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A Study for Solving Pseudo-Parabolic Viscous Diffusion, Telegraph, Poisson and Helmholtz PDE using Legendre-Collocation Method

Mohamed Fathy Emam[1], Passant Khaled Abbassi[2,*], Reda Amin Elbarkouky[3], Khaled Mohamed Abdelgaber[4]

[1] Basic and Applied Science Department, College of Engineering and Technology, Arab Academy for Science, Technology and Maritime Transport, Cairo, Egypt
[2] Mathematics Department, Faculty of Engineering, German University in Cairo, Cairo, Egypt
[3] Mathematics Department, Faculty of Engineering, Ain Shams University, Abbassia, Cairo, Egypt
[4] Department of Physics & Engineering Mathematics, Faculty of Engineering - Mataria, Helwan University, Cairo, Egypt

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ABSTRACT

Legendre collocation approach is implemented employing the operative-differential array escorted by the shifted Legendre polynomials. The significance of the proposed approach is reducing the intended partial differential equation problem into an algebraic system of equations by the aid of Legendre polynomials as orthogonal basis. Using mesh collocation points, the domain of the resulting algebraic system solution domain is discretized, and it is then solved to obtain an approximation solution. To demonstrate the accuracy of the procedure, the maximum absolute error and the estimate error are calculated. Also, Illustrative numerical examples are demonstrated to display the veracity of the potential approach in particular pseudo-parabolic viscous diffusion, telegraph, Poisson and Helmholtz partial differential equations. The generated results are contrasted with the exact solutions/other methods to highlight the efficiency of the proposed approach.

1. Introduction

There are enormous physical phenomena, engineering and applied science applications modelled mathematically as systems of higher-order linear/nonlinear partial differential equations [1]. Solving these systems analytically isn't always promising so the numerical techniques are useful in this case. Recent research used computational techniques like collocation and spectral approaches. Spectral approaches introduced solutions as an infinite series of polynomials that have orthogonality properties. Contrarily, collocation technique is a strategy for finding the coefficients of a basis functions expansion to nullify the values of a problem equation at certain points. Both methods have been utilized extensively in computer systems research with many applications. When solving differential/integral equations using a spectral approach, the proposed solution is often represented

* *Corresponding author.*
E-mail address: passant.abbassy@guc.edu.eg

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as a truncated series of smooth orthogonal polynomials. The expansion has unknown coefficients in such a form where the variables need to be determined. The orthogonal polynomials such as Chebyshev, Sinc, wavelet, Laguerre, Legendre, or Jacobi functions used for solving numerous engineering and science problems discussed in [2-15].

Collocation method has been very helpful in obtaining an accurate solution for differential/integral equations utilizing various kinds of bases. The essential principle behind this method is that a differential equation's solution may be assumed by a linear compound of basis customized to fit the specific problem [16-29].

The Legendre polynomials are an essential family of orthogonal polynomials with applications throughout the mathematical and physical sciences [30]. Among their many useful qualities, the polynomials possess their orthogonality regarding the usual inner product on the interval [-1, 1]. These intervals are commonly employed in numerical techniques for solving differential equations and in signal processing, as they are convenient for estimating functions on that interval. It turns out that the wave functions of particles in a spherically symmetric potential may be described with the help of Legendre polynomials, which are not only useful in classical mechanics but also in quantum mechanics. When taken as a whole, Legendre polynomials are a powerful tool in a variety of mathematical and physical contexts [8-13,31-35].

The main purpose of the study is proposing the Legendre collocation approach for solving linear partial differential equations using shifted Legendre polynomials as basis. The problem is modelled as a linear partial differential equation and diminished to a group of linear equations. Operative-differential array of Legendre polynomials is obtained according to the problem modelling as highlighted in section 2. The solution is approximated using the collocation method as illustrated in part 3. An analysis of the estimated error is displayed in section 4. In part 5, the pseudo-parabolic viscous diffusion, telegraph equations, Poisson equation and Helmholtz equation are solved numerically by the proposed approach to exhibit the efficiency and certainty of solution. Section 6 presents the conclusion.

The study problem is modelled as linear partial differential equation

$$\rho\frac{\partial^2 u}{\partial t^2}+\upsilon\frac{\partial u}{\partial t}+\mu\frac{\partial^3 u}{\partial x^2\partial t}+\alpha\frac{\partial^2 u}{\partial x^2}+\beta u=f(x,\, t),\ \text{in}\ Q\equiv\Omega\times I, \tag{1}$$

susceptible to

$$u(0,\, t)=u_0(t),\quad u(1,\, t)=u_1(t),\quad u(x,\, 0)=u_2(x),\quad u_t(x,0)=u_3(x), \tag{2}$$

where $u(x,t)$ is an unknown function (required solution), $f(x,t), u_0(t),\ u_1(t), u_2(x)$ and $u_3(x)$ are known functions, I is $t\in(0,\ T),\ T<\infty$, and Ω is $x\in[0,1]$.

2. Mathematical Background

The first kind Legendre polynomial of order n, $P_n(x)$, is

$$P_n(x)=2^n\sum_{k=0}^{n}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k, \tag{3}$$

where $x\in[-1,1]$. The shifted version is denoted by $\tilde{P}_n(x)=P_n(2x-1)$ such that

$$(n+1)\tilde{P}_{n+1}(x) = (2n+1)(2x-1)\tilde{P}_n(x) - n\tilde{P}_{n-1}, \qquad n = 1, 2, \dots, \tag{4}$$

where $\tilde{P}_0(x) = 1$ and $\tilde{P}_1(x) = 2x - 1$. Consequently, any role $y(x) \in L^2[0,1]$ can be approximated as sum of $\tilde{P}_n(x)$, i.e.,

$$y(x) \cong y_N(x) = \sum_{i=0}^{N} c_i \tilde{P}_i(x) = \mathbf{P}(x)\,\mathbf{C}, \;\; y'(x) \cong \mathbf{P}(x)\,\mathbf{D}^T\mathbf{C}, \;\; y''(x) \cong \mathbf{P}(x)\,(\mathbf{D}^T)^2\,\mathbf{C} \tag{5}$$

where $c_i = (2i+1)\int_0^1 y(x)\tilde{P}_i(x)\,dx$, $\mathbf{P}(x) = [\tilde{P}_0(x)\;\; \tilde{P}_1(x) \cdots \tilde{P}_N(x)]$, $\mathbf{C} = [c_0\; c_1 \cdots c_N]^t$, $\mathbf{D}$ is $(N+1) \times (N+1)$ operative array of derivative whose elements are [32]

$$d_{ij} = \begin{cases} 2(2j-1), & \text{for } j = i-k, \\ 0, & \text{otherwise} \end{cases} \text{ and } k = \begin{cases} 1,3,5,\dots,N, & \text{if } N \text{ is odd} \\ 1,3,5,\dots,N-1, & \text{if } N \text{ is even} \end{cases}$$

3. Legendre Collocation Method

To solve Eq. (1) subject to Eq. (2), it is converted into the system

$$u_t = v \;\; \text{and} \;\; \rho\, v_t + \upsilon\, v + \mu\, v_{xx} + \alpha\, u_{xx} + \beta u = f(x,\ t) \tag{6}$$

susceptible to

$$u(0,\ t) = u_0(t),\;\; u(1,\ t) = u_1(t) \text{ on } \partial\Omega, \qquad u(x,\ 0) = u_2(x),\;\; v(x,\ 0) = u_3(x) \text{on } \Omega. \tag{7}$$

The approximate solutions

$$u_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\, \tilde{P}_i(x)\tilde{P}_j(t) = \mathbf{P}(x,t)\,\mathbf{C} \;\; \text{and} \;\; v_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} \hat{c}_{ij}\, \tilde{P}_i(x)\tilde{P}_j(t) = \mathbf{P}(x,t)\,\hat{\mathbf{C}} \tag{8}$$

are invoked into the system of Eq. (6) where $\mathbf{P}(x,t) = \mathbf{P}(x) \otimes \mathbf{P}(t) = [\boldsymbol{P}_0(x)\boldsymbol{P}_0(t) \dots \boldsymbol{P}_0(x)\boldsymbol{P}_N(t) \dots \boldsymbol{P}_N(x)\boldsymbol{P}_0(t) \dots \boldsymbol{P}_N(x)\boldsymbol{P}_N(t)]$, $\otimes$ is the Kronecker product, $\mathbf{C} = [c_{00} \dots c_{0N} \dots c_{N0} \dots c_{NN}]^t$ and $\hat{\mathbf{C}} = [\hat{c}_{00} \dots \hat{c}_{0N} \dots \hat{c}_{N0} \dots \hat{c}_{NN}]^t$. Clearly, it can be said that

$$\frac{\partial u_N(x,\ t)}{\partial x} = (\mathbf{P}(x)\mathbf{D}^T \otimes \mathbf{P}(t)\mathbf{I}_{N+1})\,\mathbf{C} = \big(\mathbf{P}(x) \otimes \mathbf{P}(t)\big)\mathbf{D}_1\mathbf{C}, \tag{9}$$

$$\frac{\partial^2 u_N(x,\ t)}{\partial x^2} = (\mathbf{P}(x)(\mathbf{D}^T)^2 \otimes \mathbf{P}(t)\mathbf{I}_{N+1})\,\mathbf{C} = \big(\mathbf{P}(x) \otimes \mathbf{P}(t)\big)(\mathbf{D}_1)^2\mathbf{C}, \tag{10}$$

$$\frac{\partial u_N(x,\ t)}{\partial t} = (\mathbf{P}(x)\mathbf{I}_{N+1} \otimes \mathbf{P}(t)\mathbf{D}^T)\,\mathbf{C} = \big(\mathbf{P}(x) \otimes \mathbf{P}(t)\big)\mathbf{D}_2\mathbf{C}. \tag{11}$$

Also, in the same manner, it can be said that

$$\frac{\partial v_N(x,\ t)}{\partial t} = \big(\mathbf{P}(x) \otimes \mathbf{P}(t)\big)\mathbf{D}_2\,\hat{\mathbf{C}} \quad \text{and} \quad \frac{\partial^2 v_N(x,\ t)}{\partial x^2} = \big(\mathbf{P}(x) \otimes \mathbf{P}(t)\big)(\mathbf{D}_1)^2\,\hat{\mathbf{C}}, \tag{12}$$

where $\mathbf{D}_1 = \mathbf{D}^T \otimes \mathbf{I}_{N+1}, \mathbf{D}_2 = \mathbf{I}_{N+1} \otimes \mathbf{D}^T$ and $\mathbf{I}_{N+1}$ is the identity array of size $N+1$.
The system Eq. (6) is discretized into the form

$$\frac{\partial u_N(x_i, t_j)}{\partial t} = v_N(x_i, t_j), \tag{13}$$

$$\rho\,\frac{\partial v_N(x_i, t_j)}{\partial t} + \upsilon\, v_N(x_i, t_j) + \mu\,\frac{\partial^2 v_N(x_i, t_j)}{\partial x^2} + \alpha\frac{\partial^2 u_N(x_i, t_j)}{\partial x^2} + \beta\, u_N(x_i, t_j) = f(x_i, t_j), \tag{14}$$

where $i = 0, 1, 2, \dots., N,\ j = 1, 2, \dots., N$ and the collocation nodes are defined as follows

$$x_i = \frac{1}{2}\left(\cos\left(\frac{i\pi}{N}\right) + 1\right) \quad \text{and} \quad t_j = \frac{T}{2}\left(\cos\left(\frac{j\pi}{N}\right) + 1\right) \tag{15}$$

Substituting Eq. (8)-(12) into the system Eq. (13)-(14) obtains

$$\mathbf{P}(x_i, t_j)(\mathbf{D}_2\,\mathbf{C} - \hat{\mathbf{C}}) = \mathbf{O}_{N^2+N}, \tag{16}$$

$$\mathbf{P}(x_i, t_j)\big[(\alpha\,(\mathbf{D}_1)^2 + \beta\,\mathbf{I}_{N+1})\mathbf{C} + (\rho\mathbf{D}_2\ + \upsilon\ \mathbf{I}_{N+1}\ + \mu\,(\mathbf{D}_1)^2)\hat{\mathbf{C}}\,\big] = \mathbf{f}(x_i, t_j), \tag{17}$$

with conditions

$$\mathbf{P}(0, t_j)\mathbf{C} = u_0(t_j), \quad \mathbf{P}(1, t_j)\mathbf{C} = u_1(t_j), \quad \mathbf{P}(x_i, 0)\mathbf{C} = u_2(x_i), \quad \mathbf{P}(x_i, 0)\hat{\mathbf{C}} = u_3(x_i), \tag{18}$$

where $\mathbf{O}_{N^2+N}$ is zero vector of size $N^2 + N$. Eq. (16)-(17) is reduced to the matrix form

$\mathbf{\Lambda}\,\mathbf{\Pi} = \mathbf{\Phi}$

where $\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda_{11}} & \mathbf{\Lambda_{12}} \\ \mathbf{\Lambda_{21}} & \mathbf{\Lambda_{22}} \end{bmatrix}$, $\mathbf{\Pi} = \begin{bmatrix} \mathbf{C} \\ \hat{\mathbf{C}} \end{bmatrix}$, $\mathbf{\Phi} = \begin{bmatrix} \mathbf{f_1} \\ \mathbf{f_2} \end{bmatrix}$, $\mathbf{f_1} = [\mathbf{O}_{N^2+N}\ \ u_3(x_0)\ \ u_3(x_1)\ \cdots\ u_3(x_N)]^T$,
$\mathbf{f_2} = [f(x_1, t_1)\ \cdots\ f(x_N, t_{N-1})\ \ u_0(t_1)\ \cdots\ u_0(t_N)\ \ u_1(t_1)\ \cdots\ u_1(t_N)\ \ u_2(x_0)\ \cdots\ u_2(x_N)]^T$,

$$\mathbf{\Lambda_{11}} = \big[\mathbf{P}(x_0, t_1)\mathbf{D}_2\ \ \cdots\ \ \mathbf{P}(x_N, t_N)\mathbf{D}_2\ \ \widetilde{\mathbf{O}}_{(N+1)\times(N+1)}\ \big]^T_{(N+1)^2},$$

$$\mathbf{\Lambda_{12}} = [-\mathbf{P}(x_0, t_1)\mathbf{I}\ \ \cdots\ -\mathbf{P}(x_N, t_N)\mathbf{I}\ \ \mathbf{P}(x_0, 0)\ \ \cdots\ \ \mathbf{P}(x_N, 0)]^T,$$

$$\begin{aligned}\mathbf{\Lambda_{21}} = [&\mathbf{P}(x_1, t_1)(\alpha\,(\mathbf{D}_1)^2 + \beta\,\mathbf{I})\ \cdots\ \mathbf{P}(x_N, t_{N-1})(\alpha\,(\mathbf{D}_1)^2 \\ &+ \beta\,\mathbf{I})\ \ \mathbf{P}(0, t_1)\ \cdots\ \mathbf{P}(0, t_N)\ \ \mathbf{P}(1, t_1)\ \cdots\ \mathbf{P}(1, t_N)\ \ \mathbf{P}(x_0, 0)\ \cdots\ \mathbf{P}(x_N, 0)]^T_{(N+1)^2},\end{aligned}$$

$$\mathbf{\Lambda_{22}} = [\mathbf{P}(x_1, t_1)(\rho\,\mathbf{D}_2\ + \upsilon\,\mathbf{I}\ + \mu\,(\mathbf{D}_1)^2)\ \cdots\ \mathbf{P}(x_N, t_{N-1})(\rho\,\mathbf{D}_2\ + \upsilon\,\mathbf{I}\ + \mu\,(\mathbf{D}_1)^2)\ \ \mathbf{O}\ ]^T_{(N+1)^2}.$$

where $\mathbf{O} = [\widetilde{\mathbf{O}}, \widehat{\mathbf{O}}\,, \widehat{\mathbf{O}}]$, $\widetilde{\mathbf{O}}_{(N+1)\times(N+1)}$ is zero array of size $(N+1)\times(N+1)$ and $\widehat{\mathbf{O}}_{N\times(N+1)}$ is zero array of size $N \times (N+1)$. The resultant linear system of $2(N+1)^2$ linear equations is then solved to determine the unknown coefficients $\mathbf{C}$ and $\hat{\mathbf{C}}$. Consequently, the solution of Eq. (1) is determined.

4. Estimation of The Error

Suppose that $u(x, y)$ is a suitably smooth on Ω and its closest approximation is $u_N(x, y)$. It is enquired to get a bound for $\|u(x, y) - u_N(x, y)\|_2$. An N^{th} degree polynomial of variables x and y is $P_N(x, y)$ for this purpose as shown in the following theorem.

Theorem 1: If the mixed third partial derivative $\partial^3 u/\partial x^2 \partial t$ is limited for a continuous function $u(x, t)$ given on $[0, 1] \times [0, T]$, then the Legendre series of $u(x, t)$ converges uniformly to $u(x, t)$. The approximate error for a sufficiently smooth role $u(x, t)$ on $[0, 1] \times [0, T]$, is

$$\|u(x, y) - u_N(x, y)\|_2 \leq \left\|u(x, y) - \tilde{P}_N(x, y)\right\|_2 \leq \left(C_1 + C_2 + C_3 \left(\frac{1}{N}\right)^{N+1}\right)\left(\frac{1}{N}\right)^{N+1},$$

where

$$C_1 = \frac{1}{4} \max_{(x,t) \in \Omega} \left|\frac{\partial^{N+1} u(x, t)}{\partial x^{N+1}}\right|, C_2 = \frac{1}{4} \max_{(x,t) \in \Omega} \left|\frac{\partial^{N+1} u(x, t)}{\partial t^{N+1}}\right|, C_3 = \frac{1}{4} \max_{(x,t) \in \Omega} \left|\frac{\partial^{2N+2} u(x, t)}{\partial x^{N+1}\, \partial t^{N+1}}\right|.$$

The Legendre-collocation method's error function, e_N, is evaluated by

$$e_N(x, t) = u_{\text{exact}}(x, t) - u_N(x, t), \qquad 0 \leq x \leq 1, \qquad 0 \leq t \leq T, \tag{19}$$

Consequently, u_n should satisfy

$$\rho \frac{\partial^2 u_N}{\partial t^2} + \upsilon \frac{\partial u_N}{\partial t} + \mu \frac{\partial^3 u_N}{\partial x^2 \partial t} + \alpha \frac{\partial^2 u_N}{\partial x^2} + \beta u_N = f(x, t) + \aleph_N(x, t), \text{ in } Q \equiv \Omega \times I, \tag{20}$$

subject to the boundary/initial conditions

$$u_N(0,\, t) = u_0(t), \quad u_N(1,\, t) = u_1(t), \quad u_N(x,\, 0) = u_2(x), \quad \frac{\partial u_N}{\partial t}(x, 0) = u_3(x), \tag{21}$$

where $\aleph_n(x)$ is a perturbation term associated with u_n. Subtracting Eq. (20) and Eq. (21) from Eq. (1) and Eq. (2), respectively, yields

$$\rho \frac{\partial^2 e_N}{\partial t^2} + \upsilon \frac{\partial e_N}{\partial t} + \mu \frac{\partial^3 e_N}{\partial x^2 \partial t} + \alpha \frac{\partial^2 e_N}{\partial x^2} + \beta e_N = \aleph_N(x, t), \text{ in } Q \equiv \Omega \times I, \tag{22}$$

subject to the boundary/initial conditions

$$e_N(0,\, t) = 0, \quad e_N(1,\, t) = 0, \quad e_N(x,\, 0) = 0, \quad \frac{\partial e_N}{\partial t}(x, 0) = 0, \tag{23}$$

The resultant partial differential Eq. (22) represents the error $e_N(x)$. Consider error $\hat{e}_N(x)$ to be estimated one for $e_N(x)$ which is defined in Eq. (19). The estimate error $\hat{e}_N(x)$ is computed by

$$\hat{e}_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\, \tilde{P}_i(x)\tilde{P}_j(t).$$

After applying collocation method, it can be computed as in the previous section.

5. Numerical Examples

This portion focuses on proving the validity, efficiency, and precision of the suggested technique of the Legendre collocation technique. Our primary objective is to compare different approaches to the same problems as pseudo-parabolic viscous diffusion, Poisson, Helmholtz, and telegraph equation. The Intel Core I7, 16GB RAM personal computer running Mathematica 12 did all of the calculations.

Example 1: Consider the following pseudo-parabolic viscous diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial^2 u}{\partial x^2} - u = f(x,\, t),$$

susceptible to

$$u(0,\, t) = u(1,\, t) = 0, u(x,\, 0) = 0,$$

where $f(x,t) = (-4\pi^2 + 1)\cos t\ \sin(2\pi x) + (4\pi^2 - 1)\sin t \sin(2\pi x)$. This equation is generated from Eq. (1) by putting $\rho = 0$ and $\upsilon = \mu = -\alpha = -\beta = 1$ and is satisfied by $u(x,t) = \sin t \sin(2\pi x)$. Figure 1(a) and Figure 1(b) presents the accurate and approximate solution, respectively. The error and the estimate error for the proposed method introduced in Figure 1(c) and Figure 1(d), respectively. Table 1 reveals the highest absolute exact/estimate error for various values of N.

Table 1
Highest absolute error/estimate error of **Example 1**

N	$\|e_N(x,t)\|$	$\|\hat{e}_N(x,t)\|$
7	3.44085×10^{-2}	3.42827×10^{-2}
9	1.25294×10^{-5}	3.14315×10^{-14}
12	1.66636×10^{-7}	2.57393×10^{-14}
15	1.86581×10^{-11}	1.92238×10^{-14}

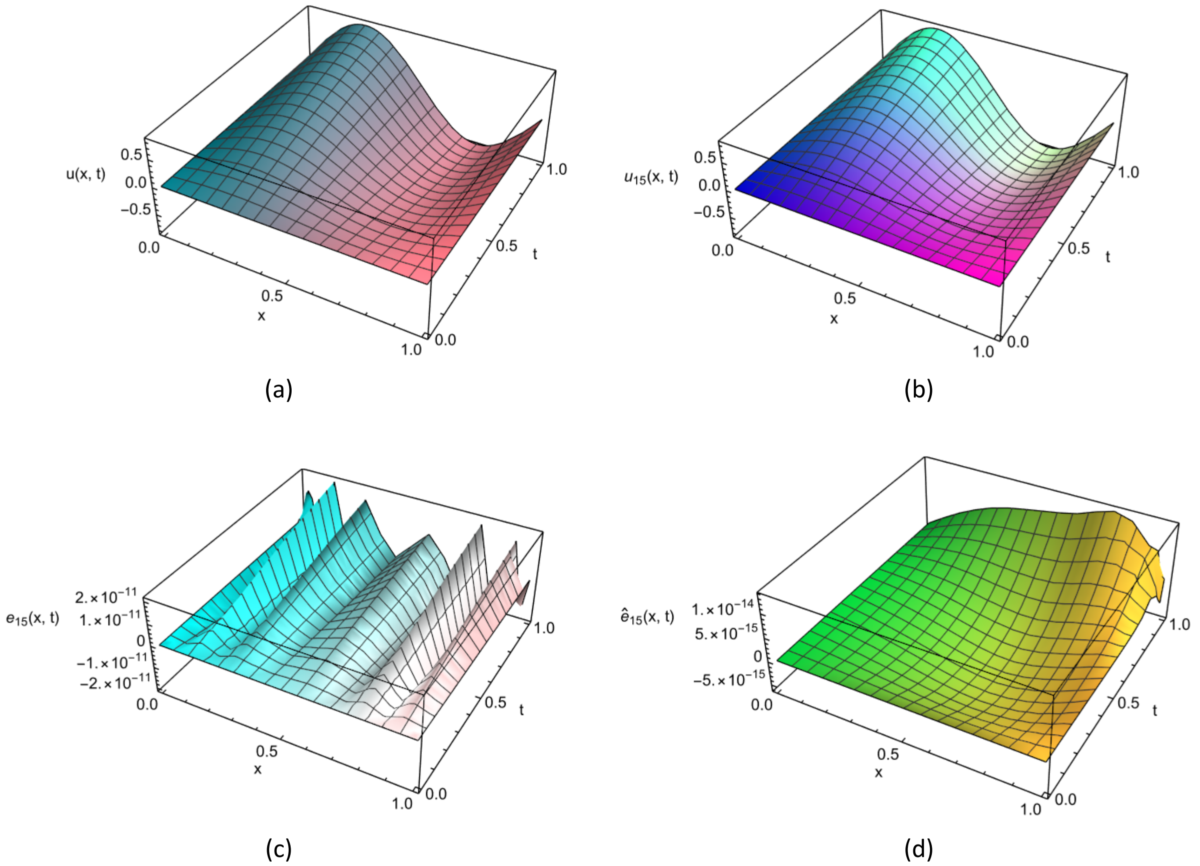

Fig. 1. The exact/approximate functions (a) the $u(x,t)$ and (b) $u_{15}(x,t)$ and the errors (c) $e_{15}(x,t)$ and (d) $\hat{e}_{15}(x,t)$ of **Example 1**

Example 2: Let the telegraph equation be

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+u=(0.91+4\pi^2)\mathrm{e}^{-0.1t}\sin(2\pi x),$$

susceptible to

$$u(0,\ t)=\ u(1,\ t)=0, u(x,\ 0)=\sin(2\pi x),\ u_t(x,\ 0)=-0.1\sin(2\pi x).$$

This equation is generated from Eq. (1) by putting $\mu=0$ and $\rho=v=-\alpha=\beta=1$ and is satisfied by $u(x,t)=\mathrm{e}^{-0.1t}\sin(2\pi x)$. The precise and approximate solution, as well as the error and estimate error, for the suggested technique are shown in Figure 2. Table 2 displays the highest absolute exact/estimate error for a range of N values.

Table 2
Highest absolute error/absolute estimate error of **Example 2**

N	$\|e_N(x,t)\|$	$\|\hat{e}_N(x,t)\|$
3	2.83393×10^{-1}	5.23667×10^{-4}
5	3.77052×10^{-2}	5.79841×10^{-4}
7	9.71648×10^{-4}	1.02844×10^{-5}
9	1.55989×10^{-5}	6.28411×10^{-7}
11	2.61997×10^{-7}	1.57665×10^{-8}
13	9.92631×10^{-9}	2.26356×10^{-10}

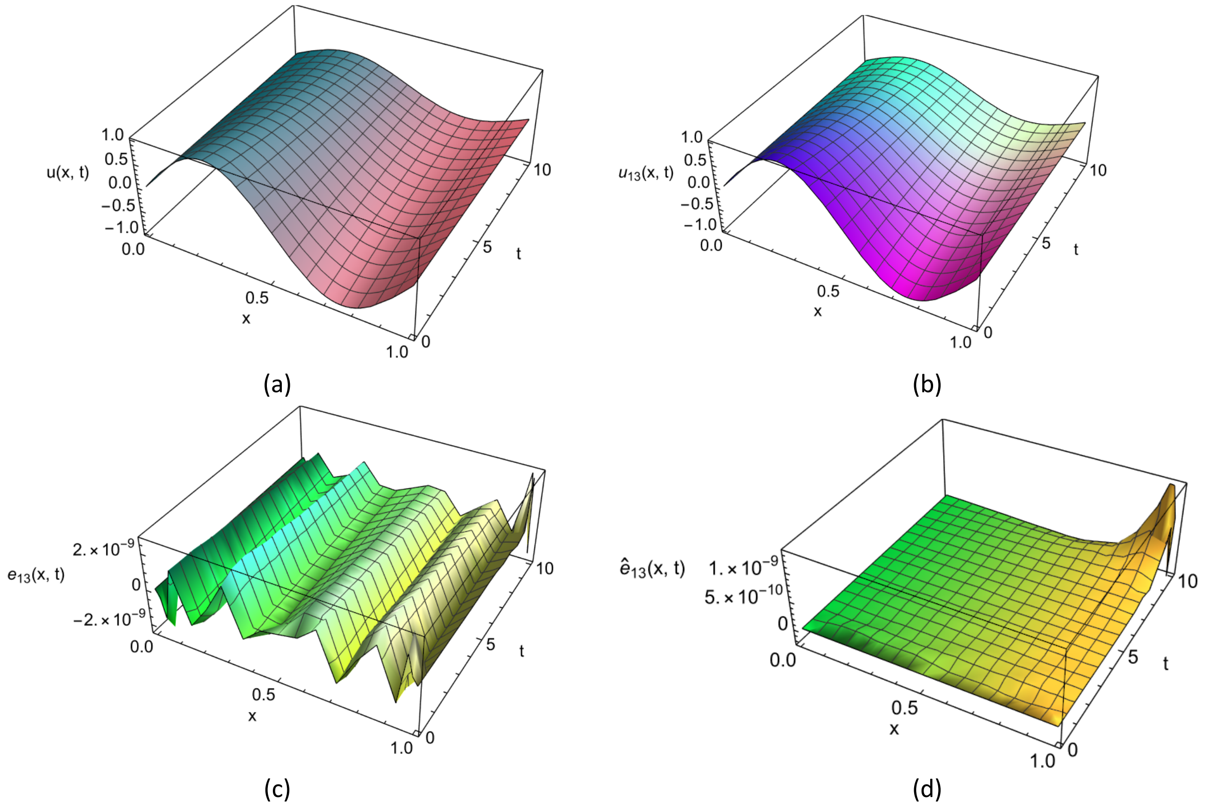

Fig. 2. The exact/approximate functions (a) the $u(x,t)$ and (b) $u_{13}(x,t)$ and the errors (c) $e_{13}(x,t)$ and (d) $\hat{e}_{13}(x,t)$ of **Example 2**

Example 3: Consider the following Poisson equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}=(-18xt+6x^2t+6xt^2+6x^2t^2)e^{x+t},$$

susceptible to

$$u(0,\ t)=\ u(1,\ t)=u(x,0)=u(x,1)=0.$$

This equation is generated from Eq. (1) by putting $\mu=v=\beta=0$ and $\rho=\alpha=1$ and is satisfied by $u(x,t)=3xte^{x+t}(1-x)(1-t)$. Figure 3 depicts the closed form and approximate solution, as well as the error, for the proposed method for $N=12$. Table 3 displays the maximum absolute error and CPU time taken to generate the approximate solution for a range of N values and compares it by two other methods used in [34, 35]. It is observed that the proposed method attains less CPU processing time than existing approaches for solving this problem. As a result, the suggested approach is more useful than other methods.

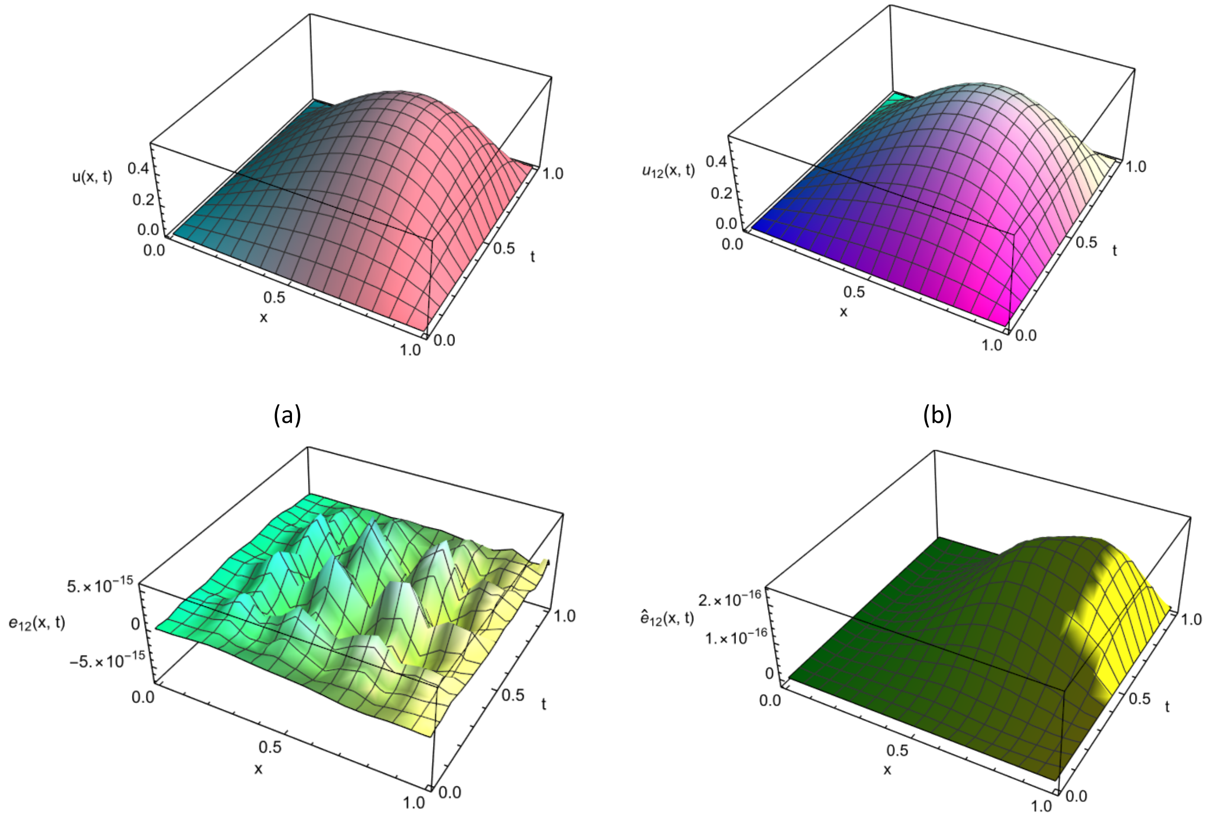

Fig. 3. The exact/approximate functions (a) the $u(x,t)$ and (b) $u_{12}(x,t)$ and the errors (c) $e_{12}(x,t)$ and (d) $\hat{e}_{12}(x,t)$ of **Example 3**

Table 3
Highest absolute errors of **Example 3**

N	Presented Method		Ali Davari Method [34]		Liu-Lin Method [35]	
	$\|e_N(x,t)\|$	CPU time(s)	$\|e_N(x,t)\|$	CPU time(s)	$\|e_N(x,t)\|$	CPU time(s)
3	4.80×10^{-2}	0.9164	4.02×10^{-2}	2.2544	4.02×10^{-2}	12.5089
4	3.76×10^{-3}	0.4297	7.10×10^{-3}	3.2077	7.10×10^{-3}	24.9837
5	1.85×10^{-4}	0.4797	2.00×10^{-4}	10.9164	2.00×10^{-4}	97.7510
6	9.55×10^{-5}	0.5369	9.49×10^{-5}	16.7807	9.49×10^{-5}	244.1086
7	4.27×10^{-6}	0.7471	5.50×10^{-6}	54.9082	5.50×10^{-6}	410.3025

Example 4: Let Helmholtz equation be in the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + 900\,u = \mathrm{e}^{xt^2}(900 + t^4 + 2x + 4x^2t^2), \qquad 0 \le x,t \le 1$$

susceptible to

$$u(0,\ t) = u(x,\ 0) = 1, u(1,\ t) = \mathrm{e}^{t^2},\ u(x,\ 1) = \mathrm{e}^{x}.$$

This equation is generated from Eq. (1) by putting $\rho = \alpha = \beta = 1$ and $\upsilon = \mu = 0$ and is satisfied by $u(x,t) = \mathrm{e}^{xt^2}$. The precise, approximate solution and absolute exact/estimate errors, for the suggested technique are shown in Figure 4. Table 4 displays the highest absolute error for a range of N values in comparison with Legendre collocation method and Liu-Lin method used in [35]. Table 4 demonstrate the highest absolute error for a range of N values.

Table 4
Absolute error of proposed approach, Legendre collocation method and Liu-Lin method

N	$\|e_n(x,t)\|$	Legendre Collocation Method [35]	Liu-Lin Method [35]
9	2.11589×10^{-7}	5.92×10^{-5}	8.67×10^{-5}
16	5.96×10^{-11}	2.53×10^{-5}	3.98×10^{-5}

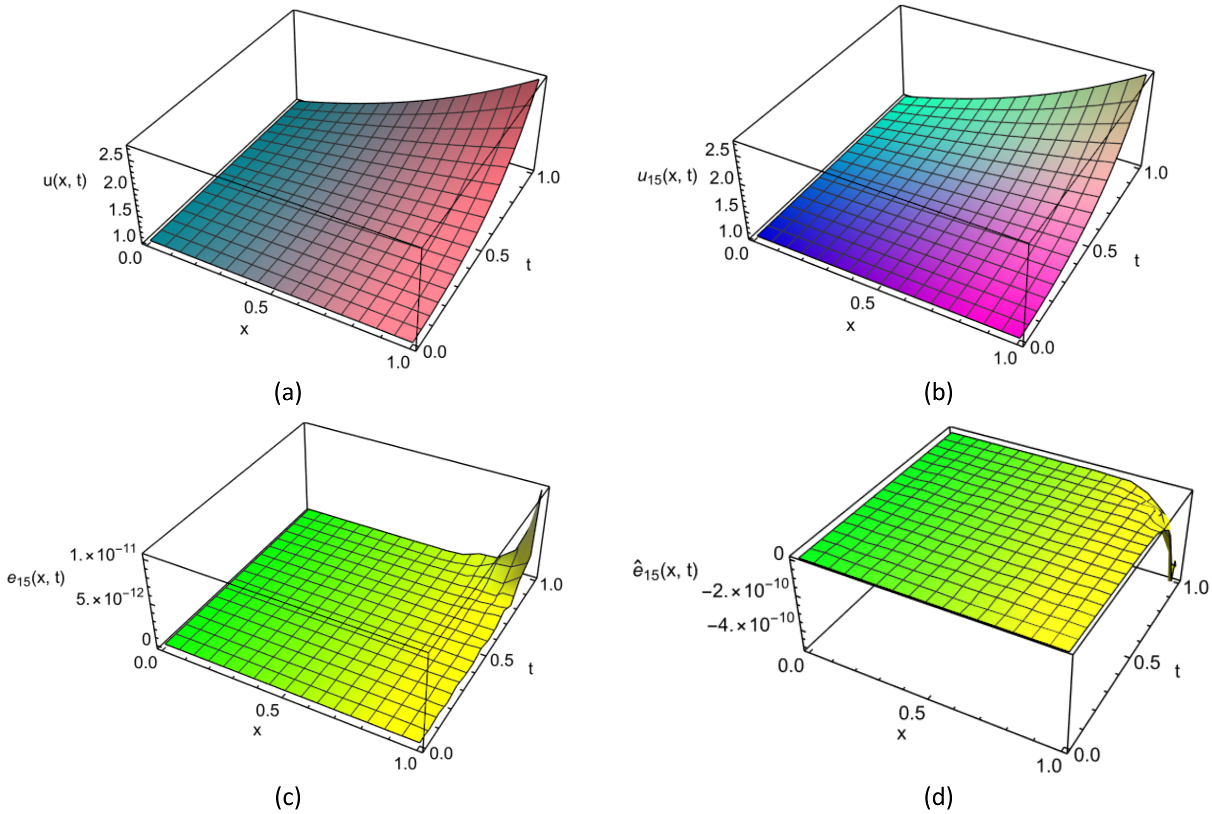

Fig. 4. The exact/approximate functions (a) the $u(x,t)$ and (b) $u_{15}(x,t)$ and the errors (c) $e_{15}(x,t)$ and (d) $\hat{e}_{15}(x,t)$ of **Example 4**

6. Conclusions

The major aspiration of this research is to promote a numerical solution using Legendre collocation approach to reduce a partial differential equation into a group of algebraic equations. Examples of second order linear partial differential equations are demonstrated to highlight the reliability and ability of the suggested approach. An analysis of the Legendre-collocation approach is introduced to approximate their solutions. The presented method displays a good approximation in contrast with closed form solution/other approaches carried using other articles over a range of N values. The main benefit of the approach is the ease with which computer programs can determine the solution of Legendre coefficients.

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References
[1] Farlow, Stanley J. *Partial differential equations for scientists and engineers*. Courier Corporation, 1993.
[2] Abbas, W., Mohamed Fathy, M. Mostafa, and AM A. Hesham. "Galerkin Method for Nonlinear Higher-Order Boundary Value Problems Based on Chebyshev Polynomials." In *Journal of Physics: Conference Series*, vol. 2128, no. 1, p. 012035. IOP Publishing, 2021. https://doi.org/10.1088/1742-6596/2128/1/012035

[3] Mostafa, M., Hesham AM A, W. Abbas, and Mohamed Fathy. "Galerkin method for nonlinear Volterra-Fredholm integro-differential equations based on Chebyshev polynomials." *Engineering Research Journal* 170 (2021): 169-183. https://doi.org/10.21608/erj.2021.177344
[4] El-Gamel, Mohamed, John Cannon, and Ahmed Zayed. "Sinc-Galerkin method for solving linear sixth-order boundary-value problems." *Mathematics of computation* 73, no. 247 (2004): 1325-1343. https://doi.org/10.1090/S0025-5718-03-01587-4
[5] Alonso III, Nicomedes, and Kenneth L. Bowers. "An alternating-direction Sinc–Galerkin method for elliptic problems." *Journal of Complexity* 25, no. 3 (2009): 237-252. https://doi.org/10.1016/j.jco.2009.02.006
[6] El-Azab, M. S., and Mohamed El-Gamel. "A numerical algorithm for the solution of telegraph equations." *Applied Mathematics and Computation* 190, no. 1 (2007): 757-764. https://doi.org/10.1016/j.amc.2007.01.091
[7] Siyyam, Hani I. "Laguerre Tau methods for solving higher-order ordinary differential equations." *Journal of Computational Analysis and Applications* 3 (2001): 173-182. https://doi.org/10.1023/A:1010141309991
[8] Fathy, Mohamed, Mohamed El-Gamel, and M. S. El-Azab. "Legendre–Galerkin method for the linear Fredholm integro-differential equations." *Applied Mathematics and Computation* 243 (2014): 789-800. https://doi.org/10.1016/j.amc.2014.06.057
[9] Fathy, Mohamed, and K. M. Abdelgaber. "Semi-analytical solutions of flow and heat transfer of fluid along expandable-stretching horizontal cylinder." *Case Studies in Thermal Engineering* 28 (2021): 101426. https://doi.org/10.1016/j.csite.2021.101426
[10] Fathy, Mohamed. "Legendre-Galerkin method for the nonlinear Volterra-Fredholm integro-differential equations." In *Journal of Physics: Conference Series*, vol. 2128, no. 1, p. 012036. IOP Publishing, 2021. https://doi.org/10.1088/1742-6596/2128/1/012036
[11] El-Gamel, Mohamed, M. S. El-Azab, and Mohamed Fathy. "An efficient technique for finding the eigenvalues and the eigenelements of fourth-order Sturm-Liouville problems." *SeMA Journal* 74 (2017): 37-56. https://doi.org/10.1007/s40324-016-0079-8
[12] Sayed, Emad A., and Mohamed Fathy. "Numerical study of flow and heat transfer of a nanofluid past a vertical cone." *Case Studies in Thermal Engineering* 34 (2022): 102038. https://doi.org/10.1016/j.csite.2022.102038
[13] Fathy, Mohamed, and Emad A. Sayed. "Maxwell Dusty Fluid Flow Past a Variable Thickness Porous Stretching Surface." *Mathematical Modelling of Engineering Problems* 9, no. 6 (2022). https://doi.org/10.18280/mmep.090606
[14] El-Gamel, M., M. S. El-Azab, and M. Fathy. "The numerical solution of sixth order boundary value problems by the Legendre-Galerkin method." *Acta Univ. Apulensis Math. Inform* 40 (2014): 145-165. https://doi.org/10.17114/j.aua.2014.40.13
[15] Fathy, Mohamed, and K. M. Abdelgaber. "Approximate solutions for the fractional order quadratic Riccati and Bagley-Torvik differential equations." *Chaos, Solitons & Fractals* 162 (2022): 112496. https://doi.org/10.1016/j.chaos.2022.112496
[16] El-Gamel, Mohamed, and Mahmoud Abd El-Hady. "A fast collocation algorithm for solving the time fractional heat equation." *SeMA Journal* (2021): 1-13. https://doi.org/10.1007/s40324-021-00245-2
[17] Fairweather, Graeme, Andreas Karageorghis, and Jon Maack. "Compact optimal quadratic spline collocation methods for the Helmholtz equation." *Journal of Computational Physics* 230, no. 8 (2011): 2880-2895. https://doi.org/10.1016/j.jcp.2010.12.041
[18] Zahra, W. K., W. A. Ouf, and M. S. El-Azab. "A robust uniform B-spline collocation method for solving the generalized PHI-four equation." *Applications and Applied Mathematics: An International Journal (AAM)* 11, no. 1 (2016): 24.
[19] Izadi, Mohammad, Şuayip Yüzbaşı, and Waleed Adel. "A new Chelyshkov matrix method to solve linear and nonlinear fractional delay differential equations with error analysis." *Mathematical Sciences* 17, no. 3 (2023): 267-284. https://doi.org/10.1007/s40096-022-00468-y
[20] El-Gamel, Mohamed, Galal I. El-Baghdady, and Mahmoud Abd El-Hady. "Highly efficient method for solving parabolic PDE with nonlocal boundary conditions." *Applied Mathematics* 13, no. 02 (2022): 101-119. https://doi.org/10.4236/am.2022.132009
[21] Dolapçi, İhsan Timuçin. "Chebyshev collocation method for solving linear differential equations." *Mathematical and Computational Applications* 9, no. 1 (2004): 107-115. https://doi.org/10.3390/mca9010107
[22] Öztürk, Yalçın, Ayşe Anapalı, Mustafa Gülsu, and Mehmet Sezer. "A collocation method for solving fractional Riccati differential equation." *Journal of Applied Mathematics* 2013 (2013). https://doi.org/10.1155/2013/598083
[23] Brink, Adam R., David A. Najera-Flores, and Cari Martinez. "The neural network collocation method for solving partial differential equations." *Neural Computing and Applications* 33 (2021): 5591-5608. https://doi.org/10.1007/s00521-020-05340-5

[24] Yüzbaşı, Şuayip, and Gamze Yıldırım. "A collocation method to solve the parabolic-type partial integro-differential equations via Pell–Lucas polynomials." *Applied Mathematics and Computation* 421 (2022): 126956. https://doi.org/10.1016/j.amc.2022.126956
[25] Samuel, F. M., and S. S. Motsa. "Solving hyperbolic partial differential equations using a highly accurate multidomain bivariate spectral collocation method." *Wave Motion* 88 (2019): 57-72. https://doi.org/10.1016/j.wavemoti.2019.01.014
[26] Zhao, Qingli, Ruiwen Wang, Zhaoqing Wang, and Xiaoping Zhang. "Barycentric rational collocation method for Burgers' equation." *Journal of Mathematics* 2022 (2022). https://doi.org/10.1155/2022/2177998
[27] Wu, Hongchun, Yulan Wang, and Wei Zhang. "Numerical solution of a class of nonlinear partial differential equations by using barycentric interpolation collocation method." *Mathematical Problems in Engineering* 2018 (2018): 1-10. https://doi.org/10.1155/2018/7260346
[28] Adel, Waleed. "A Numerical Technique for Solving a Class of Fourth-Order Singular Singularly Perturbed and Emden–Fowler Problems Arising in Astrophysics." *International Journal of Applied and Computational Mathematics* 8, no. 5 (2022): 220. https://doi.org/10.1007/s40819-022-01396-0
[29] El-Gamel, Mohamed, Nesreen Mohamed, and Waleed Adel. "Numerical study of a nonlinear high order boundary value problems using Genocchi collocation technique." *International Journal of Applied and Computational Mathematics* 8, no. 3 (2022): 143. https://doi.org/10.1007/s40819-022-01262-z
[30] Canuto, Claudio, M. Youssuff Hussaini, Alfio Quarteroni, and Thomas A. Zang. "Spectral Methods." *Spectral Methods: Fundamentals in Single Domains* (2006). https://doi.org/10.1007/978-3-540-30726-6
[31] Talib, Imran, Cemil Tunc, and Zulfiqar Ahmad Noor. "New operational matrices of orthogonal Legendre polynomials and their operational." *Journal of Taibah University for Science* 13, no. 1 (2019): 377-389. https://doi.org/10.1080/16583655.2019.1580662
[32] Ibraheem, Ghada H., and M. A. Al-Jawary. "The operational matrix of Legendre polynomials for solving nonlinear thin film flow problems." *Alexandria Engineering Journal* 59, no. 5 (2020): 4027-4033. https://doi.org/10.1016/j.aej.2020.07.008
[33] Jung, Chahn Yong, Zeqing Liu, Arif Rafiq, Faisal Ali, and Shin Min Kang. "Solution of second order linear and nonlinear ordinary differential equations using Legendre operational matrix of differentiation." *International Journal of Pure and Applied Mathematics* 93, no. 2 (2014): 285-295. https://doi.org/10.12732/ijpam.v93i2.12
[34] Davari, Ali, and Abozar Ahmadi. "New implementation of Legendre polynomials for solving partial differential equations." *Applied Mathematics* 4, no. 12 (2013): 1647. https://doi.org/10.4236/am.2013.412224
[35] Liu, Nanshan, and En-Bing Lin. "Legendre wavelet method for numerical solutions of partial differential equations." *Numerical Methods for Partial Differential Equations: An International Journal* 26, no. 1 (2010): 81-94. https://doi.org/10.1002/num.20417