SEMARAK ILMU PUBLISHING

Journal of Advanced Research in Applied Sciences and Engineering Technology

Journal homepage: https://semarakilmu.com.my/journals/index.php/applied_sciences_eng_tech/index
ISSN: 2462-1943


# Approximate Solutions to Weakly Singular Fredholm Integral Equations of the Second Kind using Shifted Legendre Polynomials of the First Kind

Emil Sobhi Shoukralla[1], Alfaisal Abdelhamid Hasan[2], Sara Abdelhamid[2,*], Ahmed Yehia Sayed[3]

[1] Faculty of Electronic, Menoufiya University, Menofia Governorate 6131567, Egypt
[2] Basic and Sciences Department, College of Engineering and Technology, Arab Academy for Science, Technology & Maritime Transport, Aswan Governorate 1241502, Egypt
[3] Physics and Engineering Mathematics Department, Faculty of Engineering Helwan University Mataria Branch, Cairo Governorate, Egypt

**ARTICLE INFO**

***Article history:***
Received 20 December 2023
Received in revised form 23 February 2024
Accepted 19 July 2024
Available online 20 August 2024

***Keywords:***
Fredholm integral equations; weakly singular kernels; Legendre polynomials; approximation; computational method

**ABSTRACT**

In this research, we presented a simple approach to approximate the second type of linear weakly singular and non-singular Fredholm integral. Shifted Legendre polynomials of the first kind in matrix-vector forms were used to construct the approach. The singularity of the kernel was removed analytically. Theorems regarding the convergence of the estimations of the error norm and the mean were proved. The numerical examples demonstrated the method's uniqueness and precision.

## 1. Introduction

One of the most crucial foundations for resolving initial, boundary, and mixed value issues is the branch on integral equations. This is due to its ability to transform each of these issues into an identical boundary integral equation. In addition to having the ability to get around the solution's singularity at specific points in the integration domain, this decreases the amount of computation time needed to solve boundary problems using conventional techniques. Numerous scientific fields, including nanotechnology, radar theory, scattering, artificial intelligence, heat equations, heat flux, and other crucial areas, have advanced along with the methods for solving singular and non-singular integral equations. In this study, we investigate a novel approach for solving the second kind Fredholm integral equations whose kernels are weakly singular, and smooth kernels. Of course, there are numerous published techniques for finding the solution of such problems. However, we will introduce a new approach based on the approximation using Legendre polynomials of the first kind. In the other hand there are numerous published methods to solve this type of equation, however the majority of them are difficult to understand and contain numerous theories and lemmas, making

* *Corresponding author.*
*E-mail address: saraD01292@m-eng.helwan.edu.eg*

https://doi.org/10.37934/araset.50.2.7689

it challenging to pinpoint the mechanism and stages for the solution. Yin Yang *et al.,* [1] applied the spectral collocation method for finding the solution of second kind Fredholm integral equations whose kernels are weakly singular. The approach was depended on the Jacobi-Gauss quadrature formula. Bijaya Laxmi Panigrahi *et al.,* [2] solved second kind Fredholm integral equations whose kernels are weakly singular. The approach was based on the Legendre multi-Galerkin technique and the corresponding eigenvalue problem. Numerical solutions for second kind Fredholm integral equations whose kernels are weakly singular were provided by Tomoaki Okayama *et al.,* [3]. Reza Behzadi *et al.,* [4] generalized the summation formula of Euler-Maclaurin to be used in second kind Fredholm integral equations whose kernels are weakly singular solution. This concept allowed them to convert the basic equations into the relevant algebraic equations system. For second kind Fredholm integral equations whose kernels are weakly singular and non-singular solution, Min Wang. *et al.,* [5] presented the Multistep Collocation method, which is applied to equations with smooth kernels under the uniform mesh, and the kernel which is weakly singular using a graded mesh. X. Y. Li *et al.,* [6] solved the linear second-kind integral equations which type is Fredholm using a globally super convergent numerical technique that makes use of the kernel functions in piecewise polynomials form in the reproducing Sobolev kernel Hilbert spaces.

Numerous publications for solving first kind Fredholm integral which contains kernel in the form of singular logarithmic and singular unknown functions have been published by Shoukralla and Shoukralla *et al.,* [7-18]. Besides, based on the enhanced formulas of the barycentric Lagrange interpolation, numerous methods *et al.,* [19-22] are established to solve weakly singular besides regular Volterra and Fredholm integral equations. All these approaches are applicable in this situation, but we wanted to propose something new. The objective of this study is to discover an approximated solution for second-type Fredholm integral equations, where the kernel is characterized by weakly singular and smooth kernel forms. The procedures for obtaining the solution are summarized in three steps, which distinguishes this method from other methods. Initially, the process involves approximating both the provided data function and the unknown function by utilizing shifted Legendre polynomials of the first kind. Thus, we create a square matrix of the Legendre polynomials that ensures that there are no imaginary values in the functional values of the given data function and the kernels. Hence, each of the unknown function and given data function is expressed in the form of the product of three matrices. The first matrix is the monomial basis functions row matrix. The second matrix is Legendre polynomials coefficients square matrix, arranged in ascending order. The third matrix is the functional values column matrix. The second stage is to approximate the kernel based on the shifted orthogonal Legendre polynomials. so that the kernel is expressed as the product of five matrices. Thus, we have obtained single approximate functions for each of the given the data and unknown functions, and a double approximate function for the kernel. The third step is to substitute the approximate unknown function in both sides of the considered integral equation simultaneously with the substitution of the double approximate kernel in the wright side. Finally, we achieve a linear algebraic system without necessitating the application of the collocation method. The solution to this system yields real values for the unknown coefficients, thereby facilitating the computation of the solution.

## 2.Methodology

### *2.1 Legendre-Interpolation Method*

Considering the following second kind Fredholm integral equation:

$$\lambda u(x)=f(x)+\int_0^1 Z(x,t)|x-t|^{-\alpha}u(t)\,dt\ ;\ 0\le x\le 1\ ,\ 0\le\alpha<1 \quad (1)$$

where $\lambda$, $f(x)$ and the kernel $k(x,t)=Z(x,t)|x-t|^{-\alpha}$ are given, the unknown function $u(x)$ is to be determined. The kernel $k(x,t)$ is defined on the square $\{(x,t):0\le x,t\le 1\}$. For $\alpha=0$, Eq. (1) is called non-singular Fredholm second kind integral equation. For $0<\alpha<1$ Eq. (1), is called weakly singular Fredholm second kind integral equation. Now, consider the shifted Legendre polynomials with orthogonal property.

$$P_n(x)=\frac{1}{n!}\frac{d^n}{dx^n}\left(x^2-x\right)^n\ ;\ \int_0^1 P_n(x)P_l(x)\,dx=\frac{1}{2n+1}\delta_{ln}\ ,\ n=\overline{0:l} \quad (2)$$

The function $u(x)$ is approximated based on $\{P_n(x)\}_0^l$, we get it in the form.

$$u_n(x)=\sum_{i=0}^{n} a_i P_i(x)=\mathrm{X}(x)V\,\mathrm{A} \quad (3)$$

where we get the known coefficients square matrix $V$ by the coefficients extracting of all polynomials $P_i(x)\ \forall\, i=\overline{0,n}$ and fulfilled the rows of $V$. Here $\mathrm{A}=[a_i]_{i=0}^n$ is the unknown coefficients column matrix to be determined, and $\mathrm{X}(x)=\left[x^i\right]_{i=0}^n$ is a row matrix of the monomial basis functions. Similarly, the given data function can be approximated in the form.

$$f_n(x)=\sum_{i=0}^{n} b_i P_i(x)=\mathrm{X}(x)V\mathrm{F} \quad (4)$$

where $\mathrm{F}=[f_i]_{i=0}^n$ is the known coefficients column matrix such that $[f_i]_{i=0}^n$ can be determined by

$$f_i=(2\mathrm{i}+1)\int_0^1 f(x)P_i(x)\,dx \quad (5)$$

The kennel $k(x,t)$ will be approximated by the same way as well as $u_n(x)$ but with the consideration of the two variables $x$ and $t$. Approximating $k(x,t)$ subjected to $x$, gives $k_n(x,t)$ via the $(n+1)\times 1$ column matrix $\mathrm{N}(t)$ in the form

$$k_n(x,t)=\mathrm{X}(x)V\mathrm{N}(t);\mathrm{N}(t)=[n_i(t)]_{i=0}^n, n_i(t)=(2\mathrm{i}+1)\int_0^1 k(x,t)P_i(x)\,dx \quad (6)$$

Moreover, each entry $n_i \; \forall \, i=\overline{0,n}$ will be approximated with respect to the variable $t$ so that we get $k_{n,n}(x,t)$ via the $(n+1)\times(n+1)$ square known coefficients matrix $\mathrm{M}$ in the form:

$$k_{n,n}(x,t)=\mathrm{X}(x)V\mathrm{K}V^T\mathrm{X}^T(t);\mathrm{K}=\left[k_{ij}\right]_{i,j=0}^{n},k_{ij}=(2i+1)\int_0^1 n_i(t)P_j(t)\,dt \tag{7}$$

Furthermore, we get.

$$k_{n,n}(x,t)u_n(t)=\mathrm{X}(x)V\,\mathrm{K}V^T\,\tilde{\mathrm{X}}(t)V\,\mathrm{A}\,;\;\tilde{\mathrm{X}}(t)=\mathrm{X}^T(t)\,\mathrm{X}(t) \tag{8}$$

Substituting $k_{n,n}(x,t)u_n(t)$ of Eq. (8) into Eq. (1), we get

$$u_n(x)=f(x)+\mathrm{X}(x)V\mathrm{K}V^T\tilde{\tilde{\mathrm{X}}}V\mathrm{A}\;;\;\tilde{\tilde{\mathrm{X}}}=\int_0^1\tilde{\mathrm{X}}(t)\,dt \tag{9}$$

Substituting $u_n(x)$ in the left side of Eq. (1), $k_{n,n}(x,t)$ and $u_n(t)$ in the wright side, we get

$$\mathrm{X}(x)V\,\mathrm{K}V^T\,\tilde{\tilde{\mathrm{X}}}V\,\mathrm{A}-\mathrm{X}(x)V\,\mathrm{K}V^T\,\tilde{\tilde{\mathrm{X}}}V\,\mathrm{K}V^T\,\tilde{\tilde{\mathrm{X}}}V\,\mathrm{A}=\mathrm{X}(x)V\,\mathrm{K}V^T\,\tilde{\tilde{\mathrm{X}}}V\mathrm{F} \tag{10}$$

Simplifying Eq. (10), yields the approximated solution $u_n(x)$ in the form.

$$u_n(x)=\mathrm{X}(x)V\left(\mathrm{I_n}-\mathrm{K}V^T\,\tilde{\tilde{\mathrm{X}}}V\right)^{-1}\mathrm{F} \tag{11}$$

Furthermore, we can get the matrix of the unknown coefficient $\mathrm{A}$ by solving the algebraic linear system Eq. (10) and thereby computed the approximate solution Eq. (3).

### *2.2 Convergence in the Mean and Error Norm Estimation*

In this section, we examine the convergence in the mean of the interpolated unknown function, as defined by formula Eq. (11), towards the precise solution. Additionally, we analyse error norm estimation and establish two theorems as part of this investigation.

**Theorem 3.1**. Assume that $\max\limits_{x,t\in[0,1]}\left|k(x,t)\right|=\varepsilon$ and $\int_0^1\int_0^1\left|k(x,t)\right|^2 dxdt\le M<\infty$ where $\varepsilon$ is a positive real number. Suppose that $f(x)$ and $u(x)$ belong to $L_2(0,1)$ with $\max\limits_{x\in[0,1]} f(x)=N$. Let $u_n(x)$ be the approximated solution of degree $n$ that approximate $u(x)$ such that $\max\limits_{x\in[0,1]} u_n(x)=L$; $N, L$ are positive real numbers. Then $\lim\limits_{n\to\infty}\left\|u(x)-u_n(x)\right\|_2=0$ . *et al.,* [23]

**Theorem 3.2.** Rewrite Eq. (1) in the form $(\kappa u)(t)=f$, where the operator $\kappa$ is defined by $\kappa u=\int_0^1 \kappa(x,t)u(t)dt$ is the Fredholm operator. Sample the error norm of the Legendre approximation by $E_n(x)$ such that $E_n(x)=\|\kappa u-\kappa u_n\|_2$ where $\|\ .\ \|_2$ denotes the Euclidean norm in $R^2$ and $\kappa u_n=f_n$. Then $E_n(x)=\|\kappa u-\kappa u_n\|_2=0$.. *et al.,* [23]

## 3. Computational Results and Discussions

Based on MATLAB2019a, we designed a MATLAB code for finding the solution of Fredholm second kind integral equations which has the kernel in the form of weakly singular or non-singular. We solved six cases, three for non-singular and the other three for weakly singular. The approximate solutions were obtained for different values of $n$. The solutions are found equal to the exact ones or strongly convergent. Table 1 shows the exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)=|u(x_i)-u_n(x_i)|$ for $n=3,5$ at $x_i=0:0.1:1$.

**Table 1**
The exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors

| $x_i$ | $u(x_i)$ | $u_3(x_i)$ | $\Re_3(x_i)$ |
|---|---|---|---|
| 0 | 1 | 0.99906 | 0.00094 |
| 0.1 | 1.1052 | 1.1054 | 0.0002 |
| 0.2 | 1.2214 | 1.2218 | 0.0004 |
| 0.3 | 1.3499 | 1.35 | 1E-04 |
| 0.4 | 1.4918 | 1.4916 | 0.0002 |
| 0.5 | 1.6487 | 1.6483 | 0.0004 |
| 0.6 | 1.8221 | 1.8219 | 0.0002 |
| 0.7 | 2.0138 | 2.0138 | 0 |
| 0.8 | 2.2255 | 2.226 | 0.0005 |
| 0.9 | 2.4596 | 2.4599 | 0.0003 |
| 1 | 2.7183 | 2.7172 | 0.0011 |

The amount of CPU calculated time for example 1 for $n=3,5$ was 3.855 s, and 11.099 s respectively. Figure 1 shows the graphs of the exact solution and the approximate solution for $n=3$.

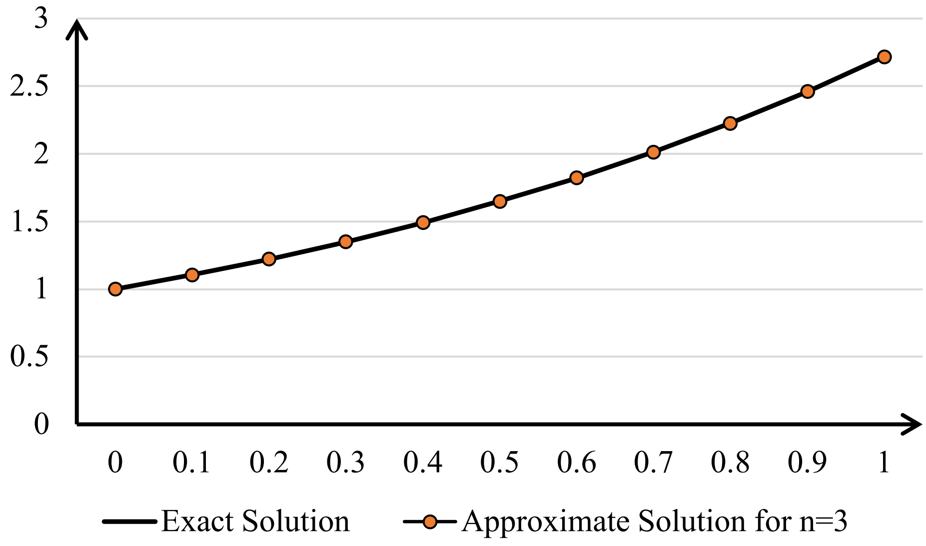


**Fig. 1**. Graphs of the exact solution and the approximate solution

Table 2 shows the exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)$ for $n=2,3$ at $x_i = 0:0.1:1$. The amount of CPU calculated time for example 2 for $n=2,3$ was 3.989 s, and 4.133 s respectively.

**Table 2**
The exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors

| $x_i$ | $u(x_i)$ | $u_2(x_i)$ | $u_3(x_i)$ | $\Re_2(x_i)$ | $\Re_3(x_i)$ |
|---|---|---|---|---|---|
| 0 | 0 | -2.31E-14 | -2.31E-14 | 2.3093E-14 | 2.3093E-14 |
| 0.1 | 0.4 | 0.4 | 0.4 | 0 | 0 |
| 0.2 | 0.8 | 0.8 | 0.8 | 0 | 0 |
| 0.3 | 1.2 | 1.2 | 1.2 | 0 | 0 |
| 0.4 | 1.6 | 1.6 | 1.6 | 0 | 0 |
| 0.5 | 2 | 2 | 2 | 0 | 0 |
| 0.6 | 2.4 | 2.4 | 2.4 | 0 | 0 |
| 0.7 | 2.8 | 2.8 | 2.8 | 0 | 0 |
| 0.8 | 3.2 | 3.2 | 3.2 | 0 | 0 |
| 0.9 | 3.6 | 3.6 | 3.6 | 0 | 0 |
| 1 | 4 | 4 | 4 | 0 | 0 |

Table 3 shows the exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)$ for $n=2,3$ at $x_i = 0:0.1:1$. The amount of CPU calculated time for example 3 for $n=2,3$ was 3.320 s, and 18.288 s respectively.

**Table 3**
The exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)$

| $x_i$ | $u(x_i)$ | $u_2(x_i)$ | $u_3(x_i)$ | $\Re_2(x_i)$ | $\Re_3(x_i)$ |
|---|---|---|---|---|---|
| 0 | 1 | 1.013 | 0.99906 | 0.013 | 0.00094 |
| 0.1 | 1.1052 | 1.1065 | 1.1054 | 0.0013 | 0.0002 |
| 0.2 | 1.2214 | 1.2168 | 1.2218 | 0.0046 | 0.0004 |
| 0.3 | 1.3499 | 1.3439 | 1.35 | 0.006 | 1E-04 |
| 0.4 | 1.4918 | 1.4877 | 1.4916 | 0.0041 | 0.0002 |
| 0.5 | 1.6487 | 1.6483 | 1.6483 | 0.0004 | 0.0004 |
| 0.6 | 1.8221 | 1.8258 | 1.8219 | 0.0037 | 0.0002 |
| 0.7 | 2.0138 | 2.02 | 2.0138 | 0.0062 | 0 |
| 0.8 | 2.2255 | 2.231 | 2.226 | 0.0055 | 0.0005 |
| 0.9 | 2.4596 | 2.4587 | 2.4599 | 0.0009 | 0.0003 |
| 1 | 2.7183 | 2.7033 | 2.7172 | 0.015 | 0.0011 |

Figure 2 shows the graphs of the exact solution and the approximate solutions for $n = 2,3$.

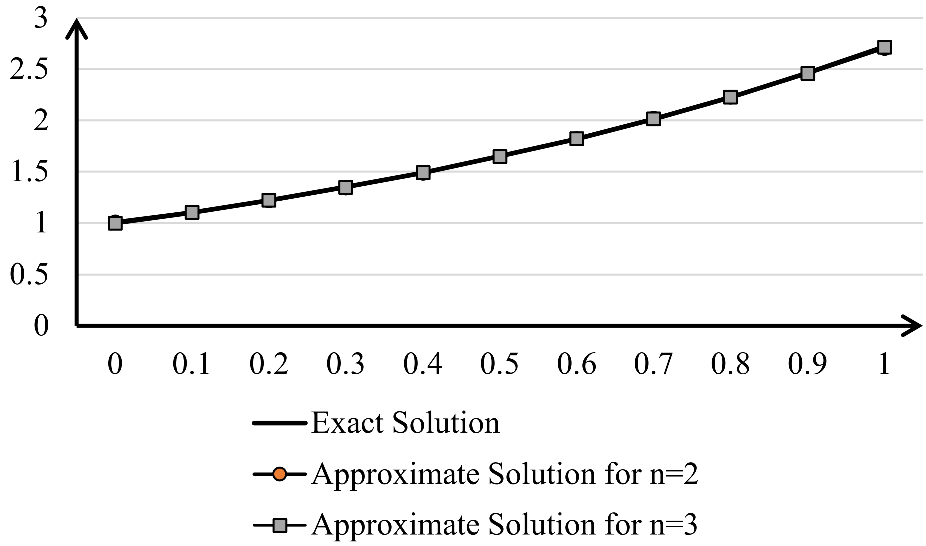


**Fig. 2.** the graphs of the exact solution and the approximate solutions for $n = 2,3$.

Table 4 shows the exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)$ for $n=3,5,10$ at $x_i = 0:0.1:1$. The amount of CPU calculated time for example 4 for $n=3,5,10$ was 4.738 s, 5.993 s and 9.012 s respectively.

**Table 4**
The exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)$

| $x_i$ | $u(x_i)$ | $u_3(x_i)$ | $u_5(x_i)$ | $u_{10}(x_i)$ | $\Re_3(x_i)$ | $\Re_5(x_i)$ | $\Re_{10}(x_i)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.12318 | 0.082773 | 0.045736 | 0.12318 | 0.082773 | 0.045736 |
| 0.1 | 0.31623 | 0.29549 | 0.31104 | 0.31698 | 0.02074 | 0.00519 | 0.00074913 |
| 0.2 | 0.44721 | 0.43505 | 0.45092 | 0.44713 | 0.01216 | 0.00371 | 0.000084856 |
| 0.3 | 0.54772 | 0.54718 | 0.54815 | 0.54855 | 0.00054 | 0.00043 | 0.0008303 |
| 0.4 | 0.63246 | 0.63721 | 0.62859 | 0.63186 | 0.00475 | 0.00387 | 0.00059335 |
| 0.5 | 0.70711 | 0.71048 | 0.7034 | 0.70773 | 0.00337 | 0.00371 | 0.00062106 |
| 0.6 | 0.7746 | 0.77232 | 0.77421 | 0.77493 | 0.00228 | 0.00039 | 0.00033343 |
| 0.7 | 0.83666 | 0.82807 | 0.83829 | 0.83632 | 0.00859 | 0.00163 | 0.00034471 |
| 0.8 | 0.89443 | 0.88305 | 0.8937 | 0.89523 | 0.01138 | 0.00073 | 0.00080541 |
| 0.9 | 0.94868 | 0.94261 | 0.94449 | 0.9482 | 0.00607 | 0.00419 | 0.00048686 |
| 1 | 1 | 1.0121 | 1.0059 | 0.99812 | 0.0121 | 0.0059 | 0.0018833 |

Figure 3 shows the graphs of the exact solution and the approximate solutions for $n=3,5,10$.

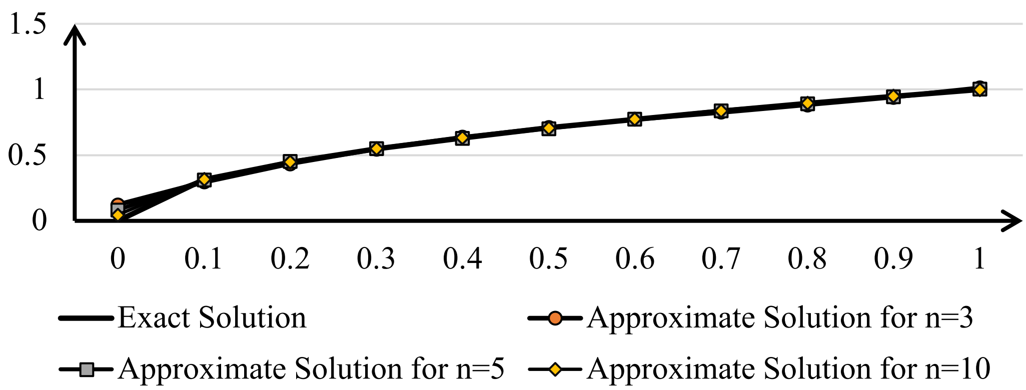


**Fig. 3.** the graphs of the exact solution and the approximate solutions for $n=3,5,10$.

Figure 4 shows the graphs of the absolute errors for $n=3,5,10$.

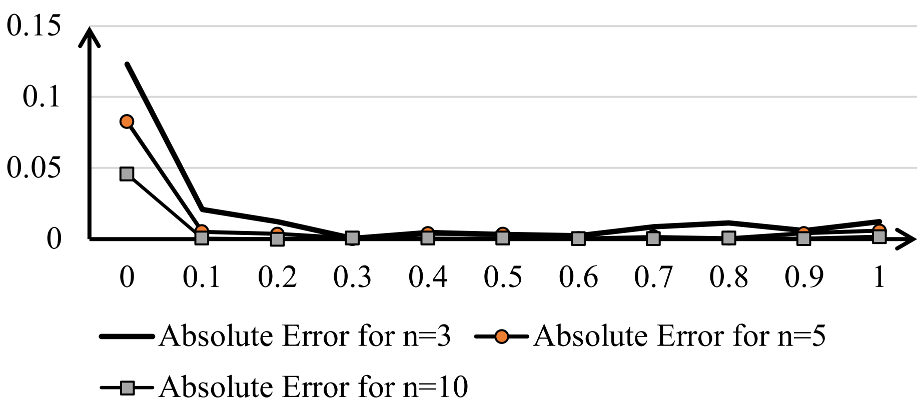


**Fig. 4.** The graphs of the absolute errors for $n=3,5,10$.

Table 5 shows the exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)$ for $n=2,5$ at $x_i = 0:0.1:1$. The amount of CPU calculated time for example 5 for $n=2,5$ was 3.262 s, and 4.681 s respectively.

**Table 5**
The exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)$

| $x_i$ | $u(x_i)$ | $u_2(x_i)$ | $u_5(x_i)$ | $\Re_2(x_i)$ | $\Re_5(x_i)$ |
|---|---|---|---|---|---|
| 0 | 0 | 2.27E-14 | 2.27E-14 | 2.2737E-14 | 2.2737E-14 |
| 0.1 | 0.01 | 0.01 | 0.01 | 0 | 0 |
| 0.2 | 0.04 | 0.04 | 0.04 | 0 | 0 |
| 0.3 | 0.09 | 0.09 | 0.09 | 0 | 0 |
| 0.4 | 0.16 | 0.16 | 0.16 | 0 | 0 |
| 0.5 | 0.25 | 0.25 | 0.25 | 0 | 0 |
| 0.6 | 0.36 | 0.36 | 0.36 | 0 | 0 |
| 0.7 | 0.49 | 0.49 | 0.49 | 0 | 0 |
| 0.8 | 0.64 | 0.64 | 0.64 | 0 | 0 |
| 0.9 | 0.81 | 0.81 | 0.81 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 |

Figure 5 shows the graphs of the exact solution and the approximate solutions for $n=2,5$.

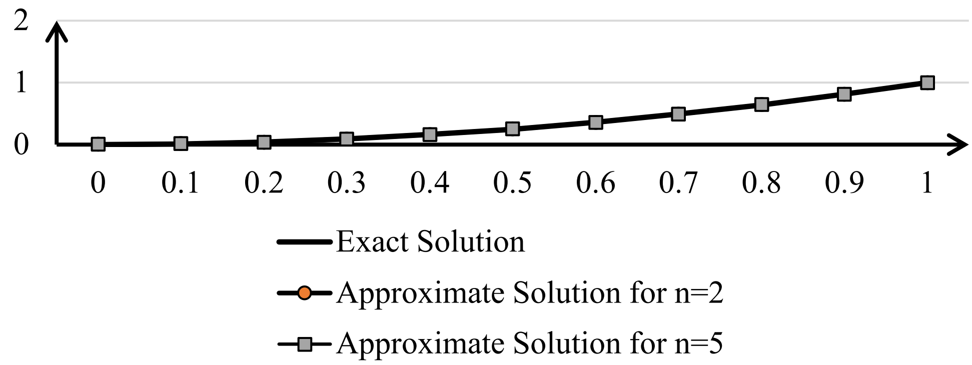


**Fig. 5.** The graphs of the exact solution and the approximate solutions for $n=2,5$.

Figure 6 shows the graphs of the absolute errors for $n=2,5$.

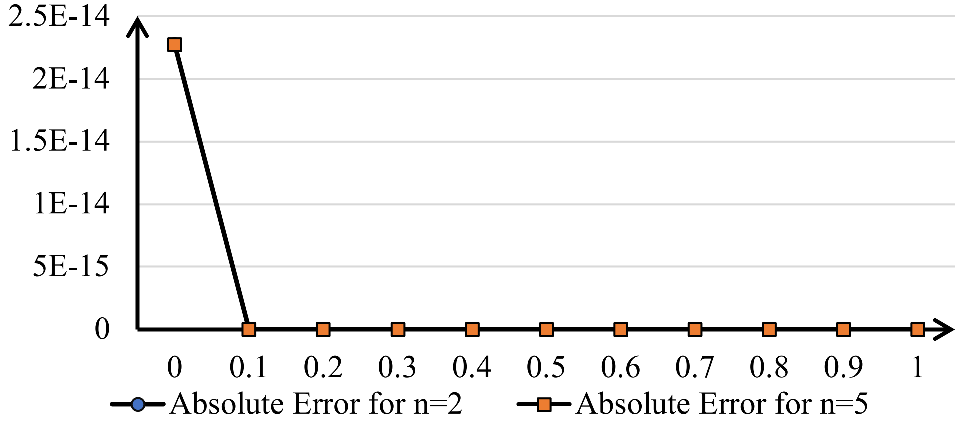


**Fig. 6.** The graphs of the absolute errors for $n=2,5$.

Table 6 shows the exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)$ for $n=2,5,8$ at $x_i=0:0.1:1$. The amount of CPU calculated time for example 6 for $n=2,5$ was 3.280 s, 3.987 s, and 5.285 s respectively.

**Table 6**
The exact solution $u(x_i)$, the approximate solutions $u_n(x_i)$, the absolute errors $\Re_n(x_i)$

| $x_i$ | $u(x_i)$ | $u_2(x_i)$ | $u_5(x_i)$ | $u_8(x_i)$ | $\Re_2(x_i)$ | $\Re_5(x_i)$ | $\Re_8(x_i)$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1.0172 | 1 | 0.99975 | 0.0172 | 0 | 0.00025 |
| 0.1 | 1.1052 | 1.1108 | 1.1052 | 1.1052 | 0.0056 | 0 | 0 |
| 0.2 | 1.2214 | 1.221 | 1.2214 | 1.2215 | 0.0004 | 0 | 1E-04 |
| 0.3 | 1.3499 | 1.3481 | 1.3499 | 1.3501 | 0.0018 | 0 | 0.0002 |
| 0.4 | 1.4918 | 1.492 | 1.4919 | 1.4918 | 0.0002 | 1E-04 | 0 |
| 0.5 | 1.6487 | 1.6526 | 1.6488 | 1.6487 | 0.0039 | 1E-04 | 0 |
| 0.6 | 1.8221 | 1.83 | 1.8222 | 1.8221 | 0.0079 | 1E-04 | 0 |
| 0.7 | 2.0138 | 2.0242 | 2.0138 | 2.0133 | 0.0104 | 0 | 0.0005 |
| 0.8 | 2.2255 | 2.2352 | 2.2256 | 2.2257 | 0.0097 | 0.0001 | 0.0002 |
| 0.9 | 2.4596 | 2.463 | 2.4596 | 2.4596 | 0.0034 | 0 | 0 |
| 1 | 2.7183 | 2.7076 | 2.7183 | 2.7185 | 0.0107 | 0 | 0.0002 |

Figure 7 shows the graphs of the exact solution and the approximate solutions for $n=2,5,8$.

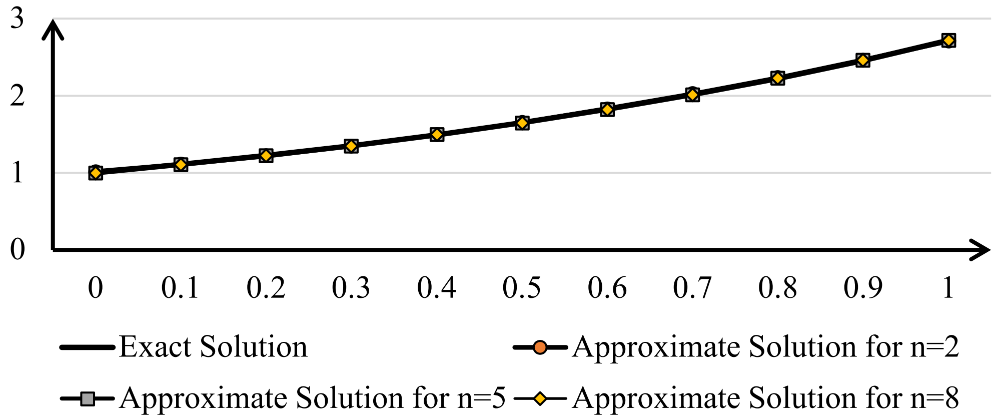


**Fig. 7.** the graphs of the exact solution and the approximate solutions for $n=2,5,8$.

Figure 8 shows the graphs of the absolute errors for $n=2,5,8$.

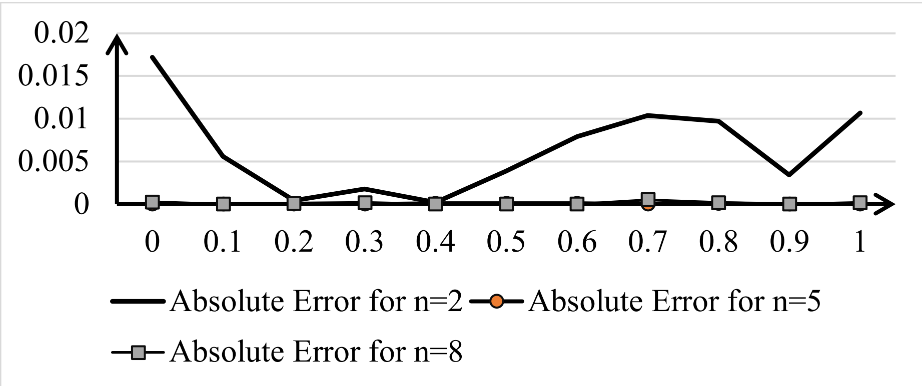


**Fig. 8.** The graphs of the absolute errors for $n=2,5,8$.

**Example 1.** For the following integral equation

$$u(x)+\int_0^1 \exp(t)\sin(x)\,dt=\exp(x)+\frac{1}{2}\left(e^2-1\right)\sin(x);\ 0\le x\le 1 \tag{12}$$

which has exact solution in the form $u(x)=e^x$ [6].

**Example 2**. For the following integral equation

$$u(x)+\int_0^1 \exp(t)\sin(x)\,dt=\exp(x)+\frac{1}{2}\left(e^2-1\right)\sin(x);\ 0\le x\le 1 \tag{13}$$

which has exact solution in the form $u(x)=e^x$ [6].

**Example 3**. For the following integral equation

$$\lambda u(x)-\int_0^1 e^{xt}u(t)\,dt=\lambda e^x-\frac{e^{x+1}-1}{x+1};\ 0\le x\le 1 \tag{14}$$

which has exact solution in the form $u(x)=e^x$ for $\lambda=50$ [5].

**Example 4.** For the following integral equation

$$u(x)-\int_0^1 \frac{u(t)}{\sqrt{1-t}}dt=-\frac{\pi}{2}+\sqrt{x};\ 0\le x\le 1 \tag{15}$$

which has exact solution in the form $u(x)=\sqrt{x}$ [4].

**Example 5.** For the following integral equation

$$u(x)-\int_0^1 \frac{u(t)}{\sqrt{1-t}}dt=x^2-\frac{16}{15};\ 0\le x\le 1 \tag{16}$$

which has exact solution in the form $u(x)=x^2$ [4].

**Example 6**. For the following integral equation

$$u(x)-\int_0^1 \frac{u(t)}{\sqrt{1-t}}dt=e^x-4.0602;\ 0\le x\le 1 \tag{17}$$

which has exact solution in the form $u(x)=e^x$ [4]

## 4. Conclusions

The shifted Legendre polynomials of the first kind have been applied for solving the second kind Fredholm integral equations which contain kernel in the form of weakly singular. The most significant contribution of this work is the idea of extracting the coefficients of Legendre polynomials to form a square matrix which contributed significantly to the abbreviations of the solution's procedures. This resulted in the solution steps being reduced to a handful of seconds. The more important contribution of this work is the idea of substituting the double approximate kernel in the right side of the integral equation simultaneously with the substituting of the single approximate unknown function in both sides. By following this process, a linear system of algebraic equations was obtained without the need for the collocation method. The six cases' solutions, as mentioned in the figures and tables, were either exact or significantly converging towards them. This supports the assertion that this approach is novel and extremely precise.

**Acknowledgement**
This research was not funded by any grant.

**References**

[1] Yang, Yin, Zhuyan Tang, and Yunqing Huang. "Numerical solutions for Fredholm integral equations of the second kind with weakly singular kernel using spectral collocation method." *Applied Mathematics and Computation* 349 (2019): 314-324. https://doi.org/10.1016/j.amc.2018.12.035
[2] Panigrahi, Bijaya Laxmi, Moumita Mandal, and Gnaneshwar Nelakanti. "Legendre multi-Galerkin methods for Fredholm integral equations with weakly singular kernel and the corresponding eigenvalue problem." *Journal of Computational and Applied Mathematics* 346 (2019): 224-236. https://doi.org/10.1016/j.cam.2018.07.010
[3] Okayama, Tomoaki, Takayasu Matsuo, and Masaaki Sugihara. "Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind." *Journal of Computational and Applied Mathematics* 234, no. 4 (2010): 1211-1227. https://doi.org/10.1016/j.cam.2009.07.049
[4] Behzadi, Reza, Emran Tohidi, and Faezeh Toutounian. "Numerical solution of weakly singular Fredholm integral equations via generalization of the Euler–Maclaurin summation formula." *Journal of Taibah University for science* 8, no. 2 (2014): 200-205. https://doi.org/10.1016/j.jtusci.2013.11.001
[5] Wang, Min. "Multistep collocation method for Fredholm integral equations of the second kind." *Applied Mathematics and Computation* 420 (2022): 126870. https://doi.org/10.1016/j.amc.2021.126870
[6] Li, X. Y., and B. Y. Wu. "Superconvergent kernel functions approaches for the second kind Fredholm integral equations." *Applied Numerical Mathematics* 167 (2021): 202-210. https://doi.org/10.1016/j.apnum.2021.05.004
[7] Shoukralla, E. S., and B. M. Ahmed. "Barycentric Lagrange interpolation methods for evaluating singular integrals." *Alexandria Engineering Journal* 69 (2023): 243-253. https://doi.org/10.1016/j.aej.2022.12.005
[8] Shoukralla, E. S., B. M. Ahmed, M. Sayed, and Ahmed Saeed. "Interpolation method for solving Volterra integral equations with weakly singular kernel using an advanced barycentric Lagrange formula." *Ain Shams Engineering Journal* 13, no. 5 (2022): 101743. https://doi.org/10.1016/j.asej.2022.101743
[9] Shoukralla, E. S., B. M. Ahmed, Ahmed Saeed, and M. Sayed. "The interpolation-Vandermonde method for numerical solutions of weakly singular Volterra integral equations of the second kind." In *Proceedings of Seventh International Congress on Information and Communication Technology: ICICT 2022, London, Volume 1*, pp. 607-614. Singapore: Springer Nature Singapore, 2022. https://doi.org/10.1007/978-981-19-1607-6_54
[10] Shoukralla, E. S., B. M. Ahmed, Ahmed Saeed, and M. Sayed. "Vandermonde-interpolation method with Chebyshev nodes for solving Volterra integral equations of the second kind with weakly singular kernels." *Eng. Lett* 30, no. 4 (2022).
[11] Shoukralla, Emil Sobhy. "Interpolation method for solving weakly singular integral equations of the second kind." *Appl Comput Math* 10, no. 3 (2021): 76-85. https://doi.org/10.11648/j.acm.20211003.14
[12] Shoukralla, E. S. "Interpolation method for evaluating weakly singular kernels." *J. Math. Comput. Sci.* 11, no. 6 (2021): 7487-7510.
[13] Shoukralla, E. S., Nermin Saber, and Ahmed Y. Sayed. "Computational method for solving weakly singular Fredholm integral equations of the second kind using an advanced barycentric Lagrange interpolation formula." *Advanced Modeling and Simulation in Engineering Sciences* 8 (2021): 1-22. https://doi.org/10.1186/s40323-021-00212-6
[14] Shoukralla, ES4174652. "A numerical method for solving Fredholm integral equations of the first kind with logarithmic kernels and singular unknown functions." *International Journal of Applied and Computational Mathematics* 6, no. 6 (2020): 172. https://doi.org/10.1007/s40819-020-00923-1
[15] Shoukralla, E. S., B. M. Ahmed, M. Sayed, and Ahmed Saeed. "Interpolation method for solving Volterra integral equations with weakly singular kernel using an advanced barycentric Lagrange formula." *Ain Shams Engineering Journal* 13, no. 5 (2022): 101743. https://doi.org/10.1016/j.asej.2022.101743
[16] Shoukralla, E. S., Nermin Saber, and Ahmed Y. Sayed. "Computational method for solving weakly singular Fredholm integral equations of the second kind using an advanced barycentric Lagrange interpolation formula." *Advanced Modeling and Simulation in Engineering Sciences* 8 (2021): 1-22. https://doi.org/10.1186/s40323-021-00212-6
[17] Shoukralla, E. S., and M. A. Markos. "Numerical solution of a certain class of singular Fredholm integral equations of the first kind via the Vandermonde matrix." *Int J Math Models Methods Appl Sci* 14 (2020): 48-53. https://doi.org/10.46300/9101.2020.14.12

[18] Shoukralla, E. S., M. Kamel, and M. A. Markos. "Numerical solution of Fredholm integral equations of the first kind with singular logarithmic kernel and singular unknown function via monic Chebyshev polynomials." *International Journal of Computing Science and Mathematics* 14, no. 1 (2021): 77-88. https://doi.org/10.1504/IJCSM.2021.118077

[19] Shoukralla, E. S., H. Elgohary, and B. M. Ahmed. "Barycentric Lagrange interpolation for solving Volterra integral equations of the second kind." In *Journal of Physics: Conference Series*, vol. 1447, no. 1, p. 012002. IOP Publishing, 2020. https://doi.org/10.1088/1742-6596/1447/1/012002

[20] Shoukralla, E. S., and B. M. Ahmed. "Numerical solutions of Volterra integral equations of the second kind using Lagrange interpolation via the Vandermonde matrix." In *Journal of Physics: Conference Series*, vol. 1447, no. 1, p. 012003. IOP Publishing, 2020. https://doi.org/10.1088/1742-6596/1447/1/012003

[21] Shoukralla, E. S., and B. M. Ahmed. "Multi-techniques method for solving Volterra integral equations of the second kind." In *2019 14th international conference on computer engineering and systems (ICCES)*, pp. 209-213. IEEE, 2019. https://doi.org/10.1109/ICCES48960.2019.9068138

[22] Cvetkovski, Zdravko. *Inequalities: theorems, techniques and selected problems*. Springer Science & Business Media, 2012. https://doi.org/10.1007/978-3-642-23792-8

[23] Shoukralla, E. S., and B. M. Ahmed. "Interpolated Solutions of Abel Integral Equations Using Barycentric Lagrange Double Interpolation." In *International Congress on Information and Communication Technology*, pp. 87-96. Singapore: Springer Nature Singapore, 2023. https://doi.org/10.1007/978-981-99-3091-3_8