.1 Modification of Grid Graph

In this subsection, we define of the graph used in this research, namely a modified of grid graph. Previously given the definition of a grid graph.

Definition 1. A graph is a grid graph if only it is an induced subgraph of a grid.

The graph used in this research is a graph construction from grid graph $4 \times n$ such that the degree of each vertices increases by one, denoted GM_n. To make it easier to carry out research, if the vertex and edge sets of grid graph are:

$V = \{a_i, b_i, c_i, d_i | 1 \leq i \leq n\}$ and (2)

$E = \{a_i a_{i+2}, b_i b_{i+2}, c_i c_{i+2}, d_i d_{i+2} | i = 1,2, \dots, n-2\} \cup \{a_i c_i, c_i d_i, d_i b_i | 1 \leq i \leq n\}$

The vertex and edge sets of GM_n are:

$V(GM_n) = \{a_i, b_i, c_i, d_i | 1 \leq i \leq n\}$ and (3)

$E(GM_n) = \{a_1 b_1, a_2 b_2, a_{n-1} a_n, b_{n-1} b_n, c_{n-1} c_n, d_{n-1} d_n\}$ (4)

$\cup \{a_i a_{i+2}, b_i b_{i+2}, c_i c_{i+2}, d_i d_{i+2} | i = 1,2, \dots, n-2\}$ (5)

$\cup \{a_i c_i, b_i d_i, c_i d_i | i = 1,2, \dots, n\}$ (6)

$\cup \left\{c_i c_j, d_i d_j | 3 \leq i \leq \frac{n+2}{2}, \frac{n+4}{2} \leq j \leq n, i = k+2, j = n-i+3, k = 1,2, \dots, \frac{n-2}{2}\right\}$ (7)

For example, in Figure 1 give GM_6.

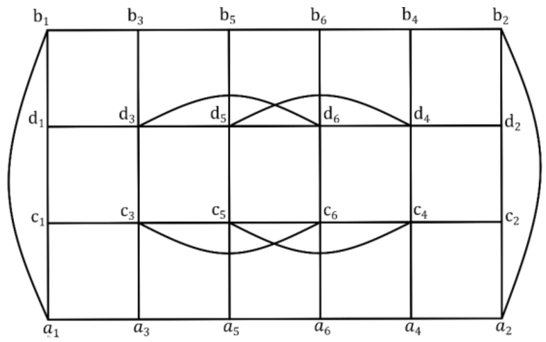

Fig. 1. Graph GM$_6$

3.2 Total Irregularity Strength of GM_n

In this subsection, we give the total vertex irregularity strength of a modified of grid graph.

Theorem 2. For even integer $\mathrm{n} \geq 6$, then:

$$tvs(GM_n) = \left\lceil \frac{4n+3}{6} \right\rceil \tag{8}$$

Proof: To find the total vertex irregularity strength of GM_n, we have to find its the lower and upper bounds. To find the lower bound, we use Theorem 1. Since order of GM_n is $4n$, minimum degree is 3, maximum degree is 5, $tvs(GM_n) \geq \left\lceil \frac{4n+3}{6} \right\rceil$.

To find the upper bound, we have to construction a total vertex irregular labelling, $f: V \cup E \rightarrow \{1, 2, 3, \dots, k\}$ for some k, in tree cases as follows. These three cases emerged because the same pattern was not found for the three cases.

Case 1: For $n = 6$, the total vertex irregular labelling, as shown in Figure 2.

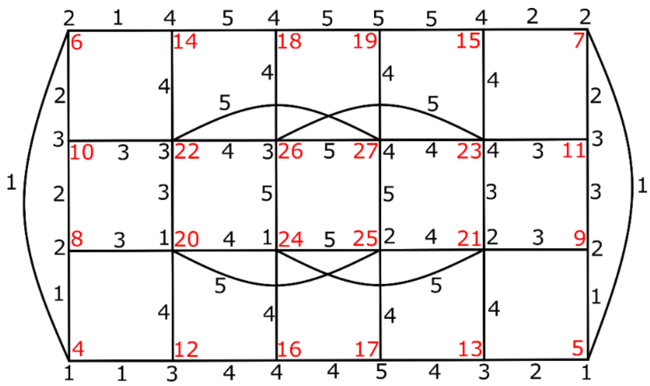

Fig. 2. The total vertex irregular 5-labelling of GM$_6$

Case 2: For $n = 8$, the total vertex irregular labelling, as shown in Figure 3.

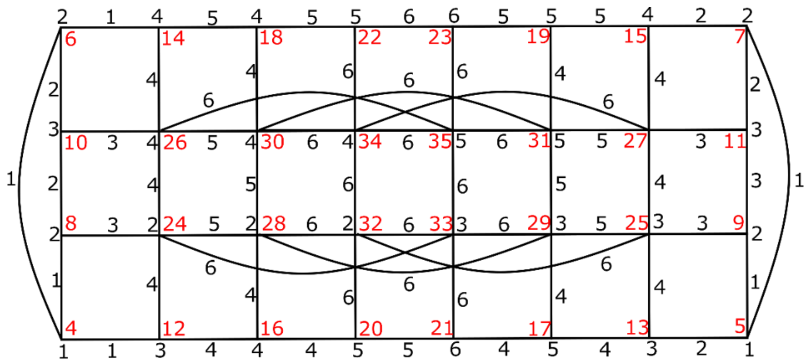

Fig. 3. The total vertex irregular 6-labelling of GM_8

Case 3: For $n \geq 10$, the total vertex irregular labelling as follows.

$$f(a_i) = \begin{cases} 1, & i = 1,2 \\ 4, & i = 3,4, \end{cases} \tag{9}$$

$$(b_i) = \begin{cases} 2, & i = 1,2 \\ 5, & i = 3,4, \end{cases} \tag{10}$$

$$f(a_i) = f(b_i) = \begin{cases} \frac{i+1}{3} + 3, & i \equiv 5 \ (mod\ 6) \\ \frac{i}{3} + 4, & i \equiv \begin{cases} 0 \ (mod\ 6) \\ 3 \ (mod\ 6) \end{cases} \\ \frac{i-1}{3} + 4, & i \equiv 1 \ (mod\ 6) \\ \frac{i+1}{3} + 4, & i \equiv 2 \ (mod\ 6) \\ \frac{i-1}{3} + 5, & i \equiv 4 \ (mod\ 6), \end{cases} \tag{11}$$

$$f(c_i) = 2, \ i = 1,2 \tag{12}$$

$$f(d_i) = 3, \ i = 1,2 \tag{13}$$

$$f(a_i b_i) = 1, \ i = 1,2 \tag{14}$$

$$f(a_i a_{i+2}) = \begin{cases} 1, & i = 1 \\ 2, & i = 2 \\ 3, & i = 3,4 \end{cases} \tag{15}$$

$$(a_i a_{i+2}) = \begin{cases} \frac{i+1}{2} + 1, & i \equiv 5 (mod\ 6) \\ \frac{i}{2} + 1, & i \equiv \begin{cases} 0 \ (mod\ 6) \\ 2 \ (mod\ 6) \\ 4 \ (mod\ 6) \end{cases} \\ \frac{i-1}{2} + 2\,, & i \equiv \begin{cases} 1 \ (mod\ 6) \\ 3 \ (mod\ 6) \end{cases} \end{cases} \tag{16}$$

$$(a_{n-1} a_n) = \frac{n}{2} + 1 \tag{17}$$

$$(a_i c_i) = \begin{cases} 1, & i = 1,2 \\ 4, & i = 3,4 \end{cases} \tag{18}$$

$$f(b_i d_i) = \begin{cases} 2, & i = 1,2 \\ 4, & i = 3,4 \end{cases} \tag{19}$$

$$f(a_i c_i) = f(b_i d_i) = \begin{cases} \frac{2i+2}{3}, & i \equiv 5 \ (mod\ 6) \\ \frac{2i}{3}, & i \equiv \begin{cases} 0 \ (mod\ 6) \\ 3 \ (mod\ 6) \end{cases} \\ \frac{2i+1}{3}, & i \equiv 1 \ (mod\ 6) \\ \frac{2i-1}{3}, & i \equiv 2 \ (mod\ 6) \\ \frac{2i-2}{3}, & i \equiv 4 \ (mod\ 6) \end{cases} \tag{20}$$

$$f(b_i b_{i+2}) = \begin{cases} 1, & i = 1 \\ 2, & i = 2 \\ 4, & i = 3,4 \end{cases} \tag{21}$$

$$(b_i b_{i+2}) = \begin{cases} \frac{i+1}{2} + 2, & i \equiv 5 (mod\ 6) \\ \frac{i}{2} + 2, & i \equiv \begin{cases} 0 \ (mod\ 6) \\ 2 \ (mod\ 6) \\ 4 \ (mod\ 6) \end{cases} \\ \frac{i-1}{2} + 3, & i \equiv \begin{cases} 1 \ (mod\ 6) \\ 3 \ (mod\ 6) \end{cases} \end{cases} \tag{22}$$

$$f(b_{n-1} b_n) = \frac{n}{2} + 2 \tag{23}$$

$$f(c_i c_{i+2}) = f(d_i d_{i+2}) = \begin{cases} 3, & i = 1,2 \\ \frac{n}{2} + 1, & i = 3,4 \end{cases} \tag{24}$$

$$f(c_i c_{i+2}) = f(d_i d_{i+2}) = \begin{cases} \frac{i+1}{6} + \frac{n}{2} + 1, & i \equiv 5 \ (mod\ 6) \\ \frac{i}{6} + \frac{n}{2} + 1, & i \equiv 0 \ (mod\ 6) \\ \frac{i-1}{6} + \frac{n}{2} + 1, & i \equiv 1 \ (mod\ 6) \\ \frac{i-2}{6} + \frac{n}{2} + 1, & i \equiv 2 \ (mod\ 6) \\ \frac{i-3}{6} + \frac{n}{2} + 1, & i \equiv 3 (mod\ 6) \\ \frac{i+2}{6} + \frac{n}{2}, & i \equiv 4 \ (mod\ 6) \end{cases} \tag{25}$$

$$f(c_i d_i) = \begin{cases} 2, & i = 1 \\ 3, & i = 2 \\ \frac{n}{2}, & i = 3,4 \end{cases} \tag{26}$$

$$(c_i d_i) = \begin{cases} \frac{2i+2}{3} + 1, & i \equiv 5\ (mod\ 6) \\ \frac{2i}{3} + 1, & i \equiv 0\ (mod\ 6) \\ \frac{2i+1}{3} + 1, & i \equiv \begin{cases} 1\ (mod\ 6) \\ 4\ (mod\ 6) \end{cases} \\ \frac{2i-1}{3} + 1, & i \equiv 2\ (mod\ 6) \\ \frac{2i}{3} + 2, & i \equiv 3\ (mod\ 6) \end{cases} \tag{27}$$

Subcase 3.1: For $n \equiv 4$ (mod 6).

$$f(c_i) = \begin{cases} \frac{n-4}{3}, & i = 3 \\ \frac{n-1}{3}, & i = 4 \end{cases} \tag{28}$$

$$f(c_i) = \begin{cases} \frac{i+1}{3} + \frac{n-1}{3} - 3, & i \equiv 5\ (mod\ 6) \\ \frac{i}{3} + \frac{n-1}{3} - 2, & i \equiv \begin{cases} 0\ (mod\ 6) \\ 3\ (mod\ 6) \end{cases} \\ \frac{i-1}{3} + \frac{n-1}{3} - 2, & i \equiv 1\ (mod\ 6) \\ \frac{i+1}{3} + \frac{n-1}{3} - 2, & i \equiv 2\ (mod\ 6) \\ \frac{i-1}{3} + \frac{n-1}{3} - 1, & i \equiv 4\ (mod\ 6), \end{cases} \tag{29}$$

$$f(d_i) = \begin{cases} \frac{n-1}{3} + 1, & i = 3 \\ \frac{n-1}{3} + 2, & i = 4, \end{cases} \tag{30}$$

$$f(d_i) = \begin{cases} \frac{i+1}{3} + \frac{n-1}{3} - 1, & i \equiv 5\ (mod\ 6) \\ \frac{i}{3} + \frac{n-1}{3}, & i \equiv \begin{cases} 0\ (mod\ 6) \\ 3\ (mod\ 6) \end{cases} \\ \frac{i-1}{3} + \frac{n-1}{3}, & i \equiv 1\ (mod\ 6) \\ \frac{i+1}{3} + \frac{n-1}{3}, & i \equiv 2\ (mod\ 6) \\ \frac{i-1}{3} + \frac{n-1}{3} + 1, & i \equiv 4\ (mod\ 6), \end{cases} \tag{31}$$

$$f(c_{n-1}c_n) = f(d_{n-1}d_n) = \frac{2n-2}{3} + 1 \tag{32}$$

$$f\left(c_i c_j\right) = f\left(d_i d_j\right) = \frac{2n+1}{3} + 1, i = 3,4, \dots, (\frac{n+2}{2}) \text{ dan } j = n - i + 3 \tag{33}$$

Subcase 3.2: For $n \equiv 0$ (mod 6).

$$f(c_i) = \begin{cases} \frac{n}{3} - 1, & i = 3 \\ \frac{n}{3}, & i = 4 \end{cases} \tag{34}$$

$$f(c_i) = \begin{cases} \frac{i+1}{3} + \frac{n}{3} - 3, & i \equiv 5 \ (mod\ 6) \\ \frac{i}{3} + \frac{n}{3} - 2, & i \equiv \begin{cases} 0 \ (mod\ 6) \\ 3 \ (mod\ 6) \end{cases} \\ \frac{i-1}{3} + \frac{n}{3} - 2, & i \equiv 1 \ (mod\ 6) \\ \frac{i-2}{3} + \frac{n}{3} - 1, & i \equiv 2 \ (mod\ 6) \\ \frac{i-1}{3} + \frac{n}{3} - 1, & i \equiv 4 \ (mod\ 6) \end{cases} \tag{35}$$

$$f(d_i) = \begin{cases} \frac{n}{3} + 1, & i = 3 \\ \frac{n}{3} + 2, & i = 4 \end{cases} \tag{36}$$

$$f(d_i) = \begin{cases} \frac{i+1}{3} + \frac{n}{3} - 1, & i \equiv 5 \ (mod\ 6) \\ \frac{i}{3} + \frac{n}{3}, & i \equiv \begin{cases} 0 \ (mod\ 6) \\ 3 \ (mod\ 6) \end{cases} \\ \frac{i-1}{3} + \frac{n}{3}, & i \equiv 1 \ (mod\ 6) \\ \frac{i+1}{3} + \frac{n}{3}, & i \equiv 2 \ (mod\ 6) \\ \frac{i-1}{3} + \frac{n}{3} + 1, & i \equiv 4 \ (mod\ 6), \end{cases} \tag{37}$$

$$(c_{n-1}c_n) = f(d_{n-1}d_n) = \frac{2n}{3} + 1 \tag{38}$$

$$\left(c_i c_j\right) = f\left(d_i d_j\right) = \frac{2n}{3} + 1, i = 3,4, \ldots, (\frac{n+2}{2}) \text{ dan } j = n - i + 3 \tag{39}$$

Subcase 3.3: For $n \equiv 2 \ (\mathrm{mod}\ 6)$.

$$f(c_i) = \begin{cases} \frac{n+1}{3} - 1, & i = 3 \\ \frac{n+1}{3}, & i = 4 \end{cases} \tag{40}$$

$$f(c_i) = \begin{cases} \frac{i+1}{3} + \frac{n+1}{3} - 3, & i \equiv 5 \ (mod\ 6) \\ \frac{i}{3} + \frac{n+1}{3} - 2, & i \equiv \begin{cases} 0 \ (mod\ 6) \\ 3 \ (mod\ 6) \end{cases} \\ \frac{i-1}{3} + \frac{n+1}{3} - 2, & i \equiv 1 \ (mod\ 6) \\ \frac{i+1}{3} + \frac{n+1}{3} - 2, & i \equiv 2 \ (mod\ 6) \\ \frac{i-1}{3} + \frac{n+1}{3} - 1, & i \equiv 4 \ (mod\ 6) \end{cases} \tag{41}$$

$$(d_i) = \begin{cases} \frac{n+1}{3} + 1, \ i = 3 \\ \frac{n+1}{3} + 2, \ i = 4 \end{cases} \tag{42}$$

$$f(d_i) = \begin{cases} \frac{i+1}{3} + \frac{n+1}{3} - 1, \ i \equiv 5 \ (mod\ 6) \\ \frac{i}{3} + \frac{n+1}{3}, \qquad i \equiv \begin{cases} 0 \ (mod\ 6) \\ 3 \ (mod\ 6) \end{cases} \\ \frac{i-1}{3} + \frac{n+1}{3}, \qquad i \equiv 1 \ (mod\ 6) \\ \frac{i+1}{3} + \frac{n+1}{3}, \qquad i \equiv 2 \ (mod\ 6) \\ \frac{i-1}{3} + \frac{n+1}{3} + 1, \ i \equiv 4 \ (mod\ 6) \end{cases} \tag{43}$$

$$f(c_{n-1}c_n) = f(d_{n-1}d_n) = \frac{2n-1}{3} + 1 \tag{44}$$

$$f(c_i c_j) = f(d_i d_j) = \frac{2n-1}{3} + 1, i = 3,4, \dots, (\frac{n+2}{2}) \text{ dan } j = n - i + 3 \tag{45}$$

Based on Eqs. 9 to 45, it can be obtained that:

$wt(a_1) < wt(a_2) < wt(b_1) < wt(b_2) < wt(c_1) < wt(c_2) < wt(d_1) < wt(d_2) < wt(a_3) < wt(a_4) < wt(b_3) < wt(b_4) < wt(a_5) < wt(a_6) < wt(b_5) < wt(b_6) < \cdots < wt(a_{n-1}) < wt(a_n) < wt(b_{n-1}) < wt(b_n) < wt(c_3) < wt(c_4) < wt(d_3) < wt(d_4) < wt(c_5) < wt(c_6) < wt(d_5) < wt(d_6) < \cdots < wt(c_{n-1}) < wt(c_n) < wt(d_{n-1}) < wt(d_n)$.

This shows that all vertices of GM_n have weight distinct. So that, the function $f: V \cup E \to \{1, 2, 3, \dots, k\}$ is total vertex irregular k-labelling of GM_n where $k = \left\lceil \frac{4n+3}{6} \right\rceil$. Therefore, $tvs(GM_n) \leq \left\lceil \frac{4n+3}{6} \right\rceil$.

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References

[1] Bača, Martin, Mirka Miller, and Joseph Ryan. "On irregular total labellings." *Discrete mathematics* 307, no. 11-12 (2007): 1378-1388. https://doi.org/10.1016/j.disc.2005.11.075
[2] Faheem, Hira. "Vertex Irregular Total Labeling Of Grid Graph." *Palestine Journal of Mathematics* 8, no. 1 (2019): 52-62.
[3] Chartrand, Gary, Linda Lesniak, and Ping Zhang. *Graphs & digraphs*. Vol. 22. London: Chapman & Hall, 1996.
[4] Fatimah, Sitti. "Nilai Total Ketidakteraturan Titik Graf Splitting." *Universitas Hasanuddin* (2018).
[5] Harianja, Salamah Fitriani. "Nilai Total Ketakteraturan Titik dari Graf Seri Paralel (m, 1, 5)." PhD diss., Universitas Islam Negeri Sultan Syarif Kasim Riau, 2021.
[6] Hinding, Nurdin, Dian Firmayasari, Hasmawati Basir, Martin Bača, and Andrea Semaničová-Feňovčíková. "On irregularity strength of diamond network." *AKCE International Journal of Graphs and Combinatorics* 15, no. 3 (2018): 291-297. https://doi.org/10.1016/j.akcej.2017.10.003
[7] Nurlindah, Nurlindah. "Nilai Total Ketidakteraturan Titik Graf Dodecahedral yang Dimodifikasi." PhD diss., Universitas Hasanuddin, 2021.

[8] Shah, Dhairya, Manoj Sahni, Ritu Sahni, Ernesto León-Castro, and Maricruz Olazabal-Lugo. "Series of floor and ceiling function-part I: Partial summations." *Mathematics* 10, no. 7 (2022): 1178. https://doi.org/10.3390/math10071178