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Solution of Volterra Integral Equations of the Second Kind with Weakly Singular Kernels Using Legendre Polynomials

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ABSTRACT

The shifted Legendre polynomials of the first kind are employed to solve second kind Volterra integral equations with weakly singular kernels. In this method the unknown function and the data function are approximated through three matrices, while the kernel function is approximated through five matrices. Regarding the kernel's two variables, it will be approximated twice, first with respect to variable x and second with respect to variable t. The singularity of the kernel is removed analytically. It is proved that the solution is equivalent to an algebraic linear system without applying the collocation method. Two numerical experiments are solved to illustrate the efficiency of the proposed method.

1. Introduction

Weakly singular Volterra integral equations of the second has many applications in various areas. For instance, mathematical physics, chemistry, electrochemistry, semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions, and population dynamics. Volterra integral equations are models of evolutionary problems arising in many applications such as electromagnetic scattering, demography, viscoelastic materials. *et al.,* [1,2].

Numerous publications describing contemporary strategies and techniques for solving Volterra integral equations with weakly singular kernels have been published *et al.,* [3-9]. Numerous publications for solving Fredholm integral equations of the first kind with singular logarithmic kernel and singular unknown functions have been published by Shoukralla *et al.,* [10-21]. Besides, based on a certain enhanced formula of the barycentric Lagrange interpolation, numerous methods *et al.,* [22-24] are established to solve regular and weakly singular Volterra and Fredholm integral equations.

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All of these approaches are applicable in this situation, but we wanted to propose something new. The aim of this research is to find the approximate solution to the Volterra integral equations of the second kind with weakly singular kernels and. However, we would like to present a new method based on shifted Legendre polynomials of the first kind. In this method, both the unknown function and the data function are approximated through three matrices, one of which is a square matrix whose elements contain the coefficients of Legendre polynomials listed in ascending order. The singularity of the kernel is treated analytically, so we get a double approximation of the kernel through five matrices. The idea of substituting the approximate unknown function on both sides of the integral equation enabled us to obtain an equivalent algebraic system for the solution without using the collocation method. Two examples were solved numerically, and the results were strongly converging to the exact solutions.

2. Methodology

Consider the Volterra integral equation of the second kind with weakly singular kernel.

$$u(x) = f(x) + \int_0^x (x-t)^{-\alpha}\phi(x,t)u(t)dt \qquad , t \in \Omega = [0,T], \alpha \in (0,1) \tag{1}$$

Here, $f(x)$ is defined on Ω, and $\varphi(x,t)$ is defined on $D := \{(x,t) \le 0 \le x \le t \le T$ such that $\varphi(x,t) \ne 0$ for $x \in \Omega$. The well-posedness of the solution $u(x)$ has attracted much attention [1,2]. Eq. (1) has a unique solution $u(x) \in C^m]0,t] \cap c[0,t]$ with $|u'(x)| \le Cx^{-\alpha}$ provided $f(x) \in C^m(\Omega),$ and $\varphi(x,t) \in c^m(D)$ for some $m \ge 1$ [3]. Now, we begin the process of initiating the implementation for solving Eq. (1). The suggested approach will rely on shifting first-kind Legendre polynomials to approximate both the unknown function and the known function. Regarding its two variables, the kernel will be approximated twice.

2.1 Definition

The set of shifted Legendre polynomials of the first kind $\{P_i(x)\}_0^n$ are orthogonal on $[0,1]$ and given by

$$p_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^2 - x)^n; \int_0^1 p_n(x)p_m(x)dx = \frac{1}{2n+}\delta_{mn}, n = \overline{0:m} \tag{2}$$

Suppose $u(x)$ is a piecewise continuous and has a finite number of maxima and minima in $[0,1]$ then the series $\sum_{i=0}^{\infty} c_i P_i(x)$, where $c_i = (2i+1)\int_0^1 u(x)P_i(x)dx$ converges to $u(x)$ if and only if x is not a point of discontinuity.

2.2 Definition

The square matrix created by extracting the coefficients of Legendre polynomials $\{P_i(x)\}_0^n$ such that the first row is the coefficents of $P_0(x)$ in ascending power of x, the second row is the coefficents of $P_1(x)$, and so on is said to be the Legendre coefficients matrix and is denoted by $\mathrm{P}_{n,n}$.

Based on definitions 2.1 and 2.2, we find the approximate unknown function of degree n, denoted by $u_n(x)$ in the form

$$u_n(x) = X(x) P_{n,n}^T U \tag{3}$$

Here $\mathrm{U}=[u_i]_{i=0}^n$ is the unknown coefficients column matrix to be determined, $\mathrm{P}_{n,n}$ can be calculated by definition 2.2, and $\mathrm{X}(x) = \left[x^i\right]_{i=0}^n$ is a row matrix of the monomial basis functions. Similarly, the given data function can be approximated in the form.

$$f_n(x) = \mathrm{X}(x)\mathrm{P}_{n,n}^T\mathrm{F} \tag{4}$$

where $\mathrm{F}=[f_i]_{i=0}^n$ is the known coefficients column matrix such that $\{f_i\}_{i=0}^n$ can be found by

$$f_i = (2\mathrm{i}+1)\int_0^1 f(x) P_i(x)\,dx\,,\ i = \overline{0,n} \tag{5}$$

The kennel $k(x,t) = \dfrac{1}{(x-t)^{\alpha}}$ will be approximated by the same way as well as $u_n(x)$ but with the consideration of the two variables x and t. Approximating $k(x,t)$ subjected to x, gives $k_n(x,t)$ via the $(n+1)\times 1$ column matrix $\mathrm{N}(t)$ in the form

$$k_n(x,t) = \mathrm{X}(x)\mathrm{P}_{n,n}^T\mathrm{N}(t); \mathrm{N}(t) = [n_i(t)]_{i=0}^n, n_i(t) = (2\mathrm{i}+1)\int_0^1 k(x,t) P_i(x)\,dx \tag{6}$$

To remove the singularity of $n_i(t)$ we consider t as a singular point and subdivide the domain of integration around this point, so we get.

$$n_i(t) = (2\mathrm{i}+1)\int_0^t k(x,t) P_i(x)\,dx + (2\mathrm{i}+1)\int_t^1 k(x,t) P_i(x)\,dx \tag{7}$$

Moreover, each entry $n_i(t)\ \forall\ i=\overline{0,n}$ will be approximated with respect to the variable t so that we get $k_{n,n}(x,t)$ via $(n+1)\times(n+1)$ square known kernel's coefficients matrix , say, $\mathrm{K}_{n,n}$ in the form

$$k_{n,n}(x,t)=\mathrm{X}(x)\mathrm{P}_{n,n}^{T}\mathrm{K}_{n,n}\mathrm{P}_{n,n}\mathrm{X}^{T}(t);\mathrm{K}_{n,n}=\left[k_{ij}\right]_{i,j=0}^{n},k_{ij}=(2i+1)\int_{0}^{1}n_i(t)P_j(t)\,dt \tag{8}$$

Furthermore, we get.

$$k_{n,n}(x,t)u_n(t)=\mathrm{X}(x)\mathrm{P}_{n,n}^{T}\mathrm{K}_{n,n}\mathrm{P}_{n,n}\tilde{\mathrm{X}}(t)\mathrm{P}_{n,n}^{T}\mathrm{U}\ ;\ \tilde{\mathrm{X}}(t)=\mathrm{X}^{T}(t)\mathrm{X}(t) \tag{9}$$

Substituting $k_{n,n}(x,t)u_n(t)$ of Eq. (9) into Eq. (1), we get

$$u_n(x)=f(x)+\mathrm{X}(x)\mathrm{P}_{n,n}^{T}\mathrm{K}_{n,n}\mathrm{P}_{n,n}\tilde{\tilde{\mathrm{X}}}(x)\mathrm{P}_{n,n}^{T}\mathrm{U}\ ;\ \tilde{\tilde{\mathrm{X}}}(x)=\int_{0}^{x}\tilde{\mathrm{X}}(t)\,dt \tag{10}$$

Substituting $u_n(x)$ given by Eq. (10) in the left side of Eq. (1), $k_{n,n}(x,t)$ given by Eq. (8), and $u_n(t)$ in the right side, we get

$$\begin{aligned} f(x)+\mathrm{X}(x)\mathrm{P}_{n,n}^{T}\mathrm{K}_{n,n}\mathrm{P}_{n,n}\tilde{\tilde{\mathrm{X}}}(x)\mathrm{P}_{n,n}^{T}\mathrm{U}&=f(x)\\ &+\int_{0}^{x}\mathrm{X}(x)\mathrm{P}_{n,n}^{T}\mathrm{K}_{n,n}\mathrm{P}_{n,n}\mathrm{X}^{T}(t)\left\{f(t)+\mathrm{X}(t)\mathrm{P}_{n,n}^{T}\mathrm{K}_{n,n}\mathrm{P}_{n,n}\tilde{\tilde{\mathrm{X}}}(x)\mathrm{P}_{n,n}^{T}\mathrm{U}\right\} \end{aligned} \tag{11}$$

Simplifying Eq. (11) and replace $f(t)$ with $f_n(t)$ given by Eq. (4), we get.

$$\begin{aligned} X(x)P_{n,n}^{T}K_{n,n}P_{n,n}\tilde{\tilde{X}}(x)P_{n,n}^{T}U &= -X(x)P_{n,n}^{T}K_{n,n}P_{n,n}\tilde{\tilde{X}}(x)P_{n,n}^{T}K_{n,n}P_{n,n}\tilde{\tilde{X}}(x)P_{n,n}^{T}U\\ &= X(x)P_{n,n}^{T}K_{n,n}P_{n,n}\tilde{\tilde{X}}(x)P_{n,n}^{T}F \end{aligned} \tag{12}$$

Consequently, we get the algebraic linear system

$$\tilde{\tilde{\mathrm{X}}}(x)\mathrm{P}_{n,n}^{T}\mathrm{U}-\tilde{\tilde{\mathrm{X}}}(x)\mathrm{P}_{n,n}^{T}\mathrm{K}_{n,n}\mathrm{P}_{n,n}\tilde{\tilde{\mathrm{X}}}(x)\mathrm{P}_{n,n}^{T}\mathrm{U}=\tilde{\tilde{\mathrm{X}}}(x)\mathrm{P}_{n,n}^{T}\mathrm{F} \tag{13}$$

Hence, we get the unknown coefficients matrix U by

$$\mathrm{U}=\left(\mathrm{I}_n-\mathrm{K}_{n,n}\mathrm{P}_{n,n}\tilde{\tilde{\mathrm{X}}}(x)\mathrm{P}_{n,n}^{T}\right)^{-1}\mathrm{F} \tag{14}$$

Finally, we find the approximate solution $u_n(x)$ by

$$u_n(x)=\mathrm{X}(x)\mathrm{P}_{n,n}^T\left(\mathrm{I}_n-\mathrm{K}_{n,n}\mathrm{P}_{n,n}\mathrm{X}(x)\mathrm{P}_{n,n}^T\right)^{-1}\mathrm{F} \tag{15}$$

3. Results

In this section, we consider two test problems corresponding to the Eq. (1) to demonstrate the efficiency of the proposed method. The computations associated with the experiments discussed above were performed in MATLAB2019a. We solved these problems for $x=0.1,0.2,0.4,0.6$, and $n=2,3,5$. The exact solution at $x_i=0.0{:}x/10{:}1.0$ is denoted by $u(x_i)$, the approximate solution polynomial of degree n is denoted by $u_n^x(x_i)$ for $x=0.1,0.2,0.4,0.6$, and the absolute error is denoted by $\Re_n^x(x_i)=\left|u(x_i)-u_n^x(x_i)\right|$. Figures 1,2,3, and 4 are related to example 1 to show the graphs of the exact solution and the approximate solutions and the related absolute errors. Figures 5,6,7, and 8 are related to example 2.

Example 1. Consider the problem.

$$u(x)=x^2+\frac{16}{15}x^{\frac{5}{2}}-\int_0^x\frac{1}{\sqrt{x-t}}u(t)\,dt\ ,\ x\in[0,1] \tag{16}$$

whose exact solution $u(x)=x^2$ [5].

Table 1
$u(x_i)$, $u_n^{0.1}(x_i)$, and $\Re_n^{0.1}(x_i)$ of example 1 for $n=2,3,5$

x_i	$u(x_i)$	$u_2^{0.1}(x_i)$	$u_3^{0.1}(x_i)$	$u_5^{0.1}(x_i)$	$\Re_2^{0.1}(x_i)$	$\Re_3^{0.1}(x_i)$	$\Re_5^{0.1}(x_i)$
0	0	0.019303	0.000559	-0.00068	0.019303	0.000559	0.000678
0.01	0.0001	0.016083	8.44E-05	-0.00091	0.015983	1.56E-05	0.001013
0.02	0.0004	0.013338	-6.1E-05	-0.00084	0.012938	0.000461	0.001237
0.03	0.0009	0.011066	0.000125	-0.00045	0.010166	0.000775	0.001347
0.04	0.0016	0.009267	0.000647	0.000262	0.007667	0.000953	0.001338
0.05	0.0025	0.007942	0.001507	0.001293	0.005442	0.000993	0.001207
0.06	0.0036	0.007091	0.002708	0.002649	0.003491	0.000892	0.000951
0.07	0.0049	0.006713	0.004254	0.004334	0.001813	0.000647	0.000566
0.08	0.0064	0.006808	0.006146	0.006351	0.000408	0.000254	4.9E-05
0.09	0.0081	0.007377	0.008389	0.008704	0.000723	0.000288	0.000604
0.1	0.01	0.00842	0.010984	0.011397	0.00158	0.000984	0.001397

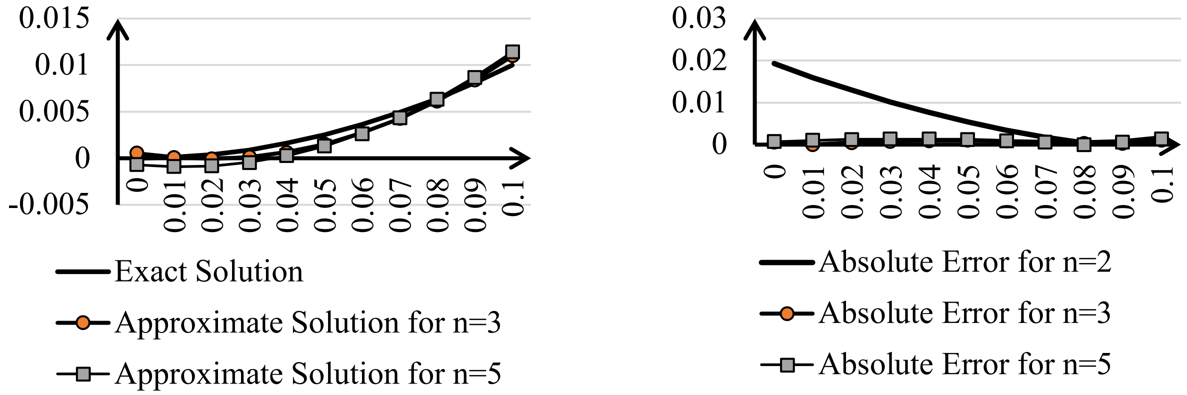


Fig. 1. Graphs of the exact solution and the approximate solutions and the related absolute errors

Table 2
$u(x_i)$, $u_n^{0.2}(x_i)$, and $\Re_n^{0.2}(x_i)$ of example 1 for $n=2,3,5$

x_i	$u(x_i)$	$u_2^{0.2}(x_i)$	$u_3^{0.2}(x_i)$	$u_5^{0.2}(x_i)$	$\Re_2^{0.2}(x_i)$	$\Re_3^{0.2}(x_i)$	$\Re_5^{0.2}(x_i)$
0	0	0.01419	-0.00274	0.001978	0.01419	0.002735	0.001978
0.02	0.0004	0.008233	-0.00436	-0.00203	0.007833	0.004761	0.002427
0.04	0.0016	0.004175	-0.00454	-0.00399	0.002575	0.006143	0.005586
0.06	0.0036	0.002014	-0.00326	-0.00397	0.001586	0.006864	0.007567
0.08	0.0064	0.001751	-0.0005	-0.00203	0.004649	0.006904	0.008433
0.1	0.01	0.003386	0.003758	0.001764	0.006614	0.006243	0.008236
0.12	0.0144	0.006918	0.009539	0.007376	0.007482	0.004861	0.007025
0.14	0.0196	0.012349	0.016859	0.014761	0.007251	0.002741	0.004839
0.16	0.0256	0.019678	0.025739	0.023886	0.005922	0.000139	0.001714
0.18	0.0324	0.028904	0.036197	0.034721	0.003496	0.003797	0.002321
0.2	0.04	0.040028	0.048253	0.047245	2.8E-05	0.008253	0.007245

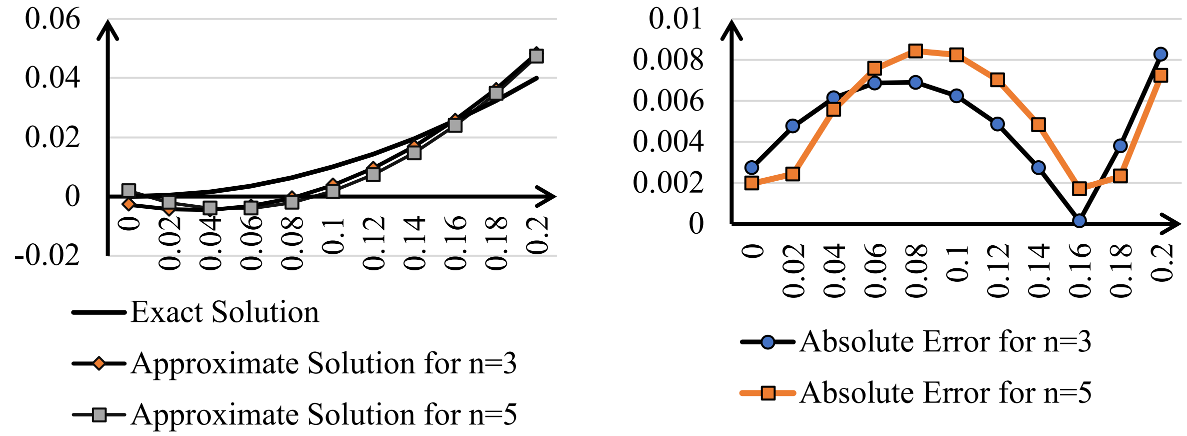


Fig. 2. Graphs of the exact solution and the approximate solutions and the related absolute errors

Table 3
$u(x_i)$, $u_n^{0.4}(x_i)$, and $\mathfrak{R}_n^{0.4}(x_i)$ of example 1 for $n=2,3,5$

x_i	$u(x_i)$	$u_2^{0.4}(x_i)$	$u_3^{0.4}(x_i)$	$u_5^{0.4}(x_i)$	$\mathfrak{R}_2^{0.4}(x_i)$	$\mathfrak{R}_3^{0.4}(x_i)$	$\mathfrak{R}_5^{0.4}(x_i)$
0	0	0.008536	0.014123	-0.02237	0.008536	0.014123	0.022367
0.04	0.0016	-0.00939	-0.00698	-0.01031	0.010987	0.008582	0.011911
0.08	0.0064	-0.01902	-0.01905	-0.00664	0.025415	0.025449	0.013036
0.12	0.0144	-0.02035	-0.02215	-0.00568	0.034749	0.036547	0.020081
0.16	0.0256	-0.01339	-0.01634	-0.00291	0.038987	0.041941	0.028511
0.2	0.04	0.001869	-0.0017	0.005182	0.038131	0.041698	0.034818
0.24	0.0576	0.025419	0.021715	0.021189	0.032181	0.035885	0.036411
0.28	0.0784	0.057265	0.053832	0.046886	0.021135	0.024568	0.031514
0.32	0.1024	0.097405	0.094586	0.083337	0.004995	0.007814	0.019063
0.36	0.1296	0.14584	0.14391	0.131	0.01624	0.01431	0.0014
0.4	0.16	0.20257	0.20174	0.18984	0.04257	0.04174	0.02984

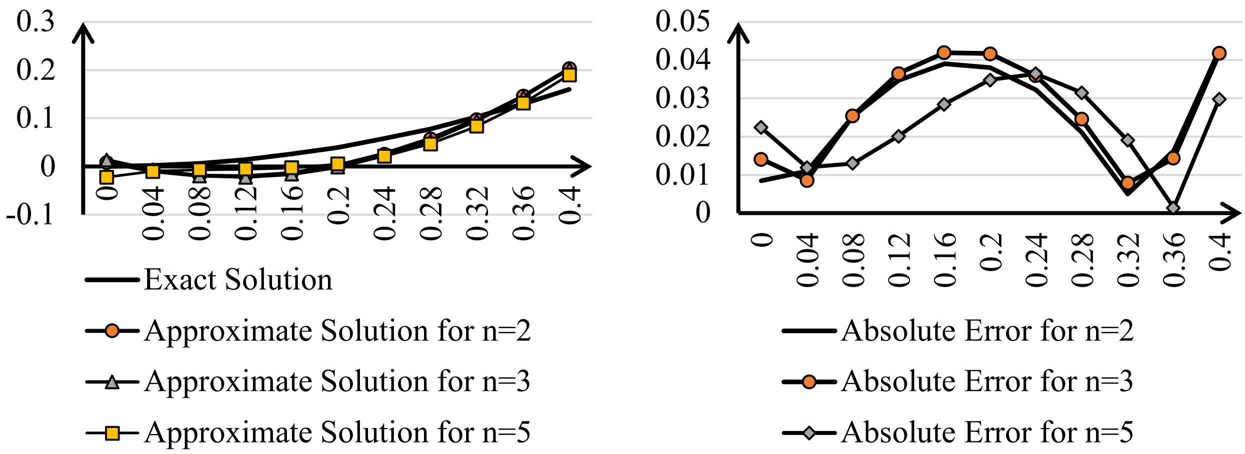
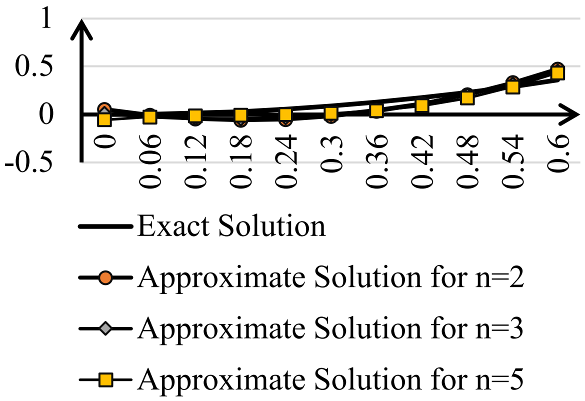


Fig. 3. Graphs of the exact solution and the approximate solutions and the related absolute errors

Table 4
$u(x_i)$, $u_n^{0.6}(x_i)$, and $\mathfrak{R}_n^{0.6}(x_i)$ of example 1 for $n=2,3,5$

x_i	$u(x_i)$	$u_3^{0.6}(x_i)$	$u_5^{0.6}(x_i)$	$u_8^{0.6}(x_i)$	$\mathfrak{R}_3^{0.6}(x_i)$	$\mathfrak{R}_5^{0.6}(x_i)$	$\mathfrak{R}_8^{0.6}(x_i)$
0	0	0.052921	0.010538	-0.0565	0.052921	0.010538	0.056497
0.06	0.0036	-0.00717	-0.02222	-0.02743	0.010773	0.025815	0.031029
0.12	0.0144	-0.04455	-0.04119	-0.01396	0.058947	0.055585	0.028363
0.18	0.0324	-0.0592	-0.04511	-0.00862	0.091599	0.07751	0.041024
0.24	0.0576	-0.05113	-0.03273	-0.00338	0.10873	0.090329	0.060978
0.3	0.09	-0.02034	-0.00278	0.009932	0.110341	0.092782	0.080068
0.36	0.1296	0.03317	0.045992	0.039143	0.09643	0.083608	0.090457
0.42	0.1764	0.1094	0.11485	0.091336	0.067	0.06155	0.085064
0.48	0.2304	0.20835	0.20506	0.17239	0.02205	0.02534	0.05801
0.54	0.2916	0.33003	0.31788	0.28655	0.03843	0.02628	0.00505
0.6	0.36	0.47442	0.45457	0.43599	0.11442	0.09457	0.07599



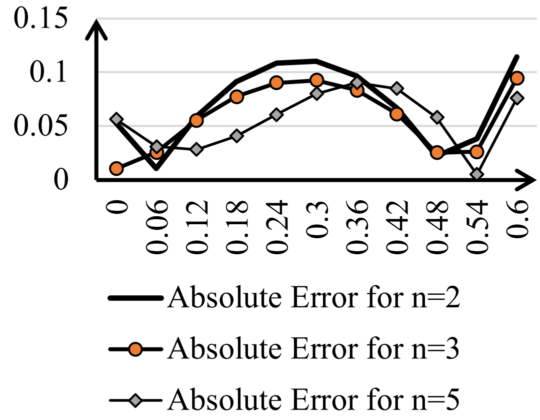


Fig. 4. Graphs of the exact solution and the approximate solutions and the related absolute errors

Example 2. Consider the problem.

$$u(x)=2\sqrt{x}-\int_0^x \frac{1}{\sqrt{x-t}}u(t)\,dt \ , \ x\in[0,2] \tag{17}$$

whose exact solution $u(x)=1-e^{\pi x}erfc\left(\sqrt{\pi x}\right)$ which is the complementary error function [Karimi].

Table 5
$u(x_i)$, $u_n^{0.1}(x_i)$, and $\Re_n^{0.1}(x_i)$ of example 2 for $n=2,3,5$

x_i	$u(x_i)$	$u_2^{0.1}(x_i)$	$u_3^{0.1}(x_i)$	$u_5^{0.1}(x_i)$	$\Re_2^{0.1}(x_i)$	$\Re_3^{0.1}(x_i)$	$\Re_5^{0.1}(x_i)$
0	0	0.11945	0.03337	-0.00257	0.11945	0.03337	0.00257
0.01	0.17233	0.15266	0.079671	0.051348	0.01967	0.092659	0.120982
0.02	0.23015	0.18556	0.12497	0.1036	0.04459	0.10518	0.12655
0.03	0.27019	0.21815	0.16927	0.15422	0.05204	0.10092	0.11597
0.04	0.30136	0.25042	0.2126	0.20326	0.05094	0.08876	0.0981
0.05	0.32704	0.28238	0.25496	0.25076	0.04466	0.07208	0.07628
0.06	0.34893	0.31402	0.29637	0.29675	0.03491	0.05256	0.05218
0.07	0.36804	0.34536	0.33686	0.34129	0.02268	0.03118	0.02675
0.08	0.38499	0.37637	0.37642	0.3844	0.00862	0.00857	0.00059
0.09	0.40023	0.40708	0.41507	0.42614	0.00685	0.01484	0.02591
0.1	0.41406	0.43747	0.45284	0.46653	0.02341	0.03878	0.05247

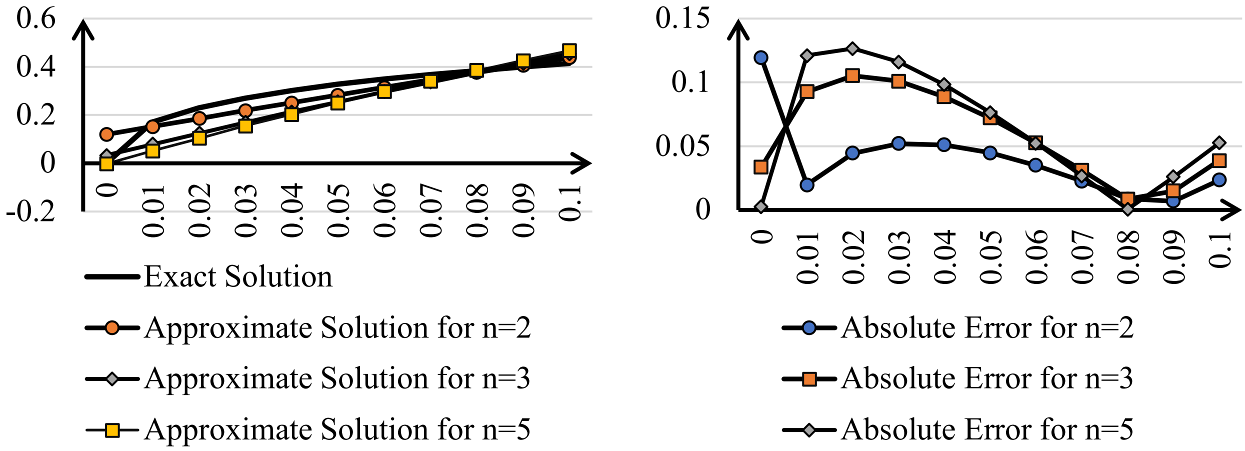


Fig. 5. Graphs of the exact solution and the approximate solutions and the related absolute errors

Table 6
$u(x_i)$, $u_n^{0.2}(x_i)$, and $\Re_n^{0.2}(x_i)$ of example 2 for $n=2,3,5$

x_i	$u(x_i)$	$u_2^{0.2}(x_i)$	$u_3^{0.2}(x_i)$	$u_5^{0.2}(x_i)$	$\Re_2^{0.2}(x_i)$	$\Re_3^{0.2}(x_i)$	$\Re_5^{0.2}(x_i)$
0	0	0.11945	0.03337	-0.00257	0.11945	0.03337	0.00257
0.01	0.17233	0.15266	0.079671	0.051348	0.01967	0.092659	0.120982
0.02	0.23015	0.18556	0.12497	0.1036	0.04459	0.10518	0.12655
0.03	0.27019	0.21815	0.16927	0.15422	0.05204	0.10092	0.11597
0.04	0.30136	0.25042	0.2126	0.20326	0.05094	0.08876	0.0981
0.05	0.32704	0.28238	0.25496	0.25076	0.04466	0.07208	0.07628
0.06	0.34893	0.31402	0.29637	0.29675	0.03491	0.05256	0.05218
0.07	0.36804	0.34536	0.33686	0.34129	0.02268	0.03118	0.02675
0.08	0.38499	0.37637	0.37642	0.3844	0.00862	0.00857	0.00059
0.09	0.40023	0.40708	0.41507	0.42614	0.00685	0.01484	0.02591
0.1	0.41406	0.43747	0.45284	0.46653	0.02341	0.03878	0.05247

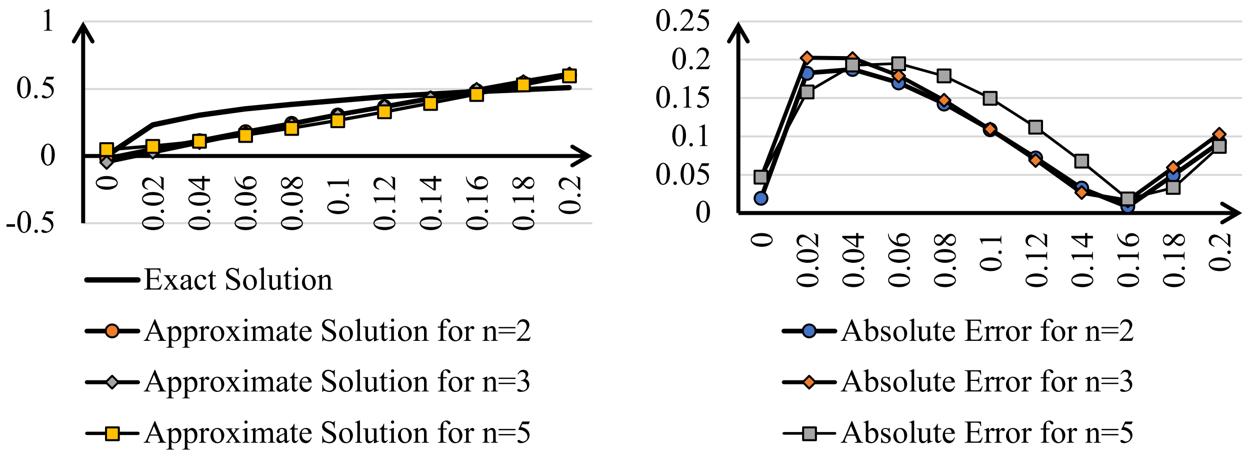


Fig. 6. Graphs of the exact solution and the approximate solutions and the related absolute errors

Table 7
$u(x_i)$, $u_n^{0.4}(x_i)$, and $\Re_n^{0.4}(x_i)$ of example 2 for $n=2,3,5$

x_i	$u(x_i)$	$u_2^{0.4}(x_i)$	$u_3^{0.4}(x_i)$	$u_5^{0.4}(x_i)$	$\Re_2^{0.4}(x_i)$	$\Re_3^{0.4}(x_i)$	$\Re_5^{0.4}(x_i)$
0	0	-0.08487	0.043977	-0.06952	0.084869	0.043977	0.069515
0.04	0.30136	0.003143	0.070079	0.064294	0.298217	0.231281	0.237066
0.08	0.38499	0.090536	0.10942	0.15029	0.294454	0.27557	0.2347
0.12	0.43839	0.17731	0.16075	0.20898	0.26108	0.27764	0.22941
0.16	0.47757	0.26346	0.22281	0.25626	0.21411	0.25476	0.22131
0.2	0.50835	0.34899	0.29436	0.30387	0.15936	0.21399	0.20448
0.24	0.53357	0.4339	0.37412	0.35989	0.09967	0.15945	0.17368
0.28	0.55483	0.51819	0.46086	0.42913	0.03664	0.09397	0.1257
0.32	0.57314	0.60186	0.55332	0.51362	0.02872	0.01982	0.05952
0.36	0.58916	0.68492	0.65024	0.61306	0.09576	0.06108	0.0239
0.4	0.60335	0.76735	0.75037	0.72526	0.164	0.14702	0.12191

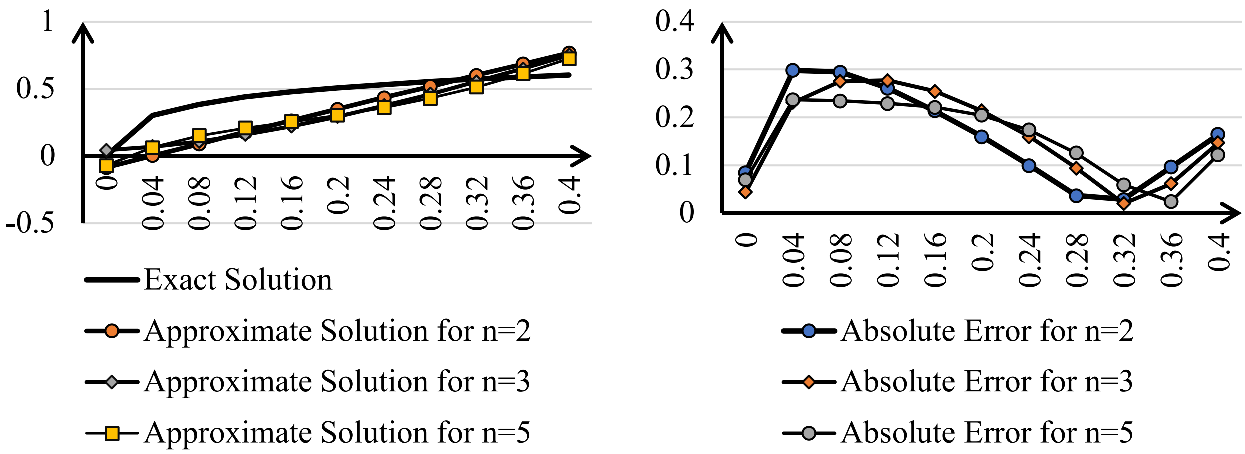


Fig. 7. Graphs of the exact solution and the approximate solutions and the related absolute errors

Table 8
$u(x_i)$, $u_n^{0.6}(x_i)$, and $\Re_n^{0.6}(x_i)$ of example 2 for $n=2,3,5$

x_i	$u(x_i)$	$u_2^{0.6}(x_i)$	$u_3^{0.6}(x_i)$	$u_5^{0.6}(x_i)$	$\Re_2^{0.6}(x_i)$	$\Re_3^{0.6}(x_i)$	$\Re_5^{0.6}(x_i)$
0	0	0.020048	0.048035	-0.17684	0.020048	0.048035	0.17684
0.06	0.34893	0.061829	0.06891	0.068084	0.287101	0.28002	0.280846
0.12	0.43839	0.11243	0.10568	0.19551	0.32596	0.33271	0.24288
0.18	0.49379	0.17184	0.15734	0.25423	0.32195	0.33645	0.23956
0.24	0.53357	0.24006	0.22293	0.28242	0.29351	0.31064	0.25115
0.3	0.56431	0.3171	0.30145	0.30856	0.24721	0.26286	0.25575
0.36	0.58916	0.40296	0.39191	0.35234	0.1862	0.19725	0.23682
0.42	0.60987	0.49763	0.49333	0.42562	0.11224	0.11654	0.18425
0.48	0.62753	0.60112	0.60472	0.53334	0.02641	0.02281	0.09419
0.54	0.64285	0.71342	0.72509	0.67445	0.07057	0.08224	0.0316
0.6	0.65632	0.83454	0.85346	0.84284	0.17822	0.19714	0.18652

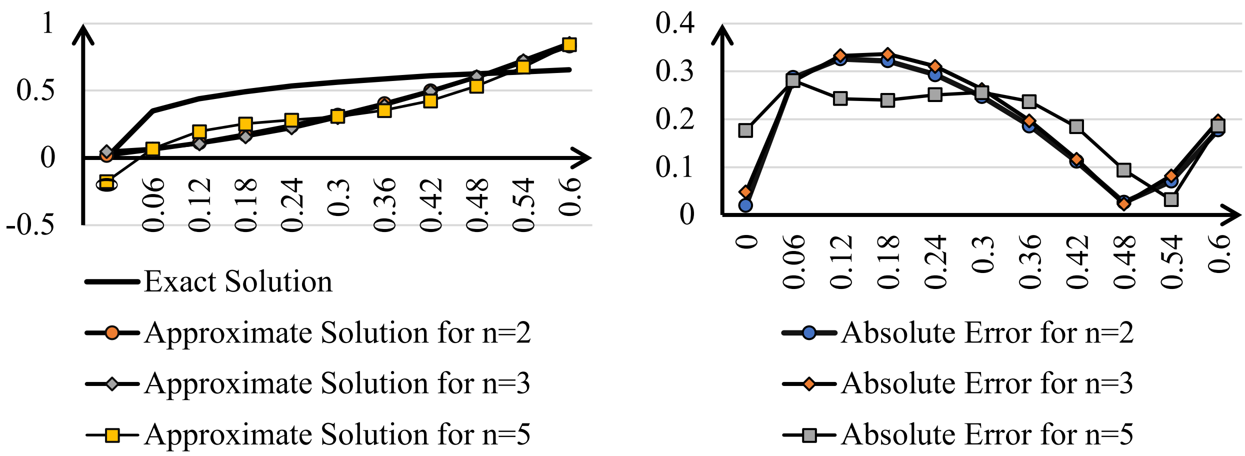


Fig. 8. Graphs of the exact solution and the approximate solutions and the related absolute errors

4. Conclusions

In this work, we investigated numerical solutions of Volterra integral equations of the second kind with weakly singular kernels using traditional Legendre polynomials. The created procedure is based on operational matrices. The main difficulty in approximating these equations is that the kernels have singularities which result in low-order convergence for any methods using traditional polynomials. To overcome this difficulty, we proposed an analytical treatment of the singularity and reduce the solution to an equivalent linear algebraic system without applying the collocation method. Two numerical examples were carried out to verify the efficiency of the proposed method.

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