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Axisymmetric Cylindrical Green's Function for the Steady-State Advection-Diffusion Operator with Uniform Velocity

Hani Kaldass[1,*], Raghda Nouralla[2], Nehad A. Zidan[2]

[1] Basic and Applied Sciences Department, Faculty of Engineering and Technology, El Nozha, Cairo Governorate 4471344, Egypt
[2] Department of Engineering Physics and Mathematics, Faculty of Engineering at El-Mataria, Helwan University, Cairo, Helwan, Cairo Governorate 4034572, Egypt

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ABSTRACT

The advection-diffusion equation has a wide range of applications as it governs energy and mass transport in a moving medium. Analytical solutions of the advection-diffusion equation for classes of boundary conditions amounts to finding exact expressions of the Green's function of the advection-diffusion operator. In this work, the Green's function of the steady-state advection-diffusion operator is obtained for an axisymmetric cylindrical problem with uniform velocity along the axis of the cylinder. The solution is exact, as axial diffusion (or conduction) is not neglected. The method of solution utilizes a formalism relating the Green's function of the diffusion operator to that of the steady-state advection-diffusion operator with uniform velocity. The Green's function obtained is applied to a Dirichlet boundary value problem. The distribution, (either particle concentration, or temperature), is plotted for various values of Péclet numbers. This work demonstrates how to invert the steady-state advection-diffusion operator with uniform velocity in a non-trivial geometry, namely the cylinder.

1. Introduction

In this paper, the Green's function of the steady-state advection-diffusion, (known also as convection-diffusion), operator is obtained for an axisymmetric cylindrical problem with uniform velocity along the axis of the cylinder. The advection-diffusion equation is the field equation for the energy and concentration scalar fields in the presence of a velocity field. These scalar fields follow the diffusive transport principle of transfer occurring from higher to lower concentration, with the current density being proportional to the negative of the field gradient. The constitutive diffusion relations are Fick's law for particle concentration and Fourier's law for energy density. In the presence of a velocity field, the total flux is the sum of the diffusive flux and the advective flux associated with the velocity field.

* *Corresponding author.*
E-mail address: hani@aast.edu

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As the advection-diffusion equation governs mass and energy transport phenomena, it has a wide range of applications. Examples of these are the study of smoke propagation in the atmosphere [1], dispersal of seeds and pollen [2], mass transfer across boundaries in blood flow [3], electron temperature in plasma [4], dendritic formation in flowing melts [5], and the use of nano fluids in various heat exchanging systems [6]. In view of its importance, it has been a subject of mathematics research, see e.g., Bazant [7] and references within.

Finding Green's functions of operators amounts to finding exact solutions to classes of boundary value problems [8,9]. The Green's function of the diffusion (conduction) operator has been extensively studied by Cole *et al*., [10]. Analytical solutions to the advection-diffusion operator are in a majority of cases limited to one-dimensional problems [11,12]. In this work, the domain of solution is a three-dimensional volume, namely a cylinder.

The problem studied in this paper has its origin in the Graetz problem [13,14]. The Graetz problem is that of a fluid at a temperature T_0, entering a pipe with wall temperature T_w. The flow is laminar. The steady-state temperature is obtained in the case where conduction in the axial direction is neglected. This approximation has been widely adopted in thermal advection-diffusion studies [15]. In this work, Green's function for the steady-state advection-diffusion operator is obtained for an axisymmetric boundary-value problem (BVP), using the result by Mitchell *et al.*, [16]. In [16], physical arguments are given to relate Green's function of the steady-state advection-diffusion operator with uniform velocity to that of the time-dependent diffusion operator. This relation is applied to obtain Green's function for a uniform channel flow with mixed boundary conditions, namely Neumann-Dirichlet. In this work, the main result in [16] is reworked. The Green's function obtained is applied to a cylindrical axisymmetric Dirichlet boundary value problem. The assumption of axisymmetric is widely used in various engineering applications due to its geometric and theoretical simplicity [17].

The assumption that the flow is uniform implies that the results obtained are exact for an inviscid fluid, or a moving solid. Studying heat propagation in moving solids is of practical applications to grinding and welding processes (see [16] and references therein). A procedure for working with a fluid with sheer in a cylinder, not from a Green's function perspective, but for a particular Dirichlet boundary condition, – the same application considered in this work, see Section 3.2 below –, is given by Papoutsakis *et al*., [18].

2. Methodology

2.1 Preliminary: Constitutive Diffusion Relations

References for the background material in this section are [19,20]. Let ψ be particle concentration (number per unit volume), with unit L^{-3}. The current density, $\mathbf{J}$, with unit $L^{-2}T^{-1}$, is given by Fick's law, as

$$\mathbf{J} = -D\nabla\psi\,, \tag{1}$$

where D is the diffusion constant, with unit L^2T^{-1}. The diffusion equation arises from the conservation law,

$$\frac{\partial\psi}{\partial t} = -\nabla\cdot\mathbf{J} + \rho\,, \tag{2}$$

where ρ represents particle source (or sink) per unit volume per unit time, with unit $L^{-3}T^{-1}$. Hence,

$$\frac{\partial\psi}{\partial t} - \nabla\cdot(D\nabla\psi) = \rho\,. \tag{3}$$

In the presence of a velocity field, $\mathbf{v}$, the advective current,

$$\mathbf{J}_{\text{adv}} = \mathbf{v}\psi\,, \tag{4}$$

is added to the diffusive one, Eq. (1); and the diffusion equation Eq. (3) becomes,

$$\frac{\partial\psi}{\partial t} - \nabla\cdot(D\nabla\psi) + \nabla\cdot(\mathbf{v}\psi) = \rho\,, \tag{5}$$

which is the advection-diffusion equation.

In the case of heat conduction, ψ is the temperature; and the current density, $\mathbf{J}$, with unit $EL^{-2}T^{-1}$, is given by Fourier's law,

$$\mathbf{J} = -k\nabla\psi\,, \tag{6}$$

with k being the thermal conductivity. The diffusion equation arises from the conservation law,

$$\frac{\partial(\rho_m c\psi)}{\partial t} = -\nabla\cdot\mathbf{J} + \rho_e = \nabla\cdot(k\nabla\psi) + \rho_e\,, \tag{7}$$

where ρ_m is the mass density, and c is the specific heat. The unit of $(\rho_m c\psi)$ is EL^{-3}, of the power density, ρ_e, is $EL^{-3}T^{-1}$, and of $(k\psi)$ is $EL^{-1}T^{-1}$. In the presence of a velocity field, the convective current,

$$\mathbf{J}_{\text{conv}} = \mathbf{v}\rho_m c\psi\,, \tag{8}$$

is added to the conduction one, Eq. (6); and the conduction equation Eq. (7), becomes,

$$\frac{\partial(\rho_m c\psi)}{\partial t} - \nabla\cdot(k\nabla\psi) + \nabla\cdot(\mathbf{v}\rho_m c\psi) = \rho_e\,, \tag{9}$$

which is the convection-diffusion equation.

Taking all physical properties as constant, the advection-diffusion equation, Eq. (5), reads,

$$\frac{\partial\psi}{\partial t} - D\nabla^2\psi + \nabla\cdot(\mathbf{v}\psi) = \rho \tag{10}$$

or, in terms of the diffusion operator, $L := \frac{\partial}{\partial t} - D\nabla^2$,

$$L\psi + \nabla\cdot(\mathbf{v}\psi) = \rho \tag{11}$$

The above equation applies also with ψ being the temperature. In this case, $D := \frac{k}{\rho_m c}$.

2.2 Diffusion Equation in a Volume V

A reference for the background material in this section is Barton [20].

The following boundary value problem with the diffusion operator, Eq. (11), in a volume V bounded by a surface S, is considered,

$$\begin{aligned}&L\psi(x,t)=\rho(x,t)\,,\quad t>t_0\,,\\&\psi(x,t_0)=\rho'(x)\,,\\&x\in S:\ \psi(x,t)=d(x,t)\,,\text{ or }\partial_n\psi(x,t)=n(x,t)\,,\quad t\geq t_0\,.\end{aligned}\tag{12}$$

In Eq. (12), x is the position vector. The Green's function method provides the means of solving Eq. (12). The Green's function associated with Eq. (12), i.e., the Green's function with the same boundary conditions as ψ albeit homogeneous, is defined by,

$$\begin{aligned}&LG(x,t;x',t')=\delta(x-x')\delta(t-t')\\&G(x,t;x',t')=0\,,\quad \text{for } t<t'\,,\\&G(x,t;x',t')=0\,,\text{ or }\partial_nG(x,t;x',t')=0\,,\quad x\in S\end{aligned}\tag{13}$$

Once Eq. (13) is solved, the solution to Eq. (12) is expressed as,

$$\begin{aligned}&\psi(x,t)=\int_{t_0}^{t}d\,t'\int_V d\,V'G(x,t;x',t')\rho(x',t')\\&+D\int_{t_0}^{t}d\,t'\int_S d\,S'[G(x,t;x',t')\,\partial_n'\psi_S(x',t')-\partial_n'G(x,t;x',t')\psi_S(x',t')]\\&+\int_V d\,V'G(x,t;x',t_0)\psi(x',t_0)\end{aligned}\tag{14}$$

To solve Eq. (13), an auxiliary function, $K(x,t;x',t')$, called the propagator, is introduced through the definition,

$$\begin{aligned}&LK(x,t;x',t')=0,\\&K|_{t=t'}=\delta(x-x')\\&K(x,t;x',t')=0\,,\text{ or }\partial_nK(x,t;x',t')=0\,,\quad x\in S\end{aligned}\tag{15}$$

The Green's function, G, and the propagator, K, are related by,

$$G(x,t;x',t')=\theta(t-t')K(x,t;x',t')\,.\tag{16}$$

The solution to Eq. (15) is given by

$$K(x,t;x',t')=\sum_p\phi_p^\star(x')\phi_p(x)\exp\left(-\lambda_pD(t-t')\right),\tag{17}$$

where λ_p and ϕ_p are the eigenvalues and normalized eigenfunctions of the $-\nabla^2$ operator under the given homogeneous Dirichlet or Neumann boundary conditions:

$$\begin{aligned}&\nabla^2\phi_p(x)=-\lambda_p\phi_p(x)\,,\\&\phi_p(x)=0\,,\text{ or }\partial_n\phi_p(x)=0\,,\quad x\in S\,.\end{aligned}$$

2.3 Green's Function of the Steady-State Advection-Diffusion Operator

The steady-state advection-diffusion equation Eq. (10) is considered with a steady velocity in the z-direction, $\mathbf{v} = v\hat{\mathbf{z}}$,

$$-D\nabla^2\psi + v\,\partial_z\psi = \rho \tag{18}$$

The Green's function associated with Eq. (18) in a volume V with boundary S is defined as,

$$\begin{aligned} &-D\nabla^2 G_v(\mathbf{r};\boldsymbol{\zeta}) + v\,\partial_z G_v(\mathbf{r};\boldsymbol{\zeta}) = \delta(\mathbf{r}-\boldsymbol{\zeta})\,, \\ &G_v(\mathbf{r};\boldsymbol{\zeta}) = 0\,, \text{ or } \partial_n G_v(\mathbf{r};\boldsymbol{\zeta}) = 0\,, \quad \mathbf{r}\in S \end{aligned} \tag{19}$$

In Eq. (19), $\mathbf{r}$ and $\boldsymbol{\zeta}$ are position vectors, field and source points respectively. In [16], in the case of constant speed v, a physical argument is given to express the solution of Eq. (19) in terms of that of Eq. (13). In the following, the result obtained in [16], namely Eq. (21) below, is derived analytically.

2.3.1 Proposition

Consider a constant velocity vector in the z-direction, $\mathbf{v} = v\hat{\mathbf{z}}$. Then, the solution of Eq. (19) is related to the solution of,

$$\begin{aligned} &-D\nabla^2 G(\mathbf{r},t;\boldsymbol{\zeta},\tau) + \partial_t G(\mathbf{r},t;\boldsymbol{\zeta},\tau) = \delta(\mathbf{r}-\boldsymbol{\zeta})\delta(t-\tau)\,, \\ &G|_{t<\tau} = 0\,, \\ &G(\mathbf{r},t;\boldsymbol{\zeta},\tau) = 0\,, \text{ or } \partial_n G(\mathbf{r},t;\boldsymbol{\zeta},\tau) = 0\,, \quad \mathbf{r}\in S\,, \end{aligned} \tag{20}$$

by

$$G_v(\mathbf{r},\boldsymbol{\zeta}) = \int_{-\infty}^{0} d\,\tau\, G(\mathbf{r},0;\boldsymbol{\zeta}-\mathbf{v}\tau,\tau)\,. \tag{21}$$

Proof.
Let $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, and define a function of $(\mathbf{r}',t;\boldsymbol{\zeta})$ by, $\tilde{G}(\mathbf{r}',t;\boldsymbol{\zeta}) := G_v(\mathbf{r}(\mathbf{r}');\boldsymbol{\zeta})$.
It is noted that, $\begin{aligned} &\nabla'^2\tilde{G} = \nabla^2 G_v\,, \\ &\partial_t\tilde{G} = v\,\partial_z G_v\,. \end{aligned}$

Then,

$$\begin{aligned} &-D\nabla'^2\tilde{G}(\mathbf{r}',t;\boldsymbol{\zeta}) + \partial_t\tilde{G}(\mathbf{r}',t;\boldsymbol{\zeta}) \\ &= -D\nabla^2 G_v(\mathbf{r};\boldsymbol{\zeta}) + v\,\partial_z G_v(\mathbf{r};\boldsymbol{\zeta}) = \delta(\mathbf{r}-\boldsymbol{\zeta}) = \delta(\mathbf{r}'+\mathbf{v}t-\boldsymbol{\zeta})\,. \end{aligned} \tag{22}$$

Consider the equation,
$$\begin{aligned} &-D\nabla^2 G(\mathbf{r},t;\boldsymbol{\zeta}-\mathbf{v}\tau,\tau) + \partial_t G(\mathbf{r},t;\boldsymbol{\zeta}-\mathbf{v}\tau,\tau) = \delta\big(\mathbf{r}-(\boldsymbol{\zeta}-\mathbf{v}\tau)\big)\delta(t-\tau)\,, \\ &G|_{t<\tau} = 0\,. \end{aligned}$$

Upon integration, $\begin{aligned} &-D\nabla^2\int_{-\infty}^{\infty} d\,\tau\, G(\mathbf{r},t;\boldsymbol{\zeta}-\mathbf{v}\tau,\tau) + \partial_t\int_{-\infty}^{\infty} d\,\tau\, G(\mathbf{r},t;\boldsymbol{\zeta}-\mathbf{v}\tau,\tau) \\ &= \int_{-\infty}^{\infty}\delta\big(\mathbf{r}-(\boldsymbol{\zeta}-\mathbf{v}\tau)\big)\delta(t-\tau)d\tau\,, \end{aligned}$

and taking account of the causality condition Eq. (20), $-D\nabla^2 \int_{-\infty}^{t} d\,\tau\, G(\mathbf{r},t;\boldsymbol{\zeta}-\mathbf{v}\tau,\tau) + \partial_t \int_{-\infty}^{t} d\,\tau\, G(\mathbf{r},t;\boldsymbol{\zeta}-\mathbf{v}\tau,\tau)$
$= \delta\big(\mathbf{r}-(\boldsymbol{\zeta}-\mathbf{v}t)\big)$.

The above equation is the same as Eq. (22). Hence, $\tilde{G}(\mathbf{r},t;\boldsymbol{\zeta}) = \int_{-\infty}^{t} d\,\tau\, G(\mathbf{r},t;\boldsymbol{\zeta}-\mathbf{v}\tau,\tau)$.

But, $G_v(\mathbf{r};\boldsymbol{\zeta}) = G(\mathbf{r},0;\boldsymbol{\zeta})$. Thus, $G_v(\mathbf{r},\boldsymbol{\zeta}) = \int_{-\infty}^{0} d\,\tau\, G(\mathbf{r},0;\boldsymbol{\zeta}-\mathbf{v}\tau,\tau)$.

2.4 Axisymmetric Cylindrical BVP
2.4.1 Green's function for an axisymmetric cylindrical problem

A cylinder of unit radius is considered. This means that the unit of length is taken as the radius of the cylinder. In the case of an axisymmetric problem, the Green's function is a function of the axial coordinate, z, and radial coordinate, r, and not of the azimuthal angle, $G = G(r,z,t;r',z',t')$. The 3-delta function in cylindrical coordinates is $\delta(x-x') = \frac{1}{r}\delta(r-r')\delta(\theta-\theta')\delta(z-z')$.

Using the form of the Laplacian in cylindrical coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}, \tag{23}$$

the BVP for the axisymmetric Green's function Eq. (13) reads

$$\begin{aligned} &\partial_t G - D\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\partial^2 G}{\partial z^2}\right\} = \frac{1}{2\pi r}\delta(r-r')\delta(z-z')\delta(t-t'), \\ &G|_{t<t'} = 0, \\ &G(r=1,z,t;r',z',t') = 0, \text{ or } \partial_r G(r,z,t;r',z',t')|_{r=1} = 0 \\ &G(r,\pm\infty,t;r',z',t') = 0. \end{aligned} \tag{24}$$

The solution to Eq. (24) is derived from a propagator function Eq. (16), $G(r,z,t;r',z',t') = \theta(t-t')K(r,z,t;r',z',t')$,

where:

$$\begin{aligned} &\partial_t K - D\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial K}{\partial r}\right) + \frac{\partial^2 K}{\partial z^2}\right\} = 0 \\ &K|_{t=t'} = \frac{1}{2\pi r}\delta(r-r')\delta(z-z') \\ &K(r=1,z,t;r',z',t') = 0, \text{ or } \partial_r K(r,z,t;r',z',t')|_{r=1} = 0, \\ &K(r,\pm\infty,t;r',z',t') = 0. \end{aligned} \tag{25}$$

The propagator Eq. (25) is separated into a radial and an axial function, $K(r,z,t;r',z',t') = K_{\mathrm{rad}}(r,t;r',t')K_{\mathrm{ax}}(z,t;z',t')$, satisfying,

$$\partial_t K_{\text{rad}} - \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial K_{\text{rad}}}{\partial r}\right) = 0\,,$$
$$K_{\text{rad}}|_{t=t\prime} = \frac{1}{2\pi r}\delta(r-r')\,, \tag{26}$$
$$K_{\text{rad}}(r=1,t;r',t') = 0\,, \text{ or } \partial_r K_{\text{rad}}(r,t;r',t')|_{r=1} = 0\,,$$

and

$$\partial_t K_{\text{ax}} - D\frac{\partial^2}{\partial z^2}K_{\text{ax}} = 0\,,$$
$$K_{\text{ax}}|_{t=t\prime} = \delta(z-z'), \tag{27}$$
$$K_{\text{ax}}(\pm\infty,t;z',t') = 0\,.$$

The solution of the axial propagator Eq. (25) is given by Barton [20],

$$K_{\text{ax}}(z,t;z',t') = \frac{1}{\sqrt{4\pi D(t-t\prime)}}\exp\left(-\frac{(z-z\prime)^2}{4D(t-t\prime)}\right). \tag{28}$$

The solution of the radial propagator Eq. (26) is given by Eq. (17),

$$K_{\text{rad}}(r,t;r',t') = \frac{1}{2\pi}\sum_n e_n^\star(r')e_n(r)\exp\left(-\lambda_n^2 D(t-t')\right). \tag{29}$$

In the above equation, the factor $1/(2\pi)$ is carried over from the equal time relation in Eq. (26); and the functions, e_n, are normalized eigenfunctions with eigenvalues $-\lambda_n^2$, of the radial part of the Laplacian operator Eq. (23) subject to the relevant Dirichlet or Neumann homogeneous radial boundary conditions:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{de_n}{dr}\right) = -\lambda_n^2 e_n\,. \tag{30}$$

With Eq. (28) and Eq. (29), the axisymmetric Green's function Eq. (24) is given by

$$G(r,z,t;r',z',t') = \frac{\theta(t-t\prime)}{\sqrt{16\pi^3 D(t-t\prime)}}\exp\left(-\frac{(z-z\prime)^2}{4D(t-t\prime)}\right)\times\sum_{n=1}^{\infty} e_n(r')e_n(r)\exp\left(-\lambda_n^2 D(t-t')\right) \tag{31}$$

2.4.2 Moving medium

As an application of the Proposition in section 2.3.1, the axisymmetric cylindrical problem in Section 2.4.1 is considered. With Eq. (31), the integrand in Eq. (21) is, $G(r,z,0;r',z'-vt',t') = \theta(-t')\sum_{n=1}^{\infty} e_n(r')e_n(r)\times\frac{1}{\sqrt{16\pi^3 D(-t\prime)}}\exp\left(\lambda_n^2 Dt' + \frac{\left(z-(z\prime-vt\prime)\right)^2}{4Dt\prime}\right)$

Hence Eq. (21) assumes the form,

$$G_v(r,z;r',z') = \frac{1}{\sqrt{16\pi^3 D}}\sum_{n=1}^{\infty} e_n(r')e_n(r)\times\int_{-\infty}^{0}\frac{1}{\sqrt{-t\prime}}\exp\left(\lambda_n^2 Dt' + \frac{(z-z\prime+vt\prime)^2}{4Dt\prime}\right)dt'\,. \tag{32}$$

The above expression evaluates to

$$G_v(r,z;r',z') = \frac{1}{4\pi\sqrt{D}} \sum_{n=1}^{\infty} e_n(r') e_n(r) \exp\left(\frac{v(z-z\prime)}{2D}\right) \times \frac{1}{\sqrt{\lambda_n^2 D + \frac{v^2}{4D}}} \exp\left(-|z-z'|\sqrt{\lambda_n^2 + \frac{v^2}{4D^2}}\right). \quad (33)$$

With the definition,

$$\gamma_{v,n} := \sqrt{4\lambda_n^2 + \frac{v^2}{D^2}}, \quad (34)$$

the Green's function Eq. (33) assumes the form,

$$G_v(r,z;r',z') = \frac{1}{2\pi D} \sum_{n=1}^{\infty} e_n(r') e_n(r) \exp\left(\frac{v(z-z\prime)}{2D}\right) \times \frac{1}{\gamma_{v,n}} \exp\left(-|z-z'|\frac{\gamma_{v,n}}{2}\right) \quad (35)$$

Eq. (35) is the exact expression, that we found, for the Green's function of the steady-state advection-diffusion operator, when applied to an axisymmetric cylindrical problem with uniform velocity along the axis, without neglection of axial diffusive or conductive current.

3. Results

3.1 Green's Function Solution Equation

In order to derive the Green's function solution equation in Proposition 3.1.1 below, the reciprocal problem to Eq. (19) is first obtained. Inspection of G_v Eq. (35) reveals that
$G_v(\mathbf{r};\mathbf{r}') = G_{-v}(\mathbf{r}';\mathbf{r})$.
The above relation is used to rewrite the BVP Eq. (19) as

$$\begin{aligned} &-D\nabla^2 G_{-v}(\mathbf{r}';\mathbf{r}) + v\,\partial_z G_{-v}(\mathbf{r}';\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}'), \\ &G_{-v}(\mathbf{r}';\mathbf{r}) = 0 \text{,or } \partial_n G_{-v}(\mathbf{r}';\mathbf{r}) = 0, \quad \mathbf{r} \in S \end{aligned} \quad (36)$$

Performing the change of variables, $\mathbf{r} \leftrightarrow \mathbf{r}'$, $v \leftrightarrow -v$, in Eq. (36), yields the problem reciprocal to Eq.(19),

$$\begin{aligned} &-D\nabla'^2 G_v(\mathbf{r};\mathbf{r}') - v\,\partial_{z\prime} G_v(\mathbf{r};\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \\ &G_v(\mathbf{r};\mathbf{r}') = 0 \text{,or } \partial_{n\prime} G_v(\mathbf{r};\mathbf{r}') = 0, \quad \mathbf{r}' \in S \end{aligned} \quad (37)$$

3.1.1 Proposition

The solution of the PDE

$$-D\nabla^2\psi(r,z) + v\,\partial_z\psi(r,z) = \rho(r,z), \quad (38)$$

subject to boundary conditions on the cylinder surface is given in terms of the Green's function G_v as

$$\begin{aligned} \psi(\mathbf{r}) &= \int_V d\,V' G_v(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}') \\ &+ D\int_S d\,S'[G_v(\mathbf{r};\mathbf{r}')\,\partial_{n\prime}\psi(\mathbf{r}') - \psi(\mathbf{r}')\,\partial_{n\prime} G_v(\mathbf{r};\mathbf{r}')]. \end{aligned} \quad (39)$$

Proof. The reciprocal equation Eq. (37) is multiplied by $\psi(\mathbf{r}')$,

$$-D\psi(\mathbf{r}')\nabla'^2 G_v(\mathbf{r};\mathbf{r}') - v\psi(\mathbf{r}')\,\partial_{z'}G_v(\mathbf{r};\mathbf{r}') = \psi(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\,, \tag{40}$$

and the equation of ψ Eq. (38), – with $\mathbf{r}$ replaced by $\mathbf{r}'$ –, is multiplied by $G_v(\mathbf{r};\mathbf{r}')$,

$$-DG_v(\mathbf{r};\mathbf{r}')\nabla'^2\psi(\mathbf{r}') + vG_v(\mathbf{r};\mathbf{r}')\,\partial_{z'}\psi(\mathbf{r}') = G_v(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}')\,. \tag{41}$$

Subtracting Eq. (40) from Eq. (41), and integrating over the cylinder volume, yields,

$$\int_V dV'\{-D[G_v(\mathbf{r};\mathbf{r}')\nabla'^2\psi(\mathbf{r}') - \psi(\mathbf{r}')\nabla'^2G_v(\mathbf{r};\mathbf{r}')] + v[G_v(\mathbf{r};\mathbf{r}')\,\partial_{z'}\psi(\mathbf{r}') + \psi(\mathbf{r}')\,\partial_{z'}G_v(\mathbf{r};\mathbf{r}')]\}$$
$$= \int_V dV' G_v(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}') - \int_V dV'\psi(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')$$

Hence,

$$\begin{aligned}&\textstyle\int_V dV'\{-D[G_v(\mathbf{r};\mathbf{r}')\nabla'^2\psi(\mathbf{r}') - \psi(\mathbf{r}')\nabla'^2G_v(\mathbf{r};\mathbf{r}')]\\ &+v\,\partial_{z'}[G_v(\mathbf{r};\mathbf{r}')\psi(\mathbf{r}')]\} = \textstyle\int_V dV' G_v(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}') - \psi(\mathbf{r})\end{aligned} \tag{42}$$

In the above equation, the integration over the $\partial_{z'}$ term is null:

$$\int_V \partial_{z'}\,[G_v(\mathbf{r};\mathbf{r}')\psi(\mathbf{r}')]2\pi r'dr'dz' = \int\left\{\int_{-\infty}^{\infty}\partial_{z'}\,[G_v(r,z;r',z')\psi(r',z')]dz'\right\}2\pi r'dr'\,.$$

Since $G_v(r,z;r',\pm\infty) = 0$, as can be seen in Eq. (35), and under the assumption that $\psi(r',\pm\infty)$ is finite, the above integration vanishes. Equation Eq. (42), in conjunction with Green's theorem[1], yields the solution Eq. (39) to Eq. (38). Green's theorem states that, for two scalar fields, ψ and ϕ, defined over a volume V, bounded by a closed surface S, the following identity holds, $\int_V(\psi\nabla^2\phi - \phi\nabla^2\psi)\,dV = \int_S(\psi\nabla\phi - \phi\nabla\psi)\cdot d\mathbf{S}$.

3.2 Application: Dirichlet BC

The BVP in a cylinder with radius R, with Dirichlet boundary conditions,

$$\begin{aligned}&-D\nabla^2\psi(r,z) + v\,\partial_z\psi(r,z) = 0,\\ &\psi(R,z) = \psi_-\theta(-z) + \psi_+\theta(z)\,,\end{aligned} \tag{43}$$

is considered. These are the same wall boundary conditions considered in [18]. Under the transformation, $\psi \rightarrow \frac{\psi-\psi_+}{\psi_--\psi_+}$,

Eq. (43) assumes the form,

$$\begin{aligned}&-D\nabla^2\psi(r,z) + v\,\partial_z\psi(r,z) = 0,\\ &\psi(R,z) = \theta(-z),\end{aligned} \tag{44}$$

where now ψ is dimensionless. In terms of the dimensionless quantity, known as the Péclet number,

$$\mathrm{Pe} := \frac{Rv}{D}, \tag{45}$$

the dimensionless form of Eq. (44) is

$$\begin{aligned} &-\nabla^2\psi(r,z) + \mathrm{Pe}\,\partial_z\psi(r,z) = 0, \\ &\psi(1,z) = \theta(-z) \end{aligned} \tag{46}$$

The Péclet number is a measure of the relative importance of advection and diffusion to the transport of the conserved scalars, where a large number indicates an advectively dominated distribution, and a small number indicates a diffuse flow [21]. The solution to Eq. (44) is given by Eq. (39) as, $\psi(\mathbf{r}) = -D\int_S \psi(\mathbf{r}')\frac{\partial G_v}{\partial r'}(\mathbf{r},\mathbf{r}')dS'$.

The boundary conditions for Eq. (30) are, $e_n(1) = 0$, with regularity at $r = 0$. In this case, the orthonormal basis is given in terms of Bessel functions of the first kind [22,23] as,

$$\{e_n\} = \left\{\frac{\sqrt{2}J_0(\lambda_n r)}{J_1(\lambda_n)}\right\}_{n=1}^{\infty}, \text{ where } J_0(\lambda_n) = 0\,. \tag{47}$$

Using Eq. (35) with the basis Eq. (47), the solution to Eq. (44) is then given by

$$\begin{aligned} \psi(r,z) &= -\sum_{n=1}^{\infty}\frac{e_n{}'(1)e_n(r)}{\gamma_{v,n}}\int_{-\infty}^{0}\exp\left(\frac{v(z-z')}{2D}\right)\exp\left(-|z-z'|\frac{\gamma_{v,n}}{2}\right)dz' \\ &= \theta(-z) + \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{J_0(\lambda_n r)}{J_1(\lambda_n)}\left(\frac{v}{\gamma_{v,n}D} - \mathrm{sgn}(-z)\right)\exp\left(\frac{v}{2D}z - \frac{\gamma_{v,n}}{2}|z|\right) \end{aligned} \tag{48}$$

In the above expressions the definitions $\theta(0) = 1$ and $\mathrm{sgn}(0) = 1$ are adopted. In terms of Péclet number, the distribution Eq. (48) is

$$\psi(r,z) = \theta(-z) + \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{J_0(\lambda_n r)}{J_1(\lambda_n)}\left(\frac{\mathrm{Pe}}{\gamma_{\mathrm{Pe},n}} - \mathrm{sgn}(-z)\right)\exp\left(\frac{\mathrm{Pe}}{2}z - \frac{\gamma_{\mathrm{Pe},n}}{2}|z|\right) \tag{49}$$

It is noted that, as $z \to \pm\infty$, $\psi(r,z) \to \theta(-z)$ for all r. This property of the distribution has been imposed as boundary condition in [18]. Eq. (49) is the steady-state distribution function, that we found, of the axisymmetric Dirichlet boundary value problem in a cylinder with a medium having uniform velocity.

The graphs below are plots of Eq. (49) for various Péclet numbers. The range of z in these plots satisfy, $\frac{1}{2}(\mathrm{Pe} + \gamma_{\mathrm{Pe},1})z_{\min} = -3$, $\frac{1}{2}(\mathrm{Pe} - \gamma_{\mathrm{Pe},1})z_{\max} = -3$

It is noted that $\mathrm{Pe} = 0$ corresponds to a static medium ($v = 0$), while a negative Péclet number refers to a flow in the negative z direction. The plots demonstrate the progression *of* $\psi(r,z)$ towards the limit $\psi(\pm\infty,z) = \theta(-z)$ in the case of no-flow as shown in Figure 3 and Figure 4, flow in the negative z direction as shown in Figure 1 and Figure 2, and flow in the positive z direction as shown in Figure 5 and Figure 6. Indeed, the range [z_{min}, z_{max}] for Pe = −9 is [−4.97, 0.31], for Pe = 0 is [−1.25, 1.25], while for Pe = 9 is [−0.31, 4.97]. With Pe = 0, at z = 0, the distribution is independent of r with an average value of 0.5. These numbers confirm the expected effect of advection/convection on the distribution ψ. For example, when the flow is in the negative z-direction, z_{max} = 0.31 compared to 1.25 in the no-flow case. This demonstrates the faster approach to ψ (r, ∞) = 0 due to advection/convection, i.e., the diminishing effect of the boundary values at z < 0 on z > 0. Also, z_{min} =

−4.97 compared to −1.25 in the no-flow case, i.e., advection/convection hampers the approach to ψ (r, −∞) = 1. This is in agreement with the expectation that the effect of z > 0 on z < 0 increases when the flow is in the negative z-direction.

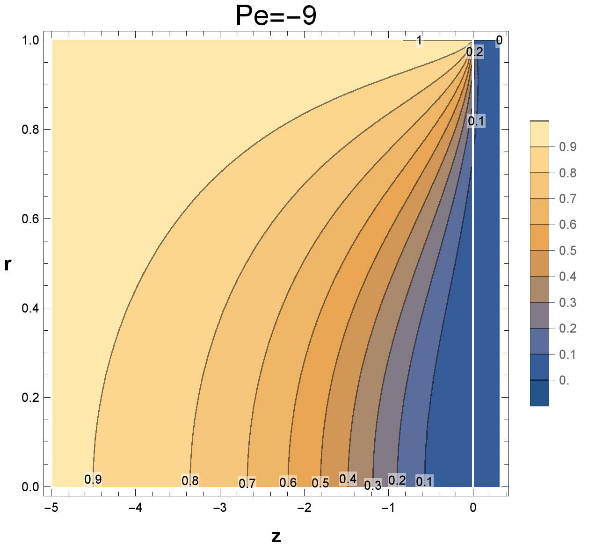

Fig. 1. Contour lines of $\psi(r, z)$ at Pe = -9

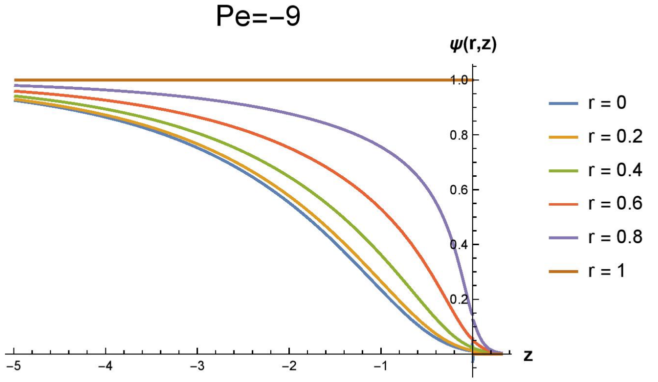

Fig. 2. Graph of $\psi(r, z)$ at Pe = -9

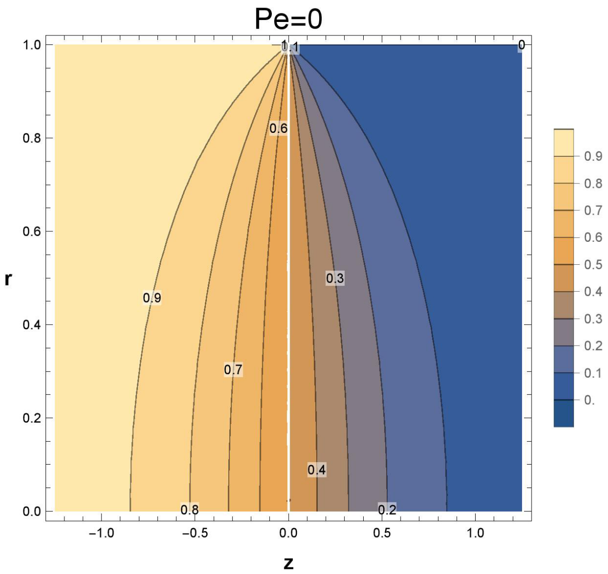

Fig. 3. Contour lines of $\psi(r, z)$ at Pe = 0

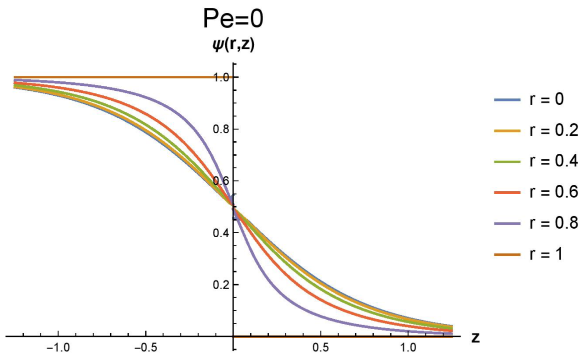

Fig. 4. Graph of $\psi(r, z)$ at Pe = 0

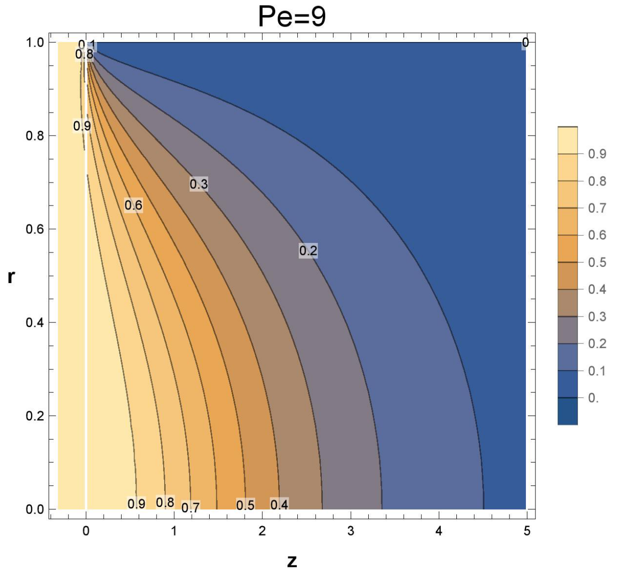

Fig. 5. Contour lines of $\psi(r, z)$ at Pe = 9

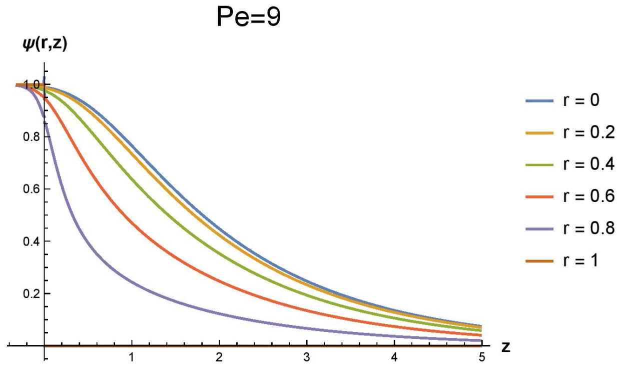

Fig. 6. Graph of $\psi(r, z)$ at Pe = 9

4. Conclusion

The formalism presented in [16] has been applied to obtain the axisymmetric cylindrical Green's function for the steady-state advection-diffusion operator with uniform velocity as shown in Eq. (35). As the diffusive or conductive component of the current in the axial direction is not neglected, the Green's function obtained is the exact inverse of the advection-diffusion operator. Exact solutions are of value as they can be used to verify numerical solutions, and they emphasize the dependence of the solution on the various physical parameters.

The Green's function obtained is applied to a cylinder with Dirichlet boundary conditions to obtain the distribution function $\psi(r, z)$ as shown in Eq. (49). This function represents temperature distribution in the case of energy or heat transfer, or concentration distribution in the case of mass transfer. We plotted this function for different Péclet numbers and this clarifies the effect of convection / advection on the distribution ψ.

The obtained results are applicable in the case of other boundary conditions, namely Neumann and Robin; and work is ongoing in this direction.

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