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Solving Linear Volterra-Fredholm Integro-Differential Equations using Chebyshev-Galerkin with Error Estimation

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ABSTRACT

The semi-analytic method based on the Galerkin technique with Chebyshev basis is presented in this paper to solve the linear integro-differential equations in the Volterra-Fredholm type. To facilitate, the use of the Chebyshev-Galerkin technique, new theorems and lemmas are established. Different numerical examples are introduced to demonstrate how the proposed method has efficacy and is easier to apply for this problem type. Some issues' domains don't match the Chebyshev's domain. So, how to adequate the solution domain is presented. The error estimate is computed to prove the method's applicability and accuracy of the proposed method for any problem, especially in the case of the exact solution is not easy to determine.

1. Introduction

Analytically solving Volterra-Fredholm integro-differential equations is often challenging. To provide approximations for the solutions to these issues, numerical and semi-analytical approaches are being introduced. For example, Maleknejad and Mahmoudi [1] solved the higher order nonlinear Volterra-Fredholm integro-differential equations using Taylor polynomials. Shahmorad [2] presented the solution of the linear Fredholm-Volterra integro-differential equations by the Tau method. A Chebyshev polynomials approach approximation technique for higher order linear Fredholm Volterra integro-differential equations was given by Akyüz. Momani *et al.,* [3] introduced the numerical solution of periodic Fredholm Volterra integrodifferential equations of first-order. Yalçinbaş and Sezer [4] developed a Taylor method to find the approximate solution of high-order linear Volterra-Fredholm integro-differential equations. Reutskiy [5] gave a new numerical method for solving multipoint boundary value problems for Volterra-Fredholm integro-differential equations with linear functional arguments. Kashkaria and Syam [6] introduced a stochastic computational intelligence technique for solving a class of nonlinear Volterra-Fredholm integro-differential equations. Fathy *et*

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al., [7,8] introduced the solution of the nonlinear Volterra-Fredholm integro-differential equations using Legendre and Chebyshev polynomials.

Any real-life problem can be modelled in a mathematical model [9]. The equations of this model may be solved easily and have an exact solution. In many situations, the equations can't be solved analytically, and we need a numerical method such as the Chebyshev-Galerkin method. The integro-differential equation boundary value issues are one of the models for various applications in mechanics, physics, chemistry, astronomy, biology, economics, potential theory, engineering challenges, and electrostatics. This article aims to introduce the Chebyshev-Galerkin approach for solving Volterra-Fredholm integro-differential equations using the form:

$$\sum_{i=0}^{\sigma} \mu_i(x)u^{(i)}(x) = f(x) + \lambda_1 \int_{-1}^{1} k_1(x,t)u(t)dt + \lambda_2 \int_{-1}^{x} k_2(x,t)u(t)dt, \qquad -1 \le x \le 1 \tag{1}$$

$$\sigma u(-1) = \sigma(\sigma-1)u'(-1) = 0, \tag{2}$$

where $\mu_i(x), f(x), k_1(x,t)$ and $k_2(x,t)$ are continuous functions in $L^2[0,1]$ space and $\sigma = 0, 1, 2$. λ_1 and λ_2 are parameters and $u(x)$ is the unknown function. To apply the Galerkin method that used Chebyshev polynomials as a basis, the inner product between the basis and its derivatives is utilized. To compute the inner product's values, new theorems, and lemmas are proved to determine the inner product's values.

Many studies have used Chebyshev techniques to probe a wide range of scientific models. These techniques allowed for the solution of systems of high-order linear differential equations with variable coefficients [10,11], second- and fourth-order equations [12], Poisson's equation [13], linear time periodic delay-differential equations [14], nonlinear Volterra integral equations of the second kind [15], computing the eigenvalues and eigenfunctions for the second-order Sturm-Liouville problems [16], solving Nonlinear Higher-order boundary value problems [17], Time-Fractional KdV-Burgers' Equation [18], time-fractional diffusion equation [19], time-fractional nonlinear Burgers' equation [20] and variational problems [21].

This paper's outline looks like this: The fundamental ideas of Legendre polynomials, together with the new lemmas and theorems that will be used throughout the study, are presented in Section 2. The suggested approach is then utilized to make a first approximation to the answer in Section 3. Modifying the solution domain to [0,1] and dealing with nonhomogeneous boundary conditions are discussed in Section 4. In Section 5, we demonstrate the precision of the proposed technique by numerical examples and comparisons to other approaches. The article concludes briefly in Section 6.

2. Chebyshev Function Preliminaries

One of the orthogonal polynomials, including Legendre and Laguerre polynomials, employed in many applications in practical mathematics is the Chebyshev polynomial. The subsequent differential equation is satisfied by Chebyshev polynomials $T_n(x)$.

$$(1-x^2)y'' - xy' + n^2 y = 0, \quad -1 \le x \le 1, \quad n \ge 0, \tag{3}$$

where

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}, \quad n \ge 0, \tag{4}$$

with inverse

$$x^n = 2^{1-n} \sum_{\substack{j=0 \\ n-j \text{ even}}}^{n} \binom{n}{\frac{n-j}{2}} T_j(x). \tag{5}$$

The main Chebyshev sequence recurrence relationships is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \tag{6}$$

The item $T_n(x)T_m(x)$ readily distinguishes from

$$T_n(x)T_m(x) = \frac{1}{2}\left[T_{n+m}(x) + T_{|n-m|}(x)\right], \tag{7}$$

According to the weight function $\frac{1}{\sqrt{1-x^2}}$, these products are orthogonal on the interval $[-1,1]$ such that

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, & \text{if} \quad n \neq m, \\ \pi, & \text{if} \quad n = m = 0, \\ \frac{\pi}{2}, & \text{if} \quad n = m \neq 0. \end{cases}, \tag{8}$$

The formula yields the first derivative of Chebyshev polynomials is

$$T'_n(x)) = \begin{cases} 2n[T_{n-1}(x) + T_{n-3}(x) + \cdots + T_1(x)], & n = \text{even}, \\ 2n[T_{n-1}(x) + T_{n-3}(x) + \cdots + T_2(x)] + nT_0(x), & n = \text{odd}. \end{cases} \tag{9}$$

Theorem 1: Given any three integer values n, m and N such that $n, m \leq N$, then

(i) $\int_{-1}^{1} T'_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} n\pi, & n-2k+1 = m, \\ 0, & \text{otherwise}. \end{cases}$

(ii) $\int_{-1}^{1} T''_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 2n\pi \sum_{k=1}^{\left[\frac{n}{2}\right]-\left[\frac{m}{2}\right]} (n-2k+1), & n-2k-2q+2 = m, \\ 0, & \text{otherwise}. \end{cases}$

where $k = 1,2,3, \ldots, \frac{n}{2}$ and $q = 1,2,3, \ldots, \frac{n-2k+1}{2}$.

Proof. (i) First, by recalling Eq. **Error! Reference source not found.**, $T'_n(x)$ writes as

$$T'_n(x) = \begin{cases} 2n \sum_{k=1}^{\left[\frac{n}{2}\right]} T_{n-2k+1}(x), & n = \text{even}, \\ 2n \sum_{k=1}^{\left[\frac{n}{2}\right]} T_{n-2k+1}(x) + nT_0(x), & n = \text{odd}. \end{cases} \tag{10}$$

substituting the results into the left side of Theorem 1(i) and using Eq. 8 yields

$$\int_{-1}^{1}\frac{T_n'(x)T_m(x)dx}{\sqrt{1-x^2}}=\begin{cases}2n\sum_{k=1}^{[\frac{n}{2}]}\int_{-1}^{1}T_{n-2k+1}(x)T_m(x)\frac{dx}{\sqrt{1-x^2}}, & n=\text{even},\\ 2n\sum_{k=1}^{[\frac{n}{2}]}\int_{-1}^{1}T_{n-2k+1}(x)T_m(x)\frac{dx}{\sqrt{1-x^2}}+n\int_{-1}^{1}T_0(x)T_m(x)\frac{dx}{\sqrt{1-x^2}}, & n=\text{odd},\end{cases}$$

$$=\begin{cases}n\pi, & \text{if} \quad n-2k+1=m,\\ 0, & \text{if} \quad \text{otherwise}.\end{cases}$$

(ii) By differentiating Eq. 10 yields

$$T_n''(x)=\begin{cases}\sum_{k=1}^{[\frac{n}{2}]}(4n)(n-2k+1)\left[\sum_{q=1}^{[\frac{n-2k+1}{2}]}T_{n-2k-2q+2}(x)+T_0(x)\right], & n=\text{even},\\ \sum_{k=1}^{[\frac{n}{2}]}(4n)(n-2k+1)\sum_{q=1}^{[\frac{n-2k+1}{2}]}T_{n-2k-2q+2}(x), & n=\text{odd}.\end{cases}$$

Substituting the results into the left side in Theorem 1(ii) yields

$$\int_{-1}^{1}\frac{T_n''(x)T_m(x)}{\sqrt{1-x^2}}dx=$$
$$\begin{cases}\sum_{k=1}^{[\frac{n}{2}]}(4n)(n-2k+1)\left[\sum_{q=1}^{[\frac{n-2k+1}{2}]}\int_{-1}^{1}(T_{n-2k-2q+2}(x)T_m(x)\right.\\ \left.\quad +T_0(x)T_m(x))\frac{dx}{\sqrt{1-x^2}}\right], & n=\text{even},\\ \sum_{k=1}^{[\frac{n}{2}]}(4n)(n-2k+1)\sum_{q=1}^{[\frac{n-2k+1}{2}]}\int_{-1}^{1}T_{n-2k-2q+2}(x)T_m(x)\frac{dx}{\sqrt{1-x^2}}, & n=\text{odd}.\end{cases}$$

and applying Eq. 8, the required is proved.

Theorem 2: Given any four integer values n, m, α and N such that $n, m \leq N$, then

(i) $\int_{-1}^{1}x^{\alpha}T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}}=\delta_1+\delta_2$

(ii) $\int_{-1}^{1}x^{\alpha}T_n'(x)T_m(x)\frac{dx}{\sqrt{1-x^2}}=\begin{cases}\delta_3+\delta_4, & n=\text{even},\\ \delta_3+\delta_4+\delta_5, & n=\text{odd}.\end{cases}$

where

$$\delta_1=\begin{cases}2^{-\alpha}\sum_{j=0,}^{\prime\,\alpha}\binom{\alpha}{\frac{\alpha-j}{2}}\pi, & m+n=j=0,\\ 2^{1-\alpha}\sum_{j=0\,,}^{\prime\alpha}\binom{\alpha}{\frac{\alpha-j}{2}}\pi, & m+n=j\neq 0,\\ 0, & \text{otherwise}.\end{cases},\quad \delta_2=\begin{cases}2^{-\alpha}\sum_{j=0,}^{\prime\alpha}\binom{\alpha}{\frac{\alpha-j}{2}}\pi, & |m-n|=j=0,\\ 2^{1-\alpha}\sum_{j=0,}^{\prime\alpha}\binom{\alpha}{\frac{\alpha-j}{2}}\pi, & |m-n|=j\neq 0,\\ 0, & \text{otherwise}.\end{cases},$$

$$\delta_3 = \begin{cases} 2^{1-\alpha} n \sum_{j=0,}^{\prime\alpha} \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{\alpha}{\frac{\alpha-j}{2}} \pi, & n-2k+m+1 = j = 0, \\ 2^{-\alpha} n \sum_{j=0,}^{\prime\alpha} \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{\alpha}{\frac{\alpha-j}{2}} \pi, & n-2k+m+1 = j \neq 0, \\ 0, & \text{otherwise.} \end{cases},$$

$$\delta_4 = \begin{cases} 2^{1-\alpha} n \sum_{j=0,}^{\prime\alpha} \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{\alpha}{\frac{\alpha-j}{2}} \pi, & |n-2k-m+1| = j = 0, \\ 2^{-\alpha} n \sum_{j=0,}^{\prime\alpha} \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{\alpha}{\frac{\alpha-j}{2}} \pi, & |n-2k-m+1| = j \neq 0 \\ 0, & \text{otherwise.} \end{cases},$$

$$\delta_5 = \begin{cases} 2^{1-\alpha} n \sum_{j=0}^{\prime\alpha} \binom{\alpha}{\frac{\alpha-j}{2}} \pi, & m = j = 0, \\ 2^{-\alpha} n \sum_{j=0}^{\prime\alpha} \binom{\alpha}{\frac{\alpha-j}{2}} \pi, & m = j \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: (i) Using Eq. 5 and Eq. 7, $x^\alpha T_n(x) T_m(x)$ term is written as

$$x^\alpha T_n(x) T_m(x) = 2^{-\alpha} \sum_{j=0}^{\prime\alpha} \binom{\alpha}{\frac{\alpha-j}{2}} \left[T_j(x) T_{n+m}(x) + T_j(x) T_{|n-m|}(x)\right].$$

By integrating the results, using the weight function $\frac{1}{\sqrt{1-x^2}}$ and Eq. 8 yields

$$\int_{-1}^{1} x^\alpha T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = 2^{-\alpha} \sum_{j=0}^{\prime\alpha} \binom{\alpha}{\frac{\alpha-j}{2}} \int_{-1}^{1} \left(T_j(x) T_{n+m}(x) + T_j(x) T_{|n-m|}(x)\right) \frac{dx}{\sqrt{1-x^2}} = \delta_1 + \delta_2$$

(ii) Recalling Eq. 5, Eq. 7 and Eq. 10, the left side of Theorem 2(ii) is written as

$$\int_{-1}^{1} \frac{x^\alpha T_n'(x) T_m(x)\, dx}{\sqrt{1-x^2}} =$$

$$2^{1-\alpha} n \begin{cases} \sum_{j=0}^{\prime\alpha} \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{\alpha}{\frac{\alpha-j}{2}} \int_{-1}^{1} \left(T_j(x) T_{n-2k+m+1}(x) \right. \\ \quad \left. + T_j(x) T_{|n-2k-m+1|}(x)\right) \frac{dx}{\sqrt{1-x^2}}, & n = \text{even}, \\ \sum_{j=0,}^{\prime\alpha} \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{\alpha}{\frac{\alpha-j}{2}} \int_{-1}^{1} \left(T_j(x) T_{n-2k+m+1}(x) \right. \\ \quad \left. + T_j(x) T_{|n-2k-m+1|}(x)\right) \frac{dx}{\sqrt{1-x^2}} + \sum_{j=0,}^{\alpha} \int_{-1}^{1} T_j(x) T_m(x) \frac{dx}{\sqrt{1-x^2}}, & n = \text{odd}, \\ 0, & \text{otherwise.} \end{cases}$$

then using Eq. 8, the Theorem 2(ii) is proved.

Legendre function preliminaries beside new theorem and lemmas are introduced in this section that is needed in the next section that describes how to solve the problem Eq. 8 using the proposed method.

3. Description of Chebyshev-Galerkin Method

The finite expansion of the Chebyshev basis function may be used to get the approximate solution of Eq. 1 using the initial values Eq. 2. First, the solution must fulfil the beginning values Eq. 2 in order to get that. Consequently, the solution to the second order problem with $\mu_2(x) = 1$ is

$$u(x) \approx \sum_{j=0}^{n} c_j T_j(x) + Z_a + (x+1)Z_b, \quad (11)$$

where

$$Z_a = -\sum_{j=0}^{n} c_j T_j(-1), \quad Z_b = -\sum_{j=0}^{n} c_j T'_j(-1). \quad (12)$$

The Volterra term in Eq. 1 is modified using the linear transformation $t = (x+1)(s+1)/2 - 1 = \aleph/2 - 1$ before the Chebyshev-Galerkin technique is used

$$\sum_{i=0}^{\sigma} \mu_i(x) u^{(i)}(x) = f(x) + \lambda_1 \int_{-1}^{1} k_1(x,t)u(t)dt + \frac{\lambda_2}{2}(x+1)\int_{-1}^{x} k_2\left(x, \frac{1}{2}\aleph - 1\right) u\left(\frac{1}{2}\aleph - 1\right) ds, \quad (13)$$

By applying Galerkin method using Chebyshev basis, the reduction of the Eq. 13 introduces as

$$\langle \mu_2(x)u''(x), T_r(x)\rangle + \langle \mu_1(x)u'(x), T_r(x)\rangle + \langle \mu_0(x)u(x), T_r(x)\rangle - \langle f(x), T_r(x)\rangle$$
$$= \left\langle \lambda_1 \int_{-1}^{1} k_1(x,t)u(t)dt, T_r(x)\right\rangle + \left\langle \frac{\lambda_2}{2}(x+1)\int_{-1}^{1} k_2\left(x, \frac{1}{2}\aleph - 1\right) u\left(\frac{1}{2}\aleph - 1\right) ds, T_r(x)\right\rangle, \quad (14)$$

where the inner product $\langle .,. \rangle$ is defined as $\langle \zeta, \chi\rangle = \int_{-1}^{1} \zeta(x)\chi(x)/\sqrt{1-x^2}dx$. Substituting Eq. 11 into Eq. 14 yields

$$\sum_{j=0}^{n} c_j [\langle T_{j''}(x), T_r(x)\rangle + \langle \mu_1(x)\, T_{j'}(x), T_r(x)\rangle - \langle \mu_1(x)\, T_{j'}(-1), T_r(x)\rangle$$
$$+\langle \mu_0(x)\, T_j(x), T_r(x)\rangle - \langle \mu_0(x)\, T_j(-1), T_r(x)\rangle - \langle (x+1)\mu_0(x)\, T_{j'}(-1), T_r(x)\rangle$$
$$- \left\langle \lambda_1 \int_{-1}^{1} k_1(x,t)T_j(t)dt, T_r(x)\right\rangle + \left\langle \lambda_1 \int_{-1}^{1} k_1(x,t)T_j(-1)dt, T_r(x)\right\rangle$$
$$+ \left\langle \lambda_1(x+1) \int_{-1}^{1} k_1(x,t)T'_j(-1)dt, T_r(x)\right\rangle - \left\langle \frac{\lambda_2}{2}(x+1)\int_{-1}^{1} k_2\left(x, \frac{1}{2}\aleph - 1\right) T_j\left(\frac{1}{2}\aleph - 1\right) ds, T_r(x)\right\rangle$$
$$+ \left\langle \frac{\lambda_2}{2}(x+1)\int_{-1}^{1} k_2\left(x, \frac{1}{2}\aleph - 1\right) T_j(-1) ds, T_r(x)\right\rangle$$

$$+\left\langle \frac{\lambda_2}{2}(x+1)\int_{-1}^{1} k_2(x,\left(x,\frac{1}{2}\aleph-1\right))T'_j(-1)ds, T_r(x)\right\rangle] = \langle f(x), T_r(x)\rangle \tag{15}$$

The following lemma is required to assess the inner product in Eq. 15.

Lemma 1**:** The following relations hold

$$\langle \mu_1(x)\, T_{j'}(x), T_r(x)\rangle = \begin{cases}\sum_{\alpha=0}^{m_1} C_\alpha(\delta_3+\delta_4), & j = \text{even},\\ \sum_{\alpha=0}^{m_1} C_\alpha(\delta_3+\delta_4+\delta_5), & j=\text{odd}\end{cases}, \tag{16}$$

$$\langle \mu_1(x)\, T_{j'}(-1), T_r(x)\rangle = \sum_{\alpha=0}^{m_2} (-1)^{j+1} C_\alpha\, j^2(\delta_1+\delta_2), \tag{17}$$

$$\langle \mu_0(x)\, T_j(x), T_r(x)\rangle = \sum_{\alpha=0}^{m_3} C_\alpha(\delta_1+\delta_2), \tag{18}$$

$$\langle \mu_0(x)\, T_j(-1), T_r(x)\rangle = \sum_{\alpha=0}^{m_4} (-1)^j C_\alpha(\delta_1+\delta_2), \tag{19}$$

$$\langle (x+1)\mu_0(x)\, T_{j'}(-1), T_r(x)\rangle = \sum_{\alpha=0}^{m_5} (-1)^j C_\alpha\, j^2(\delta_1+\delta_2), \tag{20}$$

$$\langle f(x), T_r(x)\rangle \simeq \sum_{k=1}^{\nu} \frac{\pi}{\nu} f(x_k) T_r(x_k), \tag{21}$$

$$\left\langle \int_{-1}^{1} F(x,t)dt, T_r(x)\right\rangle \simeq \sum_{i=1}^{m}\sum_{k=1}^{\nu} \frac{\pi}{\nu}\frac{F(x_k,t_i)T_r(x_k)}{(1-t_i^2)(P'_m(t_i))^2}, \tag{22}$$

We arrive to the following theorem by replacing each term in Eq. 15 with the appropriate approximation stated in Eq. 16 to Eq. 22, respectively.

Theorem 1: The discrete Chebyshev-Galerkin system for computing the unknown coefficients $\{c_j\}_0^n$ to have an approximate solution of Eq. 1 with the initial conditions Eq. 2 is provided as

$$\sum_{j=0}^{n} [h_{j,r}+e_{j,r}+\upsilon_{j,r}+\rho_{j,r}]c_j = f_r, \quad \mu_2(x)=1, \tag{23}$$

where

$$e_{j,r} = \begin{cases} 2j\pi\sum_{k=1}^{\left[\frac{j}{2}\right]-\left[\frac{r}{2}\right]}(j-2k+1), & j-2k-2q+2=r,\\ 0, & \text{otherwise.}\end{cases} \quad f_r = \sum_{k=1}^{\nu}\frac{\pi}{\nu} f(x_k)T_r(x_k),$$

$$\upsilon_{j,r} = \begin{cases}\sum_{\alpha=0}^{m_1} C_\alpha(\delta_3+\delta_4), & j = \text{even},\\ \sum_{\alpha=0}^{m_1} C_\alpha(\delta_3+\delta_4+\delta_5), & j=\text{odd}\end{cases},$$

$$\rho_{j,r} = -\sum_{\alpha=0}^{m_2}(-1)^{j+1}C_\alpha\, j^2(\delta_1+\delta_2) + \sum_{\alpha=0}^{m_3} C_\alpha(\delta_1+\delta_2) - \sum_{\alpha=0}^{m_4}(-1)^j C_\alpha(\delta_1+\delta_2) - \sum_{\alpha=0}^{m_5}(-1)^j C_\alpha\, j^2(\delta_1+\delta_2),$$

$$h_{j,r} = \sum_{i=1}^{m}\sum_{k=1}^{\nu}\frac{\pi}{\nu}\frac{\xi(x_k,t_i)T_r(x_k)}{(1-t_i^2)(P'_m(t_i))^2} =: -\left\langle \lambda_1\int_{-1}^{1} k_1(x,t)T_j(t)dt, T_r(x)\right\rangle + \left\langle \frac{\lambda_2}{2}(x+1)\int_{-1}^{1} k_2\left(x,\frac{1}{2}\aleph-1\right)T_j(-1)ds, T_r(x)\right\rangle$$

$+\left\langle \lambda_1(x+1)\int_{-1}^{1} k_1(x,t)T'_j(-1)dt, T_r(x)\right\rangle - \left\langle \frac{\lambda_2}{2}(x+1)\int_{-1}^{1} k_2\left(x,\frac{1}{2}\aleph-1\right)T_j\left(\frac{1}{2}\aleph-1\right)ds, T_r(x)\right\rangle$

$+\left\langle \lambda_1\int_{-1}^{1} k_1(x,t)T_j(-1)dt, T_r(x)\right\rangle + \left\langle \lambda_2(x+1)\int_{-1}^{1} k_2(x,\left(x,\frac{1}{2}\aleph-1\right))T'_j(-1)ds, T_r(x)\right\rangle]$

The system Eq. 23, in its following matrix form

$$\mathrm{Ac} = \mathrm{b} \tag{24}$$

where $\mathrm{c} = \{c_j\}_0^n, \mathrm{b} = \{f_r\}_0^n$ and $A = \{h_{i,j} + e_{i,j} + v_{i,j} + \rho_{i,j}\}_{n\times n}$. By solving $n+1$ equations for $n+1$ unknown coefficients, like in the Q-R approach, yields the coefficient of the linear system Eq. 24. The coefficients of the approximation Chebyshev-Galerkin solution $u(x)$ are given by the equation $c = (c_0, c_1, \ldots, c_n)$. So, problem Eq. 1 with the boundary conditions Eq. 2 is solved using Chebyshev-Galerkin method after converting it to algebraic system and solve it numerically.

4. Estimating Errors in the Chebyshev-Galerkin Technique

This section will investigate the error estimator for the Volterra-Fredholm integro differential equation solution using the Chebyshev-Galerkin approximation. The exact solution to Eq. 1 and Eq. 2 is denoted by u(x), and the error function of the Chebyshev approximation $u_n(x)$ to $u(x)$ is represented by $e_n(x) = u(x) - u_n(x)$. If so, the approximation $u_n(x)$ solves the following problem:

$$\sum_{i=0}^{\sigma} \mu_i(x)u_n^{(i)}(x) - \lambda_1\int_{-1}^{1} k_1(x,t)u_n(t)dt - \lambda_2\int_{-1}^{x} k_2(x,t)u_n(t)dt = f(x) + H_n(x), \tag{25}$$

with boundary

$$\sigma u_n(-1) = \sigma(\sigma-1)u_n(-1) = 0, \tag{26}$$

where H_n is obtained by substituting $u_n(x)$ into the Eq. 1 in the form

$$H_n(x) = \sum_{i=0}^{\sigma} \mu_i(x)u_n^{(i)}(x) - \lambda_1\int_{-1}^{1} k_1(x,t)u_n(t)dt - \lambda_2\int_{-1}^{x} k_2(x,t)u_n(t)dt - f(x). \tag{27}$$

By subtracting Eq. 25 and Eq. 26 from Eq. 1 and Eq. 2, respectively, the equation is satisfied by the error function $e_n(x)$

$$\sum_{i=0}^{\sigma} \mu_i(x)e_n^{(i)}(x) - \lambda_1\int_{-1}^{1} k_1(x,t)e_n(t)dt - \lambda_2\int_{-1}^{x} k_2(x,t)e_n(t)dt = H_n(x), \tag{28}$$

with boundary

$$\sigma e_n(-1) = \sigma(\sigma-1)e_{n'}(-1) = 0. \tag{29}$$

The approximation $e_{n,N}(x)$ is found by solving Eq. 28 and Eq. 29 using the Chebyshev-Galerkin approach discussed in the preceding section. So, we can check on the proposed method without the existence of the exact solution, or can't be obtained analytically.

5. Treatment of General Solution Domain

Previously, we saw that the Chebyshev-Galerkin technique may be used to solve Eq. 1 for the special case of a solution domain of [-1,1]. The Chebyshev-Galerkin method requires the general domain to be converted to [-1,1] in order to solve for the solution domain [a,b]. Take the following as an example of a broad problem:

$$\sum_{i=0}^{\sigma} \mu_i(y)u^{(i)}(y) = f(y) + \lambda_1 \int_a^b k_1(y,t)u(t)dt + \lambda_2 \int_a^y k_2(y,t)u(t)dt, \quad a \le y,t \le b. \tag{30}$$

By recalling the linear transformation $y = \frac{1}{2}(b-a)(x+1)+a$, Eq. (6) writes as

$$\sum_{i=0}^{\sigma} \left(\frac{2}{b-a}\right)^i \mu_i(X)u^{(i)}(x) = f(X) + \frac{\lambda_1}{2}(b-a)\int_0^1 k_1(X,S)u(s)ds$$
$$+\frac{\lambda_2}{2}(b-a)\int_0^x k_2(X,S)u(s)ds, \quad a \le x,s \le b, \tag{31}$$

where $X := \frac{1}{2}(b-a)(x+1)+a$ and $S := \frac{1}{2}(b-a)(s+1)+a$. Hence, the integral equation has different domain can be solved with the proposed method.

6. Numerical Examples

Following is a discussion of the benefits of the Chebyshev-Galerkin technique for solving the Linear Volterra-Fredholm integro-differential equation, illustrated with various numerical examples. Four test problems are provided to help evaluate your computer's performance. The first two are linear Volterra integrodifferential equations of the second order; the third is a Fredholm integro-differential equation; the fourth is Volterra-Fredholm integrodifferential equations of the first order. Mathematica 12 was used to calculate all findings.

Example 1: [22] Consider the second order Volterra integro-differential equation with derivatives of the unknown function in the integrand given by

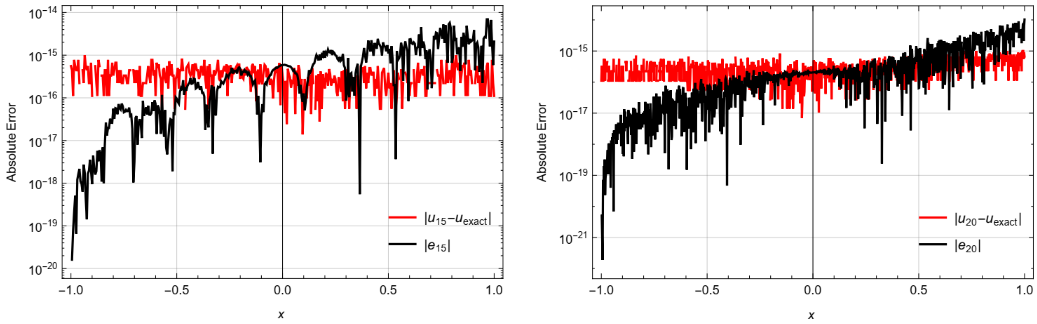
$$u''(x) - (x+1)u'(x) + u(x) = (x+1)(\sin x - \sin 1) + \int_{-1}^{x} xu(t) + u'(t) + tu''(t)dt,$$

with the boundary conditions $u(-1) = \sin 1, \quad u'(-1) = \cos 1$. The exact solution of this problem is $u(x) = \cos x$. Table 1 shows the graph of $||u_n - u_{\text{Exact}}||$ obtained by applying the spectral method described before beside Legendre spectral collocation method presented in [22].

Table 1
The maximum absolute errors for Example 1 at different n

n	Chebyshev-Galerkin	Legendre spectral collocation
4	8.1615E-04	7.9733E-05
8	5.6369E-09	9.7107E-10
12	1.1990E-14	4.5067E-14
16	7.7715E-16	4.1913E-14

Figure 1 introduces the comparison between the estimate error $|e_n|$ and the absolute error resulted from Chebyshev-Galerkin method for $n = 5, 10, 15$ and 20.



(a) (b)

Fig. 1. Absolute estimate error and absolute error for Example 1

Example 2**:** [1,2,4,23-25] Consider the linear Fredholm integro-differential equation

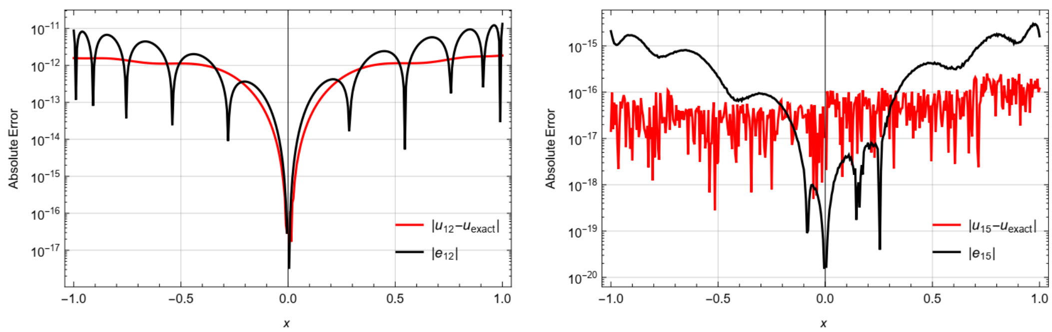
$$u''(x) + xu'(x) - xu(x) = e^x - 2\sin x + \int_{-1}^{1} \sin t\, u(t)dt, \quad -1 \le x, t \le 1,$$

with the initial conditions $u(0) = 1$ and $u'(0) = 1$. The exact solution is $u(x) = e^x$. This example comes from Eq. (3) by putting $k_2(x,t) = 0, k_2(x,t) = \sin t, \mu_0 = 1, \mu_1(x) = -\mu_2(x) = -x, \lambda_2 = 0$, and $\lambda_1 = 1$. The comparison between introduced technique, Legendre-Galerkin [23], Legendre-collocation [24], Taylor polynomials [1,5,25] and Tau [2] method listed in Table 2.

Table 2
The maximum absolute errors for Example 2

Method	$\|\|u_n - u_{\text{exact}}\|\|$
Chebyshev-Galerkin method	4.4408E-16
Taylor method [1]	1.3200E-06
Tau method [2]	7.5200E-11
Legendre-Galerkin method [23]	8.1000E-15
Legendre collocation method [24]	5.2500E-06
Taylor method [25]	4.4000E-07
Taylor method [5]	8.8000E-06

The absolute error estimation of Chebyshev-Galerkin method $|e_n|$ and the absolute error $|u_n - u_{\text{exact}}|$ is introduced in Figure 2 for $n = 5,10,12$ and 15.



(a) (b)

Fig. 2. Absolute estimate error and absolute error for Example 2

Example 3: [26,27] Consider the linear Volterra-Fredholm integral equation

$$u(x) = f(x) + \frac{1}{3}\int_0^1 \sin x \cos t \ u(t)dt + \frac{1}{3}\int_0^x \sin x \cos t \ u(t)dt, \quad 0 \le x, t \le 1,$$

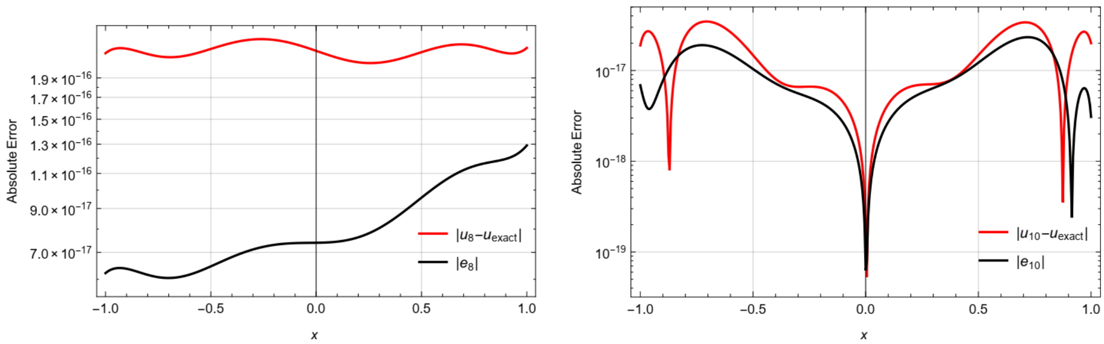
the exact solution is $u(x) = x^2$ and $f(x) = x^2 - \sin x(x^2 \sin x + 2x \cos x - 2 \sin x - \sin 1 + 2 \cos 1)/3$. We compare the numerical results with those from [13,14] in Table 3.

In [13], the authors used method called the Taylor expansion method while in [14] they presented the solution of this problem using two methods. The first is a collocation method and the second one is a fixed-point method.

Table 3
The maximum absolute errors for Example 3

Method	$\|\|u_n - u_{\text{exact}}\|\|$
Chebyshev-Galerkin method, $n = 10$	5.5511E-17
Fixed point method [14], $n = 33$	9.5000E-05
Colocation method [14], $n = 33$	4.7200E-05
Taylor expansion method [13], $n = 15$	2.0000E-15

Figure 3 introduces the maximum absolute error with the estimation error for $n = 4{,}6{,}8$ and 10.



(a) (b)

Fig. 3. Absolute estimate error and absolute error for Example 3

7. Conclusion

In this study, we estimate the solution to the linear Volterra-Fredholm integro-differential equation using the Galerkin technique with a Chebyshev basis. The linear Volterra-Fredholm integro-differential problem is reduced to a system of linear algebraic equations using the characteristics of Chebyshev polynomials in conjunction with the Galerkin technique. This method provides a rough solution to the integro-differential equations. Good agreement between numerical results and other approaches and precise solutions has been achieved. The results show that the procedure is effective, legitimate, and reasonably accurate.

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