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Cubic Trigonometric Triangular Spline with Rational Basis Functions in 3-Dimensional Space

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ABSTRACT

Digitization of an object from visual to data point will help researchers manipulate the data for further use – particularly in reconstructing a surface. Currently, surface reconstruction focuses on non-rectangular topologies such as triangular surface because of its ability to generate smooth complex surface. The use of trigonometric function to generate a triangular surface are able to offer smooth surface with flexibility because it has the advantages of derivation and cyclability. There are studies that provides trigonometric basis functions for curves and surfaces with certain range of shape parameters. This study proposed four univariate trigonometric basis function and extending it to ten trigonometric basis functions over triangular domain with smaller set of shape parameter values. The basis is proven suitable for curve and surface design because it possesses blossoms which exist if and only if the space $DT_{\lambda,\mu}$ is an EC-space on $[0, \pi/2]$. In addition, the properties of the basis are shown. Curve and surfaces are illustrated in this study using the proposed basis functions.

Keywords:

Surface reconstruction; Trigonometric basis functions; Triangular surface; Curve and surface design

1. Introduction

Three-dimensional digitization is important in these recent years, as this area has potentials for further explorations and studies as mentioned by [1]. With the advancement of 3D scanner machine, researchers can study the data point for each scanned item. It helps with the digitization of an object from visual to data point that can be stored and manipulated for further use for example the data points can be improved using computer aided design (CAD) software as shown by [2] and printed again using 3D printer.

According to [3], the early development of a system that designs curves and surfaces was led by a young mathematician named Paul de Faget de Casteljau. He used Bernstein polynomials to define

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curves and surfaces at an early stage which is now known as the de Casteljau algorithm. He used control polygon to construct a curve, where the changes of the control polygon will affect its shape.

In mathematics and its applications, approximate representations of functions are commonly used as mentioned by [4]. This representation simplifies and show closeness to the practical needs. Constructing the interpolating polynomials need to consider having a good convergence of splines to approximated objects as proposed by [5]. There is various application of the splines for example in medical sector where works of [6] shows a blood image was being improved by eliminating interference and noise using bicubic spline models.

The construction of surfaces often uses rectangular domains, which always requires reparameterization step to extract the surface points. However, with the advancement of geometric modelling technology, many researchers begin to explore more complex methods which are non-rectangular topologies such as triangular surface patches as mentioned by [7]. This method produces a mesh where the vertices of the triangle are actually the points as shown by [8]. Study by [9] suggested that surfaces that use triangular patches are powerful with the flexibility for modelling complex objects, and they have wide applications in computer aided designs, and reverse engineering.

According to [10], there are many successful methods using triangular functions such as Bezier triangular, B-spline triangular, Timmer triangular, Ball triangular.

In the previous years, several researchers had established new bases over triangular domains using trigonometric functions with shape parameters. The new bases are constructed by extending four univariate basis functions to surface fitting functions. Some of the developed bases are Bezier, Bezier-like, B-spline-like, and Bernstein-like. A trigonometric function using spline curves has geometric characteristics that offer better results because the shape is adjustable, unlike the classical spline curves as mentioned by [11]. Furthermore, the basis function constructed using trigonometric function guarantees smooth curves as proposed by [12]. A study by [13] adopted the trigonometric function spline curve which proposed C^n continuous trigonometric polynomials spline curves to interpolate data coming from a generalized Overhauser spline.

With the ability of a triangular surface patch to generate a smooth complex surface of an object with flexibility as shown by [9], researchers have shown interest on constructing a new trigonometric basis function over the triangular domain for surface fitting. For example, a study by [14] proposed a new class of trigonometric Bezier-like basis function over a triangular domain with two shape parameters. After that, work by [15] extended the study by using three shape parameters and study by [17] had six parameters with stable and efficient computation of patch using A de Casteljau-type algorithm. A study from [18] proposed four new generalized quasi cubic trigonometric Bernstein basis functions with two shape functions which includes previous work as special cases.

Studies from [14-17] proposed four univariate trigonometric basis functions and then extends to ten trigonometric basis functions. The technique of extending the four basis to ten basis helps to maintain its properties. There are few studies made for trigonometric triangular spline surface despite the advantages that it possesses. However, researchers have shown interest on studying this topic.

This study proposed four univariate trigonometric basis function and extending it to ten trigonometric basis functions over triangular domain. The basis is proven suitable for curve and surface design. In addition, the properties of the basis are shown. Curve and surfaces are illustrated in this study using the proposed basis functions.

There are 6 sections in this research paper which are, introduction, related works, cubic trigonometric basis functions, cubic trigonometric basis functions over triangular domain, surface interpolation and conclusion.

2. Related Works on Trigonometric Basis Functions

The Bernstein-like bases using spaces of trigonometric polynomial, hyperbolic polynomial and blended space has been studied in computer aided geometric design (CAGD). However, a study in [19] started to extend the trigonometric polynomial basis which corresponds for the polynomial space $T = span\{1, \sin t, \cos t\}$ to the triangular domain. Another study by [20] extend the algebraic trigonometric Bezier-like basis of order 4 to the triangular domain defined over the blended space $T = span\{1, t, \sin t, \cos t\}$.

In the previous years, several researchers had established new bases over triangular domain using trigonometric functions with shape parameters. The new Bezier bases were constructed by extending four univariate basis functions to the ten bases over a triangular domain as mentioned by [14-17].

Table 1 below shows the trigonometric space span which were characterized by existence of Bernstein-like bases.

Table 1
Control points for surface fitting

Author	Space span
[14]	$\{1, \sin^2 t, (1-\sin t)^\alpha, (1-\cos t)^\beta\}$
[15]	$\{1, \sin^2 t, (1-\sin t)^2(1-\mu \sin t), (1-\cos t)^2(1-\lambda \cos t)\}$
[16]	$\left\{1, \sin^2 t, \frac{(1-\sin t)^2}{(1+(\alpha-2)\sin t)}, \frac{(1-\cos t)^2}{(1+(\beta-2)\cos t)}\right\}$
[17]	$\{1, \sin^2 t, (1-\sin t)^\alpha(1-\lambda \sin t), (1-\cos t)^\beta(1-\mu \cos t)\}$

All of these spaces lead to a study made by [18] which constructed a generalized quasi cubic trigonometric space span $\{1, \sin^2 t, (1-\sin t)^2\alpha(t), (1-\cos t)^2\beta(t)\}$ where $\alpha(t)$ and $\beta(t)$ are the two shape functions to guarantee the new construction of Bernstein basis function.

The construction of trigonometric basis functions was explained in details by [14]. Four new trigonometric Bernstein-like basis functions with two exponential shape parameters were constructed. They were then extended to a new class of trigonometric Bezier-like basis over triangular domain with three shape parameters.

After that, work by [15] extended the research made by [14] to establish a new class of trigonometric polynomial basis function over a triangular domain with three shape parameters. The ten bases of triangular domain can degenerate to four cubic Bezier basis function when one of the three variables is taken as zero.

Another study that constructed a new trigonometric basis over a triangular domain was conducted by [16]. This study constructed a new class of trigonometric Bernstein-Bezier-like basis function with three denominator shape parameters over a triangular domain.

Study by [17] extended his study by constructing trigonometric Bezier basis functions over triangular domain with six shape parameters, while work by [14,15] only had two and three shape parameters.

The trigonometric space span for the four bases is based on the Extended Chebyshev (EC) which were characterized by existence of Bernstein-like bases as mentioned by [21,22]. The difference between them is the number of shape parameters in the basis, and the interval.

In conclusion, some of the gaps have been identified and discussed in this study, whereby a large set of shape parameter values was introduced from other studies but they did not have significant change in the shape of each patch, as the value reaches its peak. Thus, next section shows the development of the basis with a smaller set of shape parameter values.

3. The Development of Cubic Trigonometric Bezier-like Basis Functions

This section discusses the development of cubic trigonometric basis function from a trigonometric space as shown below.

For any real numbers $\lambda, \mu \in [0,1]$, $t \in [0, \pi/2]$, the work starts with the trigonometric space

$$T_{\lambda,\mu} = span\left\{1, sin^2 t, \frac{(1-\sin t)^2(1-\lambda \sin t)}{1+\lambda \sin t}, \frac{(1-\cos t)^2(1-\mu \cos t)}{1+\mu \cos t}\right\} \quad (1)$$

To prove that the space $T_{\lambda,\mu}$ is suitable for curve and surface design, it must possess blossoms which exist if and only if the space $DT_{\lambda,\mu}$ is an EC-space on $[0, \pi/2]$.

$$DT_{\lambda,\mu} = span\left\{2 \sin t \cos t\,, -\frac{2 \cos t(\sin t-1)(\lambda^2 \sin^2 t+\lambda \sin t-\lambda-1)}{(1+\lambda \sin t)^2}, \frac{2 \sin t\,(\cos t-1)(\mu^2 \cos^2 t+\mu \cos t-\mu-1)}{(1+\mu \cos t)^2}\right\} \quad (2)$$

For any $C_i \in \mathbb{R}$, $t \in [0, \pi/2]$, the linear combination is as follows

$$C_0[2 \sin t \cos t] + C_1\left[-\frac{2 \cos t(\sin t-1)(\lambda^2 \sin^2 t+\lambda \sin t-\lambda-1)}{(1+\lambda \sin t)^2}\right] + C_2\left[\frac{2 \sin t(\cos t-1)(\mu^2 \cos^2 t+\mu \cos t-\mu-1)}{(1+\mu \cos t)^2}\right] = 0 \quad (3)$$

For $t = 0$,

$$C_1[-2\lambda - 2] = 0$$

$$C_1 = 0$$

and for $t = \pi/2$,

$$C_2[2\mu + 2] = 0$$

$$C_2 = 0$$

Thus, $C_0 = 0$. Therefore, the space $DT_{\lambda,\mu}$ is a 3-dimensional space.

Next is to determine that the $DT_{\lambda,\mu}$ is ECC-space in $[0, \pi/2]$. For $t \in [a, b] \subseteq (0, \pi/2)$, let

$$u(t) = \left[-\frac{(\sin t-1)(\lambda^2 \sin^2 t+\lambda \sin t-\lambda-1)}{\sin t\,(1+\lambda \sin t)^2}\right]'$$
$$= -\cos t\left[\frac{(\lambda+1)(\lambda^2 \sin^3 t+3\lambda \sin^2 t-3\lambda \sin t-1)}{\sin^2 t\,(1+\lambda \sin t)^3}\right] > 0 \quad (4)$$

and

$$v(t) = \left[\frac{(\cos t-1)(\mu^2 \cos^2 t+\mu \cos t-\mu-1)}{\cos t\,(1+\mu \cos t)^2}\right]'$$
$$= -\sin t\left[\frac{(\mu+1)(\mu^2 \cos^3 t+3\mu \cos^2 t-3\mu \cos t-1)}{\cos^2 t(1+\mu \cos t)^3}\right] > 0 \quad (5)$$

The differential of $u(t)$ and $v(t)$ are shown below.

$$u'(t) = \frac{\lambda+1}{(1+\lambda\sin t)^4}(-\csc^3 t - 8\lambda\csc^2 t + (-3\lambda^2 - \cot^2 t)\csc t + ((\lambda^2\cos^2 t + \lambda^2 + 3)\sin t + 4\lambda\cos^2 t - 9\lambda\cos t\cot t + 4\lambda + 4)\lambda) < 0 \quad (6)$$

and

$$v'(t) = \frac{\mu+1}{(1+\mu\cos t)^4}(\mu^3\cos^3 t + 4\mu^2\cos^2 t + (-2\mu^3 - 9\mu^2 - 3\mu)\cos t + 2\sec^3 t + 8\mu\sec^2 t + (12\mu^2 - 1)\sec t - 8\mu^2 - 4\mu) > 0 \quad (7)$$

Since $u(t)$, $v(t)$ and $v'(t) > 0$, and $u'(t) < 0$, the Wronskian of $u(t)$ and $v(t)$

$$W(u,v)(t) = u(t)v'(t) - u'(t)v(t) > 0 \text{ for } \forall t \in [0,\pi/2] \quad (8)$$

This indicates that $u(t)$ and $v(t)$ are linearly independent for $\forall t \in [0,\pi/2]$. Next, the weight functions are defined.

$$\begin{aligned} w_0(t) &= 2\sin t\cos t \\ w_1(t) &= Au(t) + Bv(t) \\ w_2(t) &= C\frac{W(u,v)(t)}{[Au(t)+Bv(t)]^2} \end{aligned} \quad (9)$$

where A, B and C $A \in \mathbb{R}$.

Consider the following function space (u_0, u_1, u_2) of 3-dimension ECC-space generated by the positive weight functions $w_i(t)(i = 0,1,2)$ on $[a,b]$

$$\begin{aligned} u_0(t) &= w_0(t) \\ u_1(t) &= w_0(t)\int_a^t w_1(t_1)\,dt_1 \\ u_2(t) &= w_0(t)\int_a^t w_1(t_1)\int_a^{t_1} w_2(t_2)\,dt_2 dt_1 \end{aligned} \quad (10)$$

With a simple computation, $u_0(t)$, $u_1(t)$ and $u_2(t)$ are some linear combinations of the three functions $2\sin t\cos t$, $-\frac{2\cos t(\sin t-1)(\lambda^2\sin^2 t+\lambda\sin t-\lambda-1)}{(1+\lambda\sin t)^2}$ and $\frac{2\sin t(\cos t-1)(\mu^2\cos^2 t+\mu\cos t-\mu-1)}{(1+\mu\cos t)^2}$. These indicate that the space $DT_{\lambda,\mu}$ is an ECC-space on $[a,b]$. Therefore, the new space $T_{\lambda,\mu}$ is suitable for curve design.

For any $\lambda,\mu \in [0,1]$, the four associated trigonometric basis functions of the space $T_{\lambda,\mu}$ are given by,

$$\begin{aligned} T_0(t;\lambda,\mu) &= \frac{(1-\sin t)^2(1-\lambda\sin t)}{1+\lambda\sin t} \\ T_1(t;\lambda,\mu) &= 1 - sin^2 t - \frac{(1-\sin t)^2(1-\lambda\sin t)}{1+\lambda\sin t} \\ T_2(t;\lambda,\mu) &= 1 - cos^2 t - \frac{(1-\cos t)^2(1-\mu\cos t)}{1+\mu\cos t} \\ T_3(t;\lambda,\mu) &= \frac{(1-\cos t)^2(1-\mu\cos t)}{1+\mu\cos t} \end{aligned} \quad (11)$$

The trigonometric basis function needs to have the following properties of partition of unity, nonnegativity and total positivity. The figure below shows the plots of the basis function in Eq. (11).

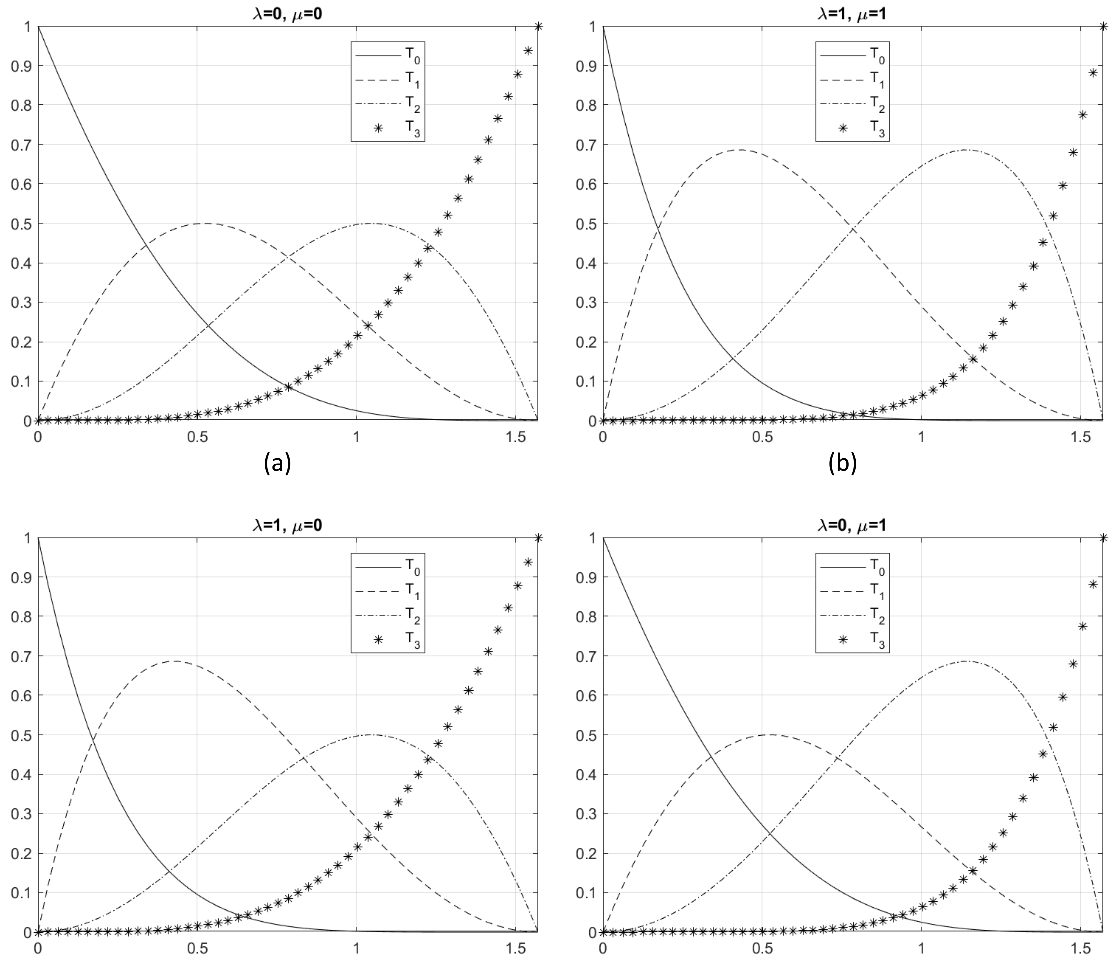


Fig. 1. The Plots of Cubic Trigonometric Basis Functions (a) $\lambda = 0, \mu = 0$ (b) $\lambda = 1, \mu = 1$ (c) $\lambda = 1, \mu = 0$ (d) $\lambda = 0, \mu = 1$

The graphical behaviours when changing the shape parameter from the proposed basis function as in Eq. (11) can be observed in the figure above. The curve is seen to be symmetrical when both shape parameters λ and μ have the same value. However, when both shape parameters λ and μ have a different value, the curve increases at the shape parameter that has a higher value than the other.

For any $t \in [0, \pi/2]$, we have

$$
\begin{aligned}
&T_2(t) + T_3(t) = \sin^2 t \\
&T_0(t) = \frac{(1-\sin t)^2(1-\lambda \sin t)}{1+\lambda \sin t} \\
&T_3(t) = \frac{(1-\cos t)^2(1-\mu \cos t)}{1+\mu \cos t}
\end{aligned}
\tag{12}
$$

Thus, the new basis has the following important end-point properties:

i. $T_0(t) = 1$ and $T_0(t)$ vanishes 3 times at $\pi/2$ (counting multiplicities as far as possible up to 3);
ii. $T_3(\pi/2) = 1$ and $T_3(t)$ vanishes 3 times at 0 (counting multiplicities as far as possible up to 3);
iii. For $i = 1, 2$, $T_i(t)$ vanishes exactly i times 0 and exactly $(3 - i)$ times at $\pi/2$.

The trigonometric basis function has the following properties:

i. Non-negativity.
For $t \in [0, \pi/2]$, $T_i(t) \geq 0$, $i = 0,1,2,3$
ii. Partition of unity.
$$\sum_{i=0}^{3} T_i(t) = 1$$
iii. Monotonicity.
For a given parameter t, $T_0(t)$ and $T_3(t)$ are monotonically decreasing and $T_1(t)$ and $T_2(t)$ are monotonically increasing.
iv. Symmetry.
$T_i(t; \lambda, \mu) = T_{3-i}(1 - t; \lambda, \mu)$ for $i = 0,1,2,3$.
Given control points P_i $(i = 0, 1, 2, 3)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. The trigonometric curve is

$$T(t; \lambda, \mu) = \sum_{i=0}^{3} T_i(t; \lambda, \mu)P_i, \quad t \in [0, \pi/2], \lambda, \mu \in [0,1] \tag{13}$$

where λ, μ are the shape parameters.

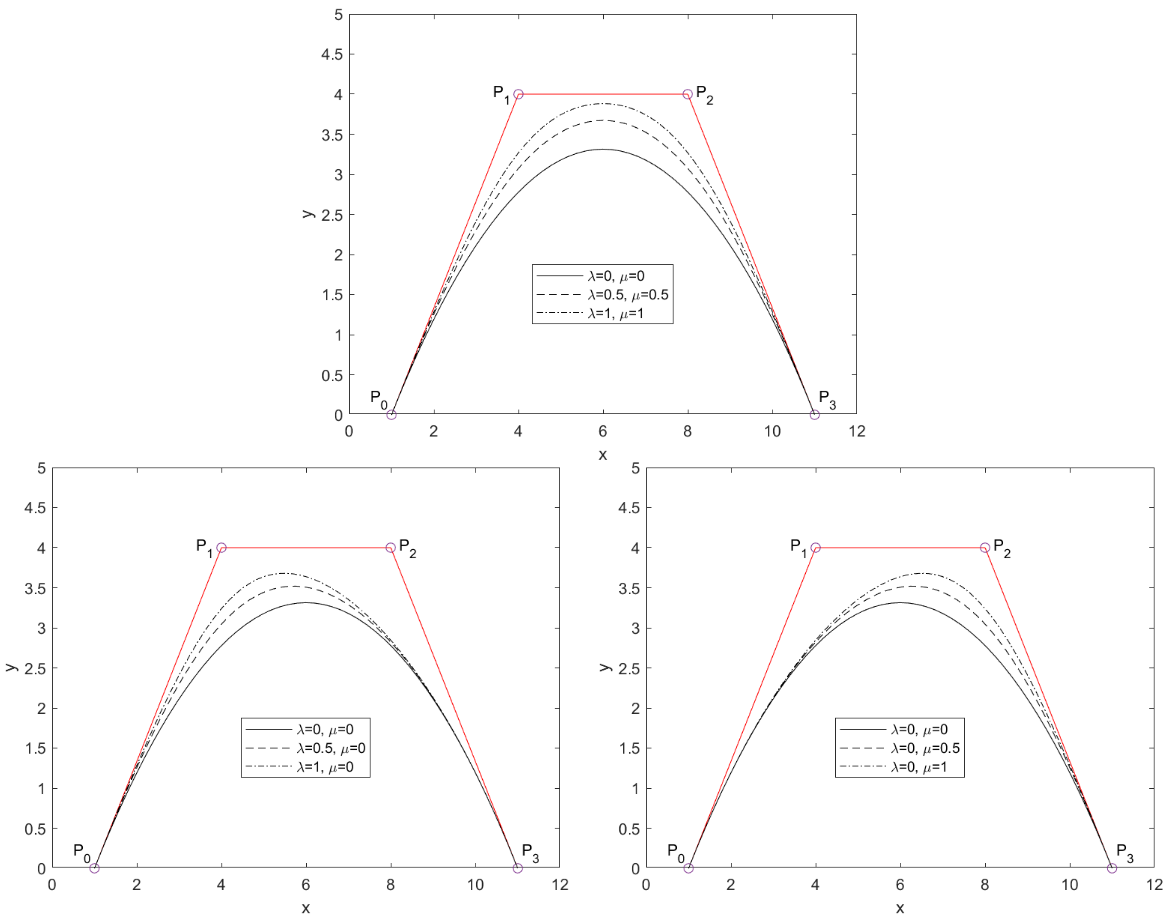


Fig. 2. The effects of shape parameters in Cubic Trigonometric curve

4. Cubic Trigonometric Basis over a Triangular Domain

Cubic trigonometric basis over a triangular domain is constructed with ten functions. Each trigonometric function is located on a particular location over the triangular domain as shown in the Figure 3 below.

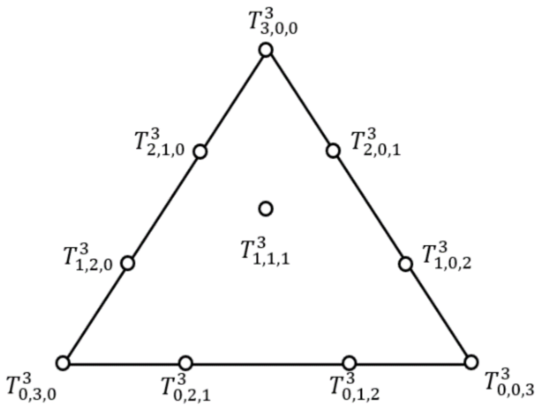


Fig. 3. Control Points over Triangular Domain

Let $\lambda, \mu, \gamma \in [0,1]$ for $D = \{(u, v, w) | u + v + w = \pi/2\,, u \geq 0, v \geq 0, w \geq 0\}$, the following ten functions are defined with three shape parameters of λ, μ and γ over the triangular domain D:

$$T^3_{3,0,0}(u,v,w;\lambda,\mu,\gamma) = \frac{(1-\cos u)^2(1-\lambda\cos u)}{1+\lambda\cos u}$$
$$T^3_{0,3,0}(u,v,w;\lambda,\mu,\gamma) = \frac{(1-\cos v)^2(1-\mu\cos v)}{1+\mu\cos v}$$
$$T^3_{0,0,3}(u,v,w;\lambda,\mu,\gamma) = \frac{(1-\cos w)^2(1-\gamma\cos w)}{1+\gamma\cos w}$$
$$T^3_{2,1,0}(u,v,w;\lambda,\mu,\gamma) = cos\, w\, sin\, v\,(1-cos\, u)\frac{2\,(1+\lambda)}{1+\lambda\, cos\, u}$$
$$T^3_{2,0,1}(u,v,w;\lambda,\mu,\gamma) = cos\, v\, sin\, w\,(1-cos\, u)\frac{2\,(1+\lambda)}{1+\lambda\, cos\, u}$$
$$T^3_{1,2,0}(u,v,w;\lambda,\mu,\gamma) = cos\, w\, sin\, u\,(1-cos\, v)\frac{2\,(1+\mu)}{1+\mu\, cos\, v}$$
$$T^3_{0,2,1}(u,v,w;\lambda,\mu,\gamma) = cos\, u\, sin\, w\,(1-cos\, v)\frac{2\,(1+\mu)}{1+\mu\, cos\, v}$$
$$T^3_{1,0,2}(u,v,w;\lambda,\mu,\gamma) = cos\, v\, sin\, u\,(1-cos\, w)\frac{2\,(1+\gamma)}{1+\gamma\, cos\, w}$$
$$T^3_{0,1,2}(u,v,w;\lambda,\mu,\gamma) = cos\, u\, sin\, v\,(1-cos\, w)\frac{2\,(1+\gamma)}{1+\gamma\, cos\, w}$$
$$\begin{aligned} T^3_{1,1,1}(u,v,w;\lambda,\mu,\gamma) &= 1-\sum_{\substack{i+j+k=3\\ ijk\neq 1}} T^3_{i,j,k}(u,v,w;\alpha,\beta,\gamma) \\ &= 2\sin u\sin v\sin w \end{aligned} \quad (14)$$

These ten multi-variable basis functions over triangular domain were constructed by extending the four univariate trigonometric basis functions as in Eq. (11). The extension is done in such a way that the ten multi-variable basis functions over triangular domain can degenerate to the four univariate trigonometric basis functions to form a partition of unity by substituting one of the three variables as zero.

The ten-basis function was constructed and validated by using the steps below:

i. Constructing $T^3_{3,0,0}(u,v,w;\lambda,\mu,\gamma)$, $T^3_{0,3,0}(u,v,w;\lambda,\mu,\gamma)$ and $T^3_{0,0,3}(u,v,w;\lambda,\mu,\gamma)$ functions are by extending $T_0(t)$ and $T_3(t)$.
If u or v or $w=0$ and $u+v+w=\pi/2$,

➔ One out of the three functions will become zero.
➔ The other two functions will degenerate to $T_0(t)$ and $T_3(t)$.

ii. $T^3_{2,1,0}(u,v,w;\lambda,\mu,\gamma)$ and $T^3_{2,0,1}(u,v,w;\lambda,\mu,\gamma)$ are constructed together. Using a similar way, the other two pairs of $T^3_{1,2,0}(u,v,w;\lambda,\mu,\gamma)$, $T^3_{0,2,1}(u,v,w;\lambda,\mu,\gamma)$ and $T^3_{1,0,2}(u,v,w;\lambda,\mu,\gamma)$, $T^3_{0,1,2}(u,v,w;\lambda,\mu,\gamma)$ are constructed.
If u or v or $w=0$ and $u+v+w=\pi/2$,
➔ Four out of the six functions will become zero.
➔ The other two functions will degenerate to $T_1(t)$ and $T_2(t)$.

The example is given on how to validate the degeneration from ten trigonometric basis functions as in Eq. (14) to four univariate trigonometric basis function as in Eq. (11).

Since,

$$u+v+w=\frac{\pi}{2} \tag{15}$$

If

$$w=0, u=\frac{\pi}{2}-v \tag{16}$$

Then,

$$T_{3,0,0}^{3}\left(\frac{\pi}{2}-v,v,0;\lambda,\mu,\gamma\right)=\frac{(1-\sin v)^2(1-\lambda\sin v)}{1+\lambda\sin v}$$
$$T_{0,3,0}^{3}\left(\frac{\pi}{2}-v,v,0;\lambda,\mu,\gamma\right)=\frac{(1-\cos v)^2(1-\mu\cos v)}{1+\mu\cos v}$$
$$T_{0,0,3}^{3}\left(\frac{\pi}{2}-v,v,0;\lambda,\mu,\gamma\right)=0$$

This implies that from the three variables with three shape parameters, it degenerates to only one variable with 2 shape parameters where $T_{3,0,0}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)=T_0(v;\lambda,\mu)$, $T_{0,3,0}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)=T_3(v;\lambda,\mu)$ and $T_{0,0,3}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)$ vanished.

Meanwhile, for the other six trigonometric functions,

$$T_{2,1,0}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)=1-sin^2 v-\frac{(1-\sin v)^2(1-\lambda\sin v)}{1+\lambda\sin v}$$
$$T_{2,0,1}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)=0$$
$$T_{1,2,0}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)=1-cos^2 v-\frac{(1-\cos v)^2(1-\mu\cos v)}{1+\mu\cos v}$$
$$T_{0,2,1}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)=0$$
$$T_{1,0,2}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)=0$$
$$T_{0,1,2}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)=0$$

Therefore, $T_{2,1,0}^{3}(\pi/2-v,v,0;\lambda,\mu,\gamma)=T_1(v;\lambda,\mu)$, $T_{1,2,0}^{3}(u,\pi/2-v,v,0;\lambda,\mu,\gamma)=T_2(v;\lambda,\mu)$ and the other four functions vanished.

The construction of cubic trigonometric basis with ten functions has the following properties:

i. Nonnegativity. $T_{i,j,k}^{3}(u,v,w;\lambda,\mu,\gamma)\geq 0, i,j,k\in\mathbb{N}, i+j+k=3$.
ii. Partition of unity. $\sum_{i+j+k=3}T_{i,j,k}^{3}(u,v,w;\lambda,\mu,\gamma)=1$.
iii. Linear independence. The set $\left\{T_{i,j,k}^{3}(u,v,w;\lambda,\mu,\gamma); i,j,k\in\mathbb{N}, i+j+k=3\right\}$ is linearly independent.
iv. Symmetry. For all $i,j,k\in\mathbb{N}, i+j+k=3$, we have
$T_{i,j,k}^{3}(u,v,w;\lambda,\mu,\gamma)=T_{j,i,k}^{3}(u,v,w;\lambda,\mu,\gamma)=T_{j,k,i}^{3}(u,v,w;\lambda,\mu,\gamma)$
$T_{i,k,j}^{3}(u,v,w;\lambda,\mu,\gamma)=T_{k,i,j}^{3}(u,v,w;\lambda,\mu,\gamma)=T_{k,j,i}^{3}(u,v,w;\lambda,\mu,\gamma)$

Let $\lambda,\mu,\gamma\in[0,1]$, given control points $P_{ijk}\in\mathcal{R}^3(i,j,k\in\mathbb{N}, i+j+k=3)$, and a domain triangle $D=\{(u,v,w)|u+v+w=\pi/2, u\geq 0, v\geq 0, w\geq 0\}$, in which (u,v,w) are the barycentric coordinates of the points in D.

$$R(u,v,w) = \sum_{i+j+k=3} T^3_{i,j,k}(u,v,w;\lambda,\mu,\gamma)\, P_{i,j,k} \quad (17)$$

where $\{T^3_{i,j,k}\}\,(i+j+k=3)$ is the basis functions and $\{P_{i,j,k}\}\,(i+j+k=3)$ are the control points.

The trigonometric patch over a triangular domain has the following geometric properties:

i. Affine invariance and convex hull property.
The basis Eq. (14) has nonnegativity and partition of unity. Therefore Eq. (17) also possess this property.

ii. Geometric property at the corner points.
$R(\pi/2\,,0,0) = P_{3,0,0}$
$R(0,\pi/2\,,0) = P_{0,3,0}$
$R(0,0,\pi/2) = P_{0,0,3}$

iii. Corner point tangent plane.
Let $w = \pi/2 - u - v$,

$$\left.\frac{\partial R}{\partial u}\right|_{(\pi/2,0,0)} = (2+2\lambda)\big(P_{3,0,0} - P_{2,0,1}\big),\quad \left.\frac{\partial R}{\partial v}\right|_{(\pi/2,0,0)} = (2+2\lambda)\big(P_{2,1,0} - P_{2,0,1}\big),$$
$$\left.\frac{\partial R}{\partial u}\right|_{(0,\pi/2,0)} = (2+2\mu)\big(P_{1,2,0} - P_{0,2,1}\big),\quad \left.\frac{\partial R}{\partial v}\right|_{(0,\pi/2,0)} = (2+2\mu)\big(P_{0,3,0} - P_{0,2,1}\big),$$
$$\left.\frac{\partial R}{\partial u}\right|_{(0,0,\pi/2)} = (2+2\gamma)\big(P_{1,0,2} - P_{0,0,3}\big),\quad \left.\frac{\partial R}{\partial v}\right|_{(0,0,\pi/2)} = (2+2\gamma)\big(P_{0,1,2} - P_{0,0,3}\big),$$

These indicate that the tangent plane at $(\pi/2\,,0,0)$ spanned by the control points $P_{3,0,0}$, $P_{2,1,0}$ and $P_{2,0,1}$. Meanwhile the tangent plane $(0,\pi/2\,,0)$ and $(0,0,\pi/2)$ spanned by the control points $P_{0,3,0}, P_{1,2,0}, P_{0,2,1}$ and $P_{0,0,3}, P_{1,0,2}, P_{0,1,2}$ respectively. Figure 4 below shows the illustration of the corner points for each tangent plane.

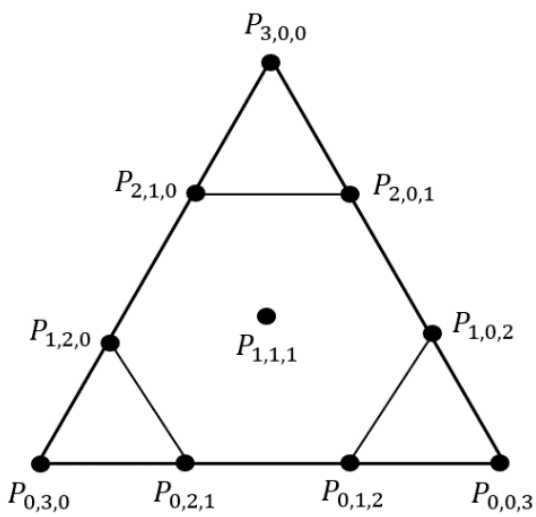


Fig. 4. Corner points for each tangent plane

iv. Boundary property.
Using the condition in (15), if $R(\pi/2 - v, v, 0)$, it will become Eq. (13) with shape parameters of λ and μ. Similar to $R(u,0,\pi/2-u)$ and $R(0,\pi/2-w,w)$, the shape parameters are λ,γ and μ,γ respectively.

v. Shape adjustable property.
vi. The basis Eq. (14) has 3 shape parameters $\lambda, \mu, \gamma \in [0,1]$ which can be changed to adjust the shape of the patch.

5. Surface Interpolation

Surface reconstruction using a trigonometric triangular spline using the ten basis functions in Eq. (14) are applied to Eq. (17) as $T^3_{i,j,k}(u, v, w; \lambda, \mu, \gamma)$. Meanwhile for $P_{i,j,k}$ in (17) is the control points using triangulated data sets which has ten points as illustrated in Figure 4.

The ten control points from [10] are used as a control polygon to generate surface points as shown in Table 2 below.

Table 2
Control points for surface fitting

x	y	z
120	400	10
70	30	10
180	65	10
100	190	30
130	220	30
50	90	20
90	50	20
180	160	20
140	70	20
100	100	30

Given the control polygon as shown in Figure 5 below.

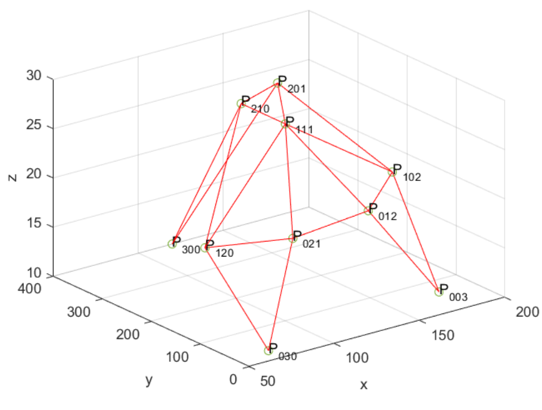


Fig. 5. Control Polygon in 3D Space

The value of the step size N, affects the number of surface points which will be used to generate a surface. The Small value of N will produce sharp edges. These sharp edges become smoother if the value of N is increased. The surface interpolation with different value of N is illustrated in Figure 6.

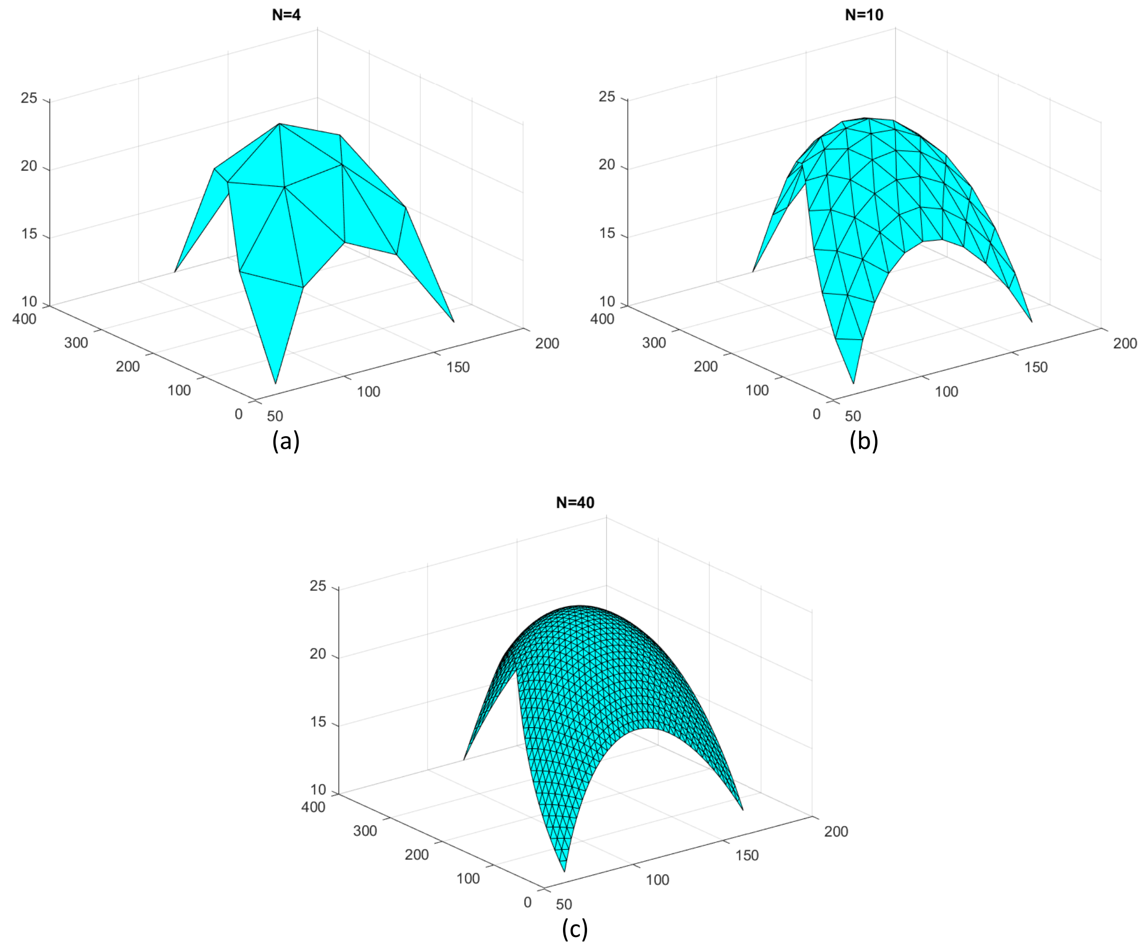


Fig. 6. Surface Interpolation with Different Value of N (a) N=4 (b) N=10 (c) N=40

The shape parameter can be used to adjust certain parts of the surface as desired. The patch approaches the control polygon as the three shape parameters increase. The parameter must be in between the given interval value, to have the surface staying inside of the control polygon. The Figure 7 below show the shape of the surface changing due to the applied values of parameters.

Figure 7(a) shows the state of the patch when using $\lambda, \mu, \gamma = 0$. By increasing the value of λ, the patch approaches the point $P_{3,0,0}, P_{2,1,0}, P_{2,0,1}$ as shown on Figure 7(b). Meanwhile, increasing the value of μ and γ, the patch approaches the point $P_{1,2,0}, P_{0,2,1}, P_{0,3,0}$ and $P_{0,1,2}, P_{1,0,2}, P_{0,0,3}$ respectively as shown in Figure 7(c) and 7(d). It shows that the shape of the patch can be adjustable without changing the control polygon.

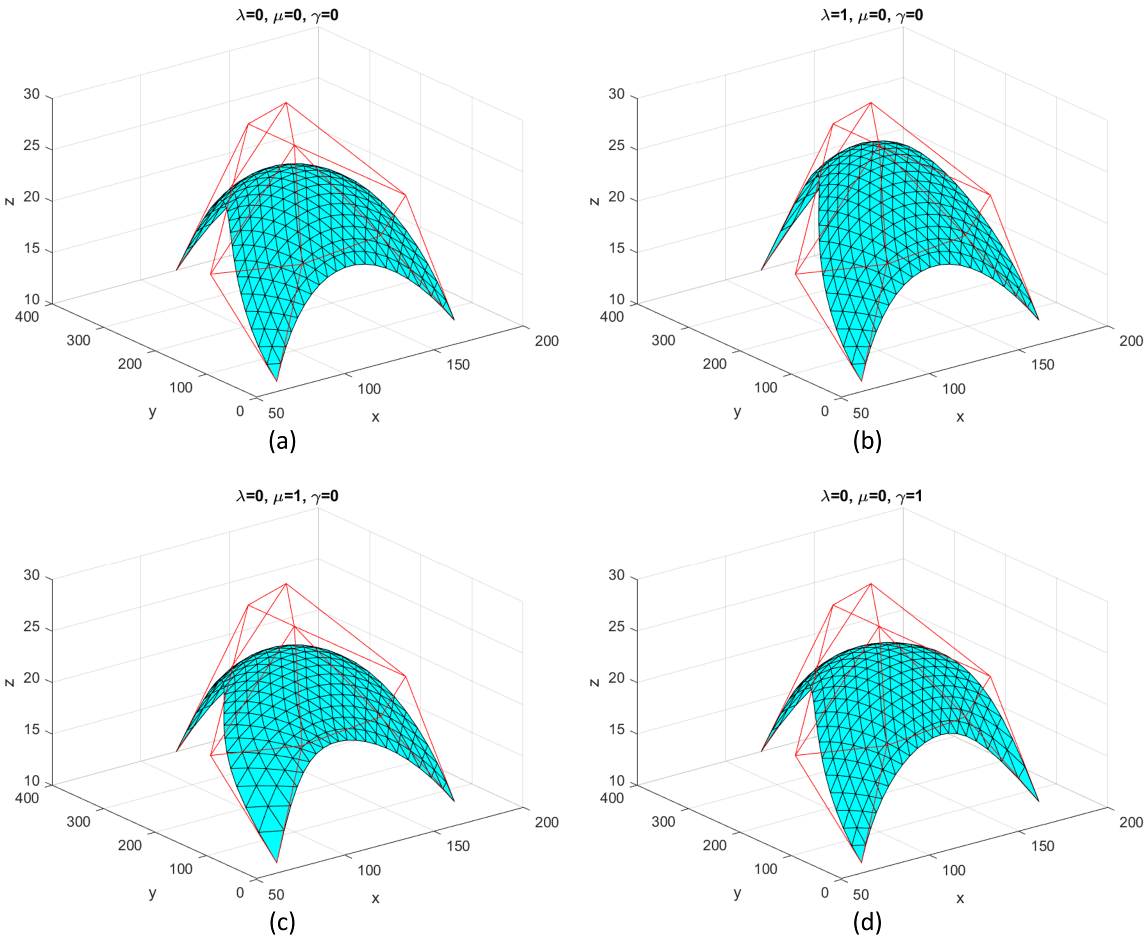
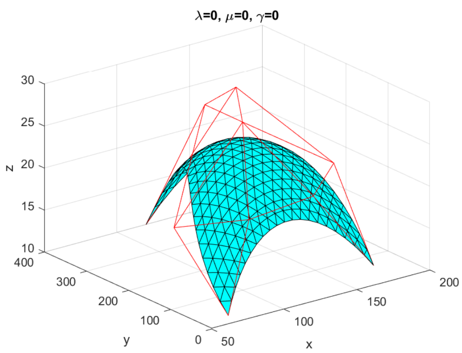
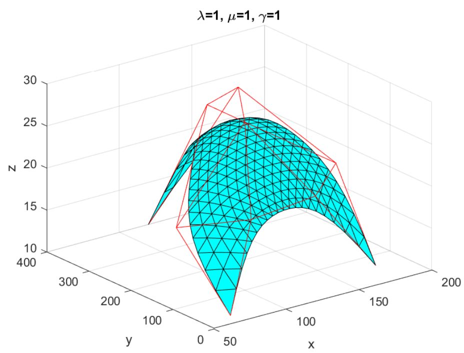


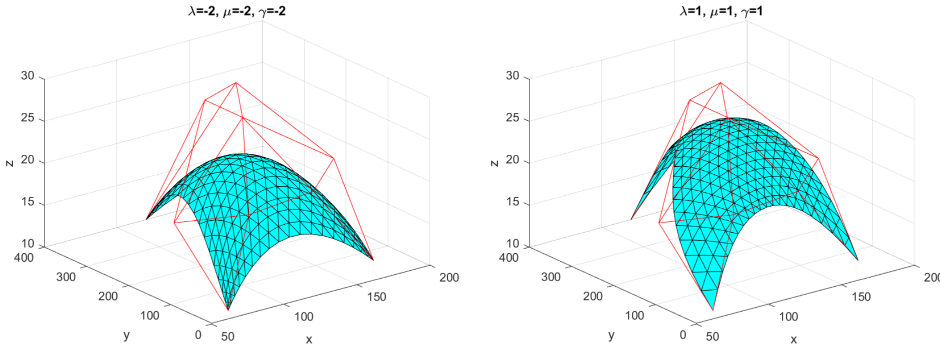
Fig. 7. The Effects of Shape Parameters in Surface Interpolation (a) $\lambda = 0, \mu = 0, \gamma = 0$ (b) $\lambda = 1, \mu = 0, \gamma = 0$ (c) $\lambda = 0, \mu = 1, \gamma = 0$ (d) $\lambda = 0, \mu = 0, \gamma = 1$

Next, table below shows the comparison of patches generated by using established trigonometric basis functions from other researchers.

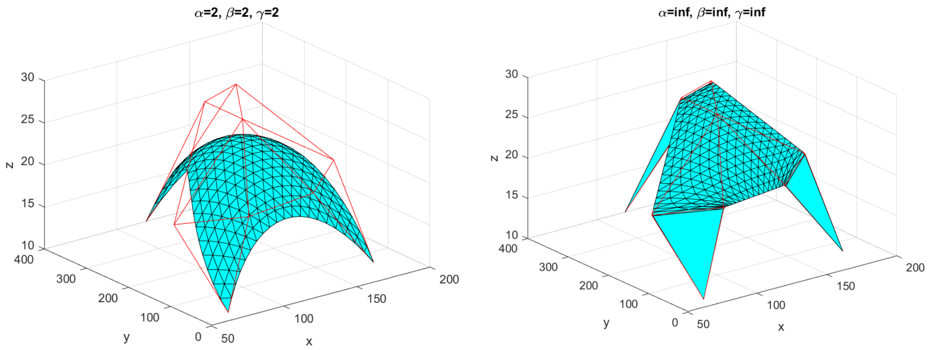
Table 3
Patch comparison with other studies

	Shape parameters (Minimum value)	Shape parameters (Maximum value)
Proposed basis		

X. Han, *et al.*, [15]



Y. Zhu, *et al.*, [14]



The proposed basis offers better shape parameters because it uses small interval values which is [0,1] rather than [-2,1] and the patch is closer to control polygon than study by [15] as shown in Figure 8 below. Meanwhile, basis in work by [14] generates the same patch as the proposed basis using minimum values of shape parameters. However, when using the maximum values, patch of study by [14] tends to be at each side of the triangle in the control polygon and the surface is not smooth as shown in Figure 9. The proposed basis still maintains its smoothness when using maximum values of parameters.

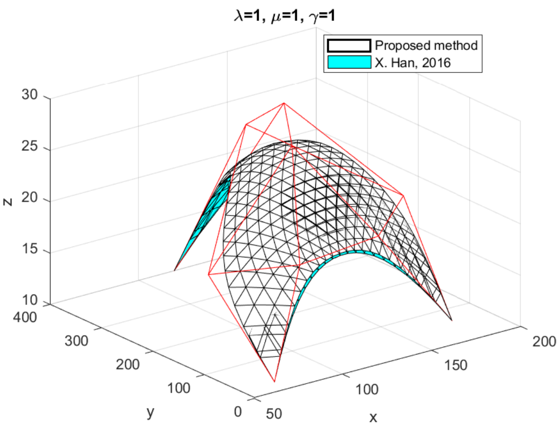


Fig. 8. Comparison of patches using proposed basis and work by [15]

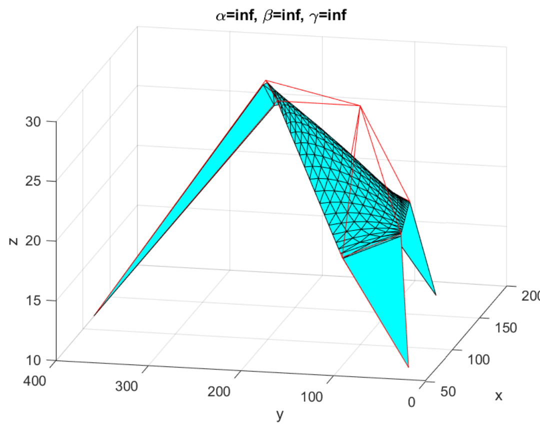


Fig. 9. Side view of a patch as proposed by [14]

6. Conclusion

Trigonometric polynomials and splines have been studied by many researchers to construct curves. However, very few put focus on constructing surface over a triangular domain as mentioned by [14]. In this study, the proposed cubic trigonometric basis for curve and surface has been constructed. The trigonometric space $T_{\lambda,\mu}$ that was used to construct the basis for a curve has been proven suitable for curve and surface design because it possesses blossoms which exist if and only if the space $DT_{\lambda,\mu}$ is an EC-space on $[0, \pi/2]$. The four univariate basis functions for curves are then extended to construct the ten basis functions for surface using triangular domain as its control points. The proposed basis for curve and surface has the end-point property and it is also proven to have the properties of non-negativity, partition of unity, monotonicity, linear independence and symmetry.

There are two shape parameters for curve and three shape parameters for surface as introduced in this study. These parameters can control the shapes easily without modifying the control points. This study also proposed shape parameter values of $[0,1]$ to give flexibility to interpolating surfaces.

The Triangular domain of control points offer flexibility in surface design. However, the computation is complex, because when the value of the step size N is higher, the surface point will increase. From the results given, the surface is smoother with a higher number of surface points.

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