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The Numerical Solutions of weakly singular Fredholm integral equations of the Second kind Using Chebyshev Polynomials of the Second Kind

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ABSTRACT

In this study, the second kind Chebyshev Polynomials were utilized to acquire interpolated solutions for the second kind Fredholm integral equations with weakly singular kernel. To accomplish this, the data, unknown, and kernel functions were converted into matrix form, and consequently we completely isolated the singularity of the kernel. The primary benefit of this method is the ability to change the form of integral equation to an equivalent algebraic system, which is easier to solve. The effectiveness of our technique was evaluated by applying it to three illustrated examples, and it was observed that the solutions obtained exhibit strong convergence towards the exact solutions.

1. Introduction

Industrial applications of weakly singular integral equations are very important in magnificent fields such as transportation, aerospace, radiating, automobile, magnetic disk drives, energy harvesting, micro-electromechanical systems [1-4], and others. In 1980, Bakirtas [5] conducted a study on the rigid punch interaction on an elastic half-space. Building upon this research, in 2015, İsa Çömez [6] introduced a significant breakthrough by addressing a novel problem where the punch is influenced by a concentrated normal force, which can be mathematically represented as the second kind integral equations with weakly singular kernel. Subsequently, in 2022, Ahmed Sayed *et al.,* [7] investigated the graded coating flat stamp contact mechanics by employing a modified Barycentric Lagrange interpolation formula as a key methodology in their study. This leads to very marvelous applications such as studying bridging cracks and studying a punch with constant friction of a plane tire. Thus, many scientists investigated various methods to solve the second kind Fredholm integral equations with weakly singular kernel [8-13]. In [8], Yin Yang utilizes the formula of Jacobi-Gauss quadrature for approximating the operator of the integral in the spectral collocation method numerical form. This approach is employed to solve the second kind Fredholm integral equations with weakly singular kernel. In their work, Behzadi *et al.,* [9] used Bernoulli functions to modify the

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Euler-Maclaurin summation formula and then, applied it to the trapezoidal rule This modification enabled them to find a solution numerically for solving integral equations with weakly singular kernels. Shoukralla *et al.,* [14-20] developed numerical and computational solutions to solve first-kind Fredholm integral equations that have weakly singular kernels. They discussed several ways and methods for using analytical methods to deal with kernels and singularities of the unknown functions.

This study uses the Chebyshev polynomials of the second kind to establish a powerful technique for solving the second kind Fredholm integral equations with weakly singular kernel. In 1959, The Chebyshev Polynomials was first used for solving linear integral equations by David Ellott [21]. In 1998, Rawitscher [22] investigated a solution of the one-dimensional quantum mechanical Lippmann- Chhwinger integral equation using Chebyshev expansion. Yucheng [23] converted Fredholm integral equation of the second kind into an algebraic system using Chebyshev polynomials which make it very easy to solve. The properties of Chebyshev wavelets were used to convert the second kind Fredholm integral equation to algebraic system [24]. Most of the recent applications are concerned with integral equations, ordinary differential equations, and partial differential equations. Doan Thi Hong Hai and Nguyen Minh Phu As [25] made a Critical Review of three Mathematical Models concerned with ordinary and partial differential equations for Solar Air Heater Analysis. As a result, researchers tried to find marvelous techniques for solving different kinds of equations. Nuran Guzel and Mustafa Bayram [26] investigated numerical solution of differential–algebraic equations (DAEs) with index-2 using Padé approximation method which appeared in different applications. In this paper we will discuss a magnificent technique using the second kind Chebyshev Polynomials for solving weakly singular Fredholm integral equations of the second kind.

2. The Technique

Consider the second kind Fredholm integral equations with weakly singular kernel, which commonly have the form:

$$u(\xi) = f(\xi) + \int_a^b k(\xi,t)u(t)\,dt; \mathrm{a} \le \xi \le b\,, \tag{1}$$

where the function $f(\xi)$ and the unknown function $u(\xi)$ belong to the space $\mathrm{L}^2[a,b]$. Here $k(\xi,t) = \dfrac{\phi(\xi,t)}{|\xi - t|^{\alpha}}$; $0 < \alpha < 1$ is the kernel and $\phi(\xi,t)$ is a continuous function defined on the square $\{(\xi,t), a \le \xi, t \le b\}$ for some constants a and b. Moreover, it is assumed that $\max\limits_{\xi\in[a,b]} |u(\xi)| \le L$, $\max\limits_{\xi\in[a,b]} |f(\xi)| \le M$, $\max\limits_{\xi,t\in[a,b]} |k(\xi,t)| \le N$. Where N, M, L are real numbers. and $u(\xi)$ is a sufficiently smooth exact solution.

Using the linear transformation $\xi = \frac{1}{2}(x+1)$; $|x| \le 1$, we can transform the domain of the integral Eq. (1) so that it becomes,

$$u(x) = f(x) + \int_0^1 k(x,t)u(t)\,dt; 0 \le x \le 1, \tag{2}$$

$u(\xi)$ is transformed into $u(x)$ for $|x| \le 1$. Consider the set of orthogonal shifted second kind polynomials of Chebyshev $\{U_n(x)\}_0^m$; $m \ge 0$ on the domain $[0,1]$.

$$U_n(x) = 2^n \prod_{k=1}^{n} \left[(2x-1) - \cos\left(\frac{k\pi}{n+1}\right)\right]; \int_0^1 \sqrt{x - x^2}\, U_i(x) U_j(x)\,dx = \begin{cases} 0 & ; i \ne j \\ \pi/8 ; & i = j \end{cases} \tag{3}$$

$$u_0(x)=1,\ u_1(x)=4x-2,\ u_2(x)=16x^2-16x+3,\ \ldots,\ u_n(x). \tag{4}$$

We now begin by applying the second kind shifted Chebyshev polynomials of Chebyshev $U_n(x)$ for approximating the unknown function $u(x)$.

Denoting the approximate unknown function of degree at most n by $u_n(x)$, then we have

$$u_n(x)=\sum_{i=0}^{n} a_i \mathrm{U}_i(x);\ a_i=\frac{8}{\pi}\int_0^1 \sqrt{x-x^2}\,u(x)\mathrm{U}_i(x)dx. \tag{5}$$

By extracting the coefficients of $\mathrm{U}_i(x)$ and putting them in a square known matrix, say, B, we get $u_n(x)$ and $u_i(x)$ in the matrix form

$$u_i(x)=\mathrm{BX}(x). \tag{6}$$

$$u_n(x)=\mathrm{ABX}(x), \tag{7}$$

where $\mathrm{A}=[a_i]_{i=0}^{n}$ is row matrix of the unknown coefficients, B is the square known matrix of the coefficients of the $\{\mathrm{U}_i(x)\}_{i=0}^{n}$, and $\mathrm{X}(x)=\left[x^i\right]_{i=0}^{n}$ is the monomial basis functions column matrix. Similarly, by approximating the given data function $f(x)$ to obtain the n^{th} degree approximant $f_n(x)$ through the known coefficients row matrix $\mathrm{F}=[f_i]_{i=0}^{n}$ such that

$$f_n(x)=\mathrm{FBX}(x);\ f_i=\frac{8}{\pi}\int_0^1 \sqrt{x-x^2}\,f(x)\mathrm{U}_i(x)dx. \tag{8}$$

The kernel $k(x,t)$ will be approximated by the same way as well as $f_n(x)$ but with the consideration of the two variables x and t. Approximating $k(x,t)$ subjected to x with reversing the matrices order, gives $k_n(x,t)$ via the $(n+1)\times 1$ column matrix $\mathrm{N}(t)$ in the form,

$$k_n(x,t)=\mathrm{X}^T(x)\mathrm{B}^T\mathrm{N}(t);n_i(t)=\frac{8}{\pi}\int_0^1 \sqrt{x-x^2}\,k(x,t)\mathrm{U}_i(x)dx,\mathrm{N}(t)=[n_i(t)]_{i=0}^{n}.. \tag{9}$$

Moreover, each entry $n_i\ \forall\ i=\overline{0,n}$ will be approximated with respect to the variable t so that we get $k_{n,n}(x,t)$ via the $(n+1)\times(n+1)$ square known coefficients matrix M in the form,

$$k_{n,n}(x,t)=\mathrm{X}^T(x)\mathrm{B}^T\mathrm{MBX}(t)=\mathrm{X}^T(x)\mathrm{PX}(t);\mathrm{M}=\left[m_{ij}\right]_{i,j=0}^{n}, \tag{10}$$

$$m_{ij}=\frac{8}{\pi}\int_0^1 \sqrt{t-t^2}\,n_i(t)\mathrm{U}_j(t)dt,$$

where, $V=\mathrm{B}^T\mathrm{MB}$ is a square matrix of $(n+1)\times(n+1)$. Furthermore, we get

$$k_{n,n}(x,t)u_n(t)=\mathrm{X}^T(x)\mathrm{B}^T\mathrm{MBX}(t)AB\mathrm{X}(t)=\mathrm{X}^T(x)\mathrm{B}^T\mathrm{MBX}(t)\mathrm{X}^T(t)\mathrm{B}^T\mathrm{A}^T, \tag{11}$$

$$\therefore k_{n,n}(x,t)u_n(t)=\mathrm{X}^T(x)V\tilde{\mathrm{X}}(t)\mathrm{B}^T\mathrm{A}^T, \tag{12}$$

where $\tilde{\mathrm{X}}(t)=\mathrm{X}(t)\mathrm{X}^T(t)$.

Hence, we can get $u_n(x)$ by replacing it with $u(x)$ given by Eq. (2) upon using Eq. (12) to get

$$u_n(x)=f(x)+\int_0^1 \mathrm{X}^T(x)V\tilde{\mathrm{X}}(t)\mathrm{B}^T\mathrm{A}^T dt=f(x)+\mathrm{X}^T(x)V\Phi\mathrm{B}^T\mathrm{A}^T;\ Z=\int_0^1 \tilde{\mathrm{X}}(t)dt, \tag{13}$$

where $Z=\int_0^1 \tilde{X}(t)\,dt$ is $(n+1)\times(n+1)$ square known matrix. Consequently, by substituting $u_n(x)$ of Eq. (10) in both sides of Eq. (2) in virtue of Eq. (8) and (10), we get

$$\begin{aligned}&\mathrm{FBX}(x)+\mathrm{X}^T(x)V\mathrm{ZB}^T\mathrm{A}^T=\\&\mathrm{FBX}(x)+\int_0^1 \mathrm{X}^T(x)V\mathrm{X}(t)\left[\mathrm{FBX}(t)+\mathrm{X}^T(t)V\mathrm{ZB}^T\mathrm{A}^T\right]dt\end{aligned} \tag{14}$$

$$\begin{aligned}&\mathrm{FBX}(x)+\mathrm{X}^T(x)V\mathrm{ZB}^T\mathrm{A}^T=\\&\mathrm{FBX}(x)+\int_0^1 \mathrm{X}^T(x)V\mathrm{X}(t)\left[\mathrm{X}^T(t)\mathrm{B}^T\mathrm{F}^T+\mathrm{X}^T(t)V\mathrm{ZB}^T\mathrm{A}^T\right]dt\end{aligned} \tag{15}$$

$$\mathrm{X}^T(x)V\mathrm{ZB}^T\mathrm{A}^T=\mathrm{X}^T(x)V\mathrm{ZB}^T\mathrm{F}^T+\mathrm{X}^T(x)VZV\mathrm{ZB}^T\mathrm{A}^T;Z=\int_0^1 \tilde{X}(t)\,dt \tag{16}$$

$$\mathrm{X}^T(x)V\mathrm{ZB}^T\mathrm{A}^T-\mathrm{X}^T(x)VZV\mathrm{ZB}^T\mathrm{A}^T=\mathrm{X}^T(x)V\mathrm{ZB}^T\mathrm{F}^T \tag{17}$$

$$\mathrm{ZB}^T\mathrm{A}^T-ZV\mathrm{ZB}^T\mathrm{A}^T=\mathrm{ZB}^T\mathrm{F}^T \tag{18}$$

$$(Z-ZVZ)\mathrm{B}^T\mathrm{A}^T=\mathrm{ZB}^T\mathrm{F}^T \tag{19}$$

Simplifying Eq. (19),

$$\mathrm{A}^T=\left((Z-ZVZ)\mathrm{B}^T\right)^{-1}\mathrm{ZB}^T\mathrm{F}^T=\left(\mathrm{B}^T\right)^{-1}(Z-ZVZ)^{-1}\mathrm{ZB}^T\mathrm{F}^T \tag{20}$$

The solution of the algebraic linear system of Eq. (17) gives the unknown coefficients matrix A which is defined by Eq. (20) and thereby we can get the unknown function $u_n(x)$ which is described by Eq. (7) as follows

$$\mathrm{A}^T=\left((Z-ZVZ)\mathrm{B}^T\right)^{-1}\mathrm{ZB}^T\mathrm{F}^T=\left(\mathrm{B}^T\right)^{-1}(Z-ZVZ)^{-1}\mathrm{ZB}^T\mathrm{F}^T \tag{21}$$

$$u_n(x)=\mathrm{ABX}(x)=\mathrm{X}^T(x)\mathrm{B}^T\mathrm{A}^T=\mathrm{X}^T(x)\mathrm{B}^T\left(\mathrm{B}^T\right)^{-1}(Z-ZVZ)^{-1}\mathrm{ZB}^T\mathrm{F}^T \tag{22}$$

$$u_n(x)=\mathrm{X}^T(x)(Z-ZVZ)^{-1}\mathrm{ZB}^T\mathrm{F}^T=\mathrm{X}^T(x)(I-ZV)^{-1}Z^{-1}\mathrm{ZB}^T\mathrm{F}^T \tag{23}$$

$$u_n(x)=\mathrm{X}^T(x)(I-ZV)^{-1}\mathrm{B}^T\mathrm{F}^T \tag{24}$$

3. Computational Results and Discussions

Using the provided technique, we developed a MATLAB R2014b code to solve three examples of the second kind Fredholm integral equations with weakly singular kernel. By applying the presented technique, we obtained interpolant solutions for the three examples. To evaluate the accuracy of our results, we compared them with the exact solutions as well as the solutions obtained using the second Barycentric Lagrange Interpolation method [14]. The obtained interpolant solutions demonstrate strong and rapid convergence towards the exact solutions, as indicated by the presented tables and figures. This superior convergence validates the effectiveness and efficiency of the method employed in this study. The exact solution is given by $u_{ex}(x)$, and the interpolant solution resulted from applying the presented technique using Chebyshev Polynomials is denoted by

$\tilde{u}_n^{Ch}(x)$ and the solution resulted from applying the second Barycentric Lagrange Interpolation method is denoted by $\tilde{u}_n^{B}(x)$, where n represents the interpolant degree.

Example 1. For the following integral equation,

$$u(x)=x^2-\frac{16}{15}+\int_0^1 \frac{u(t)}{\sqrt{1-t}}dt;\ 0\le x\le 1, \tag{25}$$

where the exact solution takes the form $u_{ex}(x)=x^2$ [14]. Using the presented technique using Chebyshev Polynomials for $n=3$, we obtain $\tilde{u}_3^{Ch}(x)$. Considering the nodes and substituting into $u_{ex}(x)$ and $\tilde{u}_3^{Ch}(x)$ and by using the second Barycentric Lagrange Interpolation technique for $n=4$, we obtain $\tilde{u}_4^{B}(x)$, Table 1, includes the absolute errors $R_3^{Ch}(x_i)=\left|u_{ex}(x_i)-\tilde{u}_3^{Ch}(x_i)\right|$ and $R_4^{B}(x_i)=\left|u_{ex}(x_i)-\tilde{u}_4^{B}(x_i)\right|$. In Figure 1, plotted is the graph of the interpolant error $R_3^{Ch}(x_i)$. In Figure 2, plotted is the graph of the interpolant error $R_4^{B}(x_i)$.

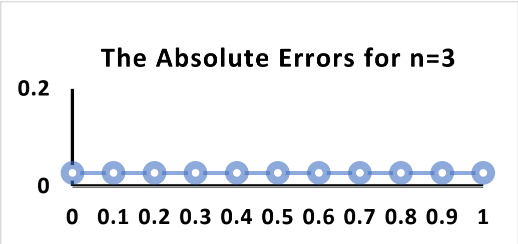

Fig. 1. The absolute Error $R_3^{Ch}(x_i)$

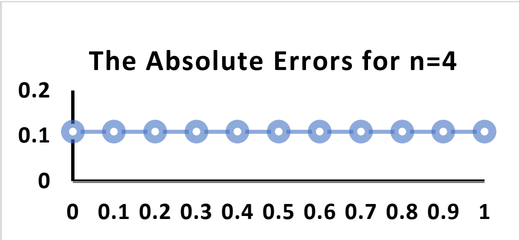

Fig. 2. The absolute Error $R_4^{B}(x_i)$

Table 1
The Absolute Errors $R_3^{Ch}(x_i)$ and $R_4^{B}(x_i)$

x_i	$R_3^{Ch}(x_i)$	$R_4^{B}(x_i)$
0	0.02649	0.10881
0.1	0.02649	0.10881
0.2	0.02649	0.10881
0.3	0.02649	0.10881
0.4	0.02649	0.10881
0.5	0.02649	0.10881
0.6	0.02649	0.10881
0.7	0.02649	0.10881
0.8	0.02649	0.10881
0.9	0.02649	0.10881
1.0	0.02649	0.10881

Example 2. For the following integral equation,

$$u(x)=\sqrt{x}-\frac{\pi}{2}+\int_0^1 \frac{u(t)}{\sqrt{1-t}}dt;\ 0\le x\le 1, \tag{26}$$

where the exact solution takes the form $u_{ex}(x)=\sqrt{x}$ [14]. Using the presented technique using Chebyshev Polynomials for $n=5$, we obtain $\tilde{u}_5^{Ch}(x)$. Considering the nodes and substituting into $u_{ex}(x)$ and $\tilde{u}_5^{Ch}(x)$ and by using the second Barycentric Lagrange Interpolation technique for $n=12$, we obtain $\tilde{u}_{12}^{B}(x)$, Table 2, includes the absolute errors $R_5^{Ch}(x_i)=\left|u_{ex}(x_i)-\tilde{u}_5^{Ch}(x_i)\right|$ and $R_{12}^{B}(x_i)=\left|u_{ex}(x_i)-\tilde{u}_{12}^{B}(x_i)\right|$. In Figure 3, plotted is the graph of the interpolant error $R_5^{Ch}(x_i)$. In Figure 4 plotted is the graph of the interpolant error $R_{12}^{B}(x_i)$.

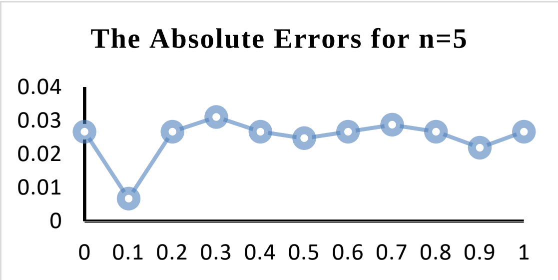

Fig. 3. The absolute Error $R_5^{Ch}(x_i)$

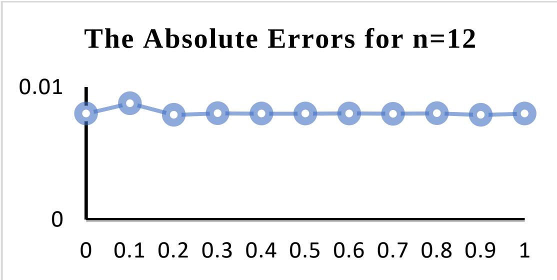

Fig. 4. The absolute Error $R_{12}^{B}(x_i)$

Table 2
The Absolute Errors $R_5^{Ch}(x_i)$ and $R_{12}^{B}(x_i)$

x_i	$R_5^{Ch}(x_i)$	$R_{12}^{B}(x_i)$
0	0.02668	0.00799
0.1	0.00670	0.32498
0.2	0.02668	0.45511
0.3	0.03105	0.55574
0.4	0.02668	0.64044
0.5	0.02472	0.71510
0.6	0.02668	0.78259
0.7	0.02870	0.84464
0.8	0.02668	0.90245
0.9	0.02192	0.95658
1.0	0.02668	1.00800

Example 3. For the following integral equation,

$$u(x)=e^{x}-4.0602+\int_0^1 \frac{u(t)}{\sqrt{1-t}}dt;\ 0\le x\le 1, \tag{27}$$

where the exact solution takes the form $u_{ex}(x)=e^{x}$ [14]. Using the presented technique using Chebyshev Polynomials for $n=3$, we obtain $\tilde{u}_3^{Ch}(x)$. Considering the nodes $x_i=0{:}0.1{:}1.0$ and substituting into $u_{ex}(x)$ and $\tilde{u}_3^{Ch}(x)$ and by using the second Barycentric Lagrange Interpolation technique for $n=8$, we obtain $\tilde{u}_8^B(x)$, Table 1, include the absolute errors $R_3^{Ch}(x_i)=\left|u_{ex}(x_i)-\tilde{u}_3^{Ch}(x_i)\right|$ and $R_8^B(x_i)=\left|u_{ex}(x_i)-\tilde{u}_8^B(x_i)\right|$. In Figure 5, plotted is the graph of the interpolant error $R_3^{Ch}(x_i)$. In Figure 6, plotted is the graph of the interpolant error $R_8^B(x_i)$.

Table 3
The Absolute Errors $R_3^{Ch}(x_i)$ and $R_8^B(x_i)$

x_i	$R_3^{Ch}(x_i)$	$R_8^B(x_i)$
0	0.02668	0.00799
0.1	0.00670	0.32498
0.2	0.02668	0.45511
0.3	0.03105	0.55574
0.4	0.02668	0.64044
0.5	0.02472	0.71510
0.6	0.02668	0.78259
0.7	0.02870	0.84464
0.8	0.02668	0.90245
0.9	0.02192	0.95658
1.0	0.02668	1.00800

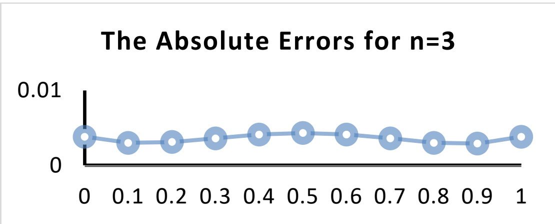

Fig. 5. The absolute Error $R_3^{Ch}(x_i)$

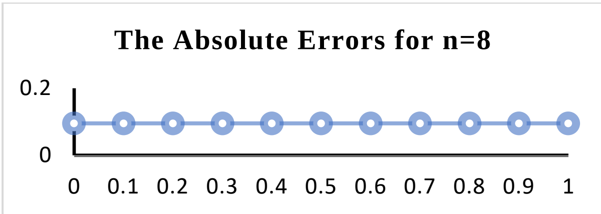

Fig. 6. The absolute Error $R_8^B(x_i)$

4. Conclusions

We applied the shifted second kind Chebyshev polynomials on the data, unknown and the kernel functions which isolates the singularity completely and converts the integral equation to easy algebraic system in the form of matrices, which makes our method more powerful than the other methods. As clarified in the numerical analysis, our method is strongly and ferociously converging as the other methods we mentioned.

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