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Direct and Indirect Methods for the Non-Standard Optimal Control Problem

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ABSTRACT

This paper comprehensively investigated direct and indirect methods to address the non-standard Optimal Control (OC) problem, specifically focusing on a case study involving piecewise function. Non-standard OC problems, deviating from standard formulations, present unique challenges that require specialized solution techniques. The primary aim of this research is to maximize the objective function. However, the presence of a piecewise function introduced non-differentiability at certain timeframes during the optimization process. Additionally, the unknown final state necessitates innovative approaches to find optimal solutions, as it leads to a non-zero costate value at the terminal time. The indirect approach was applied to handle these complexities; specifically, the shooting method, implemented in the C++ programming language, to compute the optimal solution. The case study centred around a non-standard OC problem concerning a two-stage piecewise function. To overcome the challenge of the non-differentiable piecewise function, a continuous approach was implemented by using hyperbolic tangent (tanh) modelling. We validated the performances with the direct methods, such as Euler and Runge-Kutta, through numerical experiments and comparisons—the validation of optimal results with direct methods involved utilizing the AMPL programming language with the MINOS solver. The direct method consists in parameterizing the control and state trajectories, transforming the OC problem into a nonlinear programming (NLP) problem. On the other hand, the indirect method utilizes Pontryagin's Minimum Principle to derive the necessary conditions for optimality by solving the associated Two-Point Boundary Value Problem (TPBVP). The findings from the case study demonstrate that the indirect method yields a more accurate solution. The insights gained from this investigation contribute to a better understanding of OC techniques in non-standard scenarios, guiding researchers and practitioners in selecting appropriate methods for similar real-world problems. Moreover, this research enhances the comprehension of OC methods' applicability to non-standard challenges, particularly in the domain of piecewise function problems. The outcomes of this study offer valuable insights for addressing complex optimization challenges in various disciplines and pave the way for further advancements in solving real-world non-standard OC problems.

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1. Introduction

Optimal Control (OC) theory plays a pivotal role in a wide range of disciplines, providing powerful tools for optimizing the behaviour of dynamic systems. Traditionally, OC problems are formulated within the framework of the standard OC, where the objective is to find control strategies that minimize an objective function subject to dynamic system constraints. However, real-world applications often present challenges that deviate from the standard setting, leading to non-standard OC problems.

In this study, the non-standard OC scenarios encompass a complexity involving an unknown final state, leading to a non-zero final costate value. Addressing these challenges requires innovative solution techniques tailored to the specific problem characteristics. Among the different methodologies available, the direct and indirect methods stand out as widely adopted approaches to solving OC problems.

The direct method involves approximating both the state and control variables, effectively transforming the OC problem into a nonlinear programming (NLP) problem. On the other hand, the indirect method formulates the optimization task as a Two-Point Boundary Value Problem (TPBVP), employing Pontryagin's Minimum Principle to derive the necessary conditions for optimality.

This paper aims to delve into the realm of non-standard OC problems and compare the effectiveness of direct and indirect methods in addressing such challenges. As a case study, we focus on a specific real-world application involving stage's function. In this context, the stages' function follows two-stage piecewise functions, and the final state is said to be unknown, making the problem particularly suitable for examining the capabilities of direct and indirect methods.

The primary objective of this research is to maximize the objective function while analyzing the performance of direct and indirect methods in the context involving a two-stage piecewise function. However, the presence of the piecewise function results in non-differentiable at a certain time frame when the piecewise level is increasing. Therefore, hyperbolic tangent (tanh) modelling was implemented to overcome the difficulty.

The structure of this paper is as follows: Section 2 provides a comprehensive review of related literature on OC methods and their applications to non-standard problems. Section 3 presents the details of the non-standard OC problem. In Section 4, we outline the problem formulation. Section 5 describes the direct and indirect methods, highlighting their underlying principles and solution procedures. Section 6 showcases the numerical experiments and comparative analyses of the methods along with discussions of the obtained results. Finally, Section 7 and Section 8 conclude the paper with a summary of the key findings, contributions and implications for future research.

2. Overview of Optimal Control Methods

OC theory has a rich history and finds extensive applications in various fields, including engineering, economics, biology and aerospace. Fayyadh *et al.,* [6] presented a mathematical model for optimizing heat transfer in the context of Carreau fluid conveying magnetized nanoparticles over a permeable surface. Zinober *et al.,* [27] addressed a non-standard OC problem that arises in an economics application. Trélat [23] highlighted significant results and challenges associated with using OC techniques in aerospace engineering. This covers various aspects of OC theory, including theoretical foundations, solution methods and practical applications. The finding delves into the complexities and specific requirements of aerospace applications, discussing how OC contributes to the design and operation of aerospace systems. Spence [22] explored the concept of learning curves and their implications for competition. The study delves into the relationship between learning curves, which represent the decrease in production costs as experience accumulates, and competition within industries. The research sheds light on the intricate interplay between learning, production costs, and market competitiveness, contributing to the understanding of industrial economics.

The standard OC problem deals with determining control strategies that minimize or maximize a predefined objective function, subject to dynamic system constraints described by ordinary differential equations (ODEs) or differential-algebraic equations (DAEs). The optimization task involves finding the optimal trajectories of state and control variables that satisfy the system dynamics and boundary conditions while optimizing the objective function.

Two widely used approaches to solving non-standard OC problems are the direct and indirect methods. The direct method discretizes the time domain, representing both the state and control variables over a finite number of intervals. This transforms the OC problem into an NLP problem, amenable to various optimization algorithms. Examples of direct methods in this study include the Euler and Runge-Kutta methods.

Efimov *et al.,* [5] discussed the discretization of homogeneous systems through the application of the Euler method with a state-dependent step. The proposed Euler method solved ODE with a step size that varies based on the state of the system. Chauhan *et al.,* [2] focused on computational techniques using the Runge-Kutta method to solve differential equations by comprehensively examining Runge-Kutta methods, their characteristics, and their applicability in solving a wide range of differential equations.

Conversely, the indirect method employs the principle of Pontryagin's Minimum Principle, which characterizes the necessary conditions for optimality in the form of a set of differential equations called the costate variable. The solution of the original OC problem involves solving both the system's dynamics and the costate equations, typically using techniques such as the Euler-Lagrange equation or the Hamiltonian formalism [19]. This study applies the shooting method as the indirect method to solve the non-standard OC problem.

Spada *et al.,* [21] integrated both direct and indirect methods to enhance the efficiency and accuracy for optimizing powered descent and landing trajectories. The research showcases the benefits of this hybrid strategy in achieving optimal trajectories, providing insights into the synergy between different optimization techniques. Nasresfahani *et al.,* [16] focused on the numerical solution of OC problems related to atherosclerosis dynamics using both direct and indirect methods, incorporating a shooting or collocation approach. Karbowski [11] presented the MATLAB implementation of direct and indirect shooting methods for solving an OC problem with state constraints.

3. Non-Standard Optimal Control Problems

While the standard OC framework has been extensively studied and applied, real-world problems often present challenges that deviate from the standard setting. These deviations give rise to non-standard OC problems, which require innovative solution techniques to address their unique complexities.

Cruz *et al.,* [3] contributed to a non-standard class of variational problems. Investigating the characteristics and solution techniques for these non-standard variational problems contributes to a broader understanding of optimization in unconventional scenarios. The analysis offers insights into the unique challenges posed by this class of problems and proposes approaches for addressing them effectively.

Non-standard OC scenarios encompass a variety of characteristics, such as non-smooth functions, discontinuities, and unconventional boundary conditions. Non-smooth functions can introduce non-differentiability, making the optimization problem challenging for methods relying on derivatives. Discontinuities can arise due to switching control strategies or changes in system dynamics, posing additional complexities for traditional optimization methods.

Unconventional boundary conditions can include free or unknown final states, requiring the optimization algorithm to determine both the optimal control and the final state variables. Furthermore, piecewise functions and hybrid systems can complicate the dynamics and control of the system, necessitating specialized methods capable of handling such complexities.

In this paper, we focus on investigating the effectiveness of direct and indirect methods in solving non-standard OC problems. Specifically, we conduct a case study involving piecewise function, a real-world application that exemplifies the challenges of non-standard scenarios. The presence of piecewise function and unknown final state in objective function necessitates novel approaches for finding optimal solutions.

4. Problem Formulation

The boundary conditions define the initial and final states of the system. For the piecewise function problem, we encounter non-standard boundary conditions, such as an unknown final state $y\left(t_f\right)$ and a non-differentiable two-stage piecewise function at a certain timeframe—the unknown final state value results in the non-zero final costate value. Mathematically, a new boundary condition is needed to continue the investigation. Therefore, the work of Malinowska and Torres [14] was referred to in order to prove a new boundary condition. Malinowska and Torres [14] delve into the concept of natural boundary conditions within the Calculus of Variations (CoV) by exploring its role in variational problems. In this study context, natural boundary conditions arise as optimality conditions.

In recent years, the study related to boundary conditions has been widely discussed. Jena and Gairola [10] contributed to innovative boundary conditions for studying how elevated terrains influence wind patterns in the environment and subsequently affect pedestrian winds. Duan *et al.,* [4] contributed to the approach for boundary-free mesh parameterization. Introducing the boundary-free approach enhanced the mesh parameterization process, which is essential in computer graphics and geometric modelling. Noori Skandari *et al.,* [17] designed a specific class of singular boundary value problems (BVPs) that exhibit singularity. The advancement of computational methods for differential equations provides insights into handling problems characterized by singularities at their boundaries. Heidari and Malek **Error! Reference source not found.** deal with the optimal boundary control for the hyperdiffusion equation, a partial differential equation (PDE) type. Investigating the problem of determining the OC strategy applied at the domain's boundary to guide the evolution of the hyperdiffusion process towards a desired state.

In this study, the piecewise function presents a challenging non-standard OC scenario with difficulty in differentiation when the piecewise level increases and an unknown final state. Our objective is to find the optimal strategy $u(t)$ that maximizes the objective function $J\left[u(t)\right]$ while satisfying the system dynamics ODE, the constraints, and the boundary conditions.

5. Solution Approaches - Direct and Indirect Methods

In this section, we delve into the two solution approaches employed to tackle the non-standard OC problem by implementing a two-stage piecewise function. The direct method is a widely used approach for solving OC problems. It involves discretizing the time domain, representing both the state and control variables over a finite number of intervals. By doing so, the original OC problem is transformed into an NLP problem.

Noori Skandari *et al.,* [18] investigated a direct method for solving fractional OC problems based on the Clenshaw-Curtis formula. This method is designed to handle OC problems involving fractional derivatives, a complex mathematical concept that arises in various scientific and engineering contexts. Hilli and Pennanen (2008) [8] focused on discretizing multistage stochastic programs by investigating different discretization approaches' accuracy, efficiency, and convergence properties.

To apply the direct method, we divide the time horizon $\left[t_i, t_f\right]$ into N intervals, each of size $h = \frac{t_f}{N}$. The state variable $y(t)$ and control variable $u(t)$ are approximated by discrete values at time instants $t_0, t_1, \ldots, t_N$, denoted as $y_0, y_1, \ldots, y_N$ and $u_0, u_1, \ldots, u_N$, respectively. The state and control variables are approximated using Euler and Runge-Kutta methods. The objective function $J\left[u(t)\right]$ is approximated by discretizing the integral term. This leads to a finite sum of instantaneous cost terms over discrete time intervals.

The indirect method, also known as the variational or adjoint method, is another widely used approach for solving OC problems. It involves formulating the optimization task as a TPBVP and leveraging Pontryagin's Minimum Principle to derive the necessary conditions for optimality. The core of the indirect method lies in the Hamiltonian function, which combines the cost function L with the system dynamics $u(t)$ and introduces a set of Lagrange multiplier functions (adjoint variables) $p(t)$ associated with the state dynamics in Eq. (1):

$$H(t, y, u, p) = L(t, y, u) + p(t)u(t) \tag{1}$$

Pontryagin's Minimum Principle states that an OC strategy $u^*(t)$ exists if there exists a set of adjoint variables $p^*(t)$ satisfying the following Eqs. (2)-(4) conditions [13]:

Hamiltonian stationarity $$\frac{\partial H}{\partial u} = 0 \tag{2}$$

Transversality condition $$p(t) = \frac{\partial H}{\partial y(t_f)} \tag{3}$$

Adjoint dynamic $$\frac{dp}{dt} = -\frac{\partial H}{\partial y} \tag{4}$$

The indirect method formulates the OC problem as TPBVP, involving both the system and adjoint dynamics. The solution of the TPBVP provides the OC strategy $u^*(t)$ and the associated adjoint variables $p^*(t)$. In the subsequent section, we will apply the direct and indirect methods to the two-stage piecewise function problem and assess their performances.

6. Illustrative Example, Numerical Experiments and Comparative Analysis

This section presents the numerical experiments conducted to solve the two-stage piecewise function problem using direct and indirect methods. We compared the performances of these solution approaches in terms of convergence, computational efficiency and accuracy.

To ensure a comprehensive evaluation, we set up a series of test cases with varying problem complexities, including time horizons, step size for the iteration process and system dynamics. The test cases cover scenarios where the direct and indirect methods may demonstrate distinct advantages and challenges.

We implemented the direct method using AMPL programming language with MINOS solver for solving the transformed NLP problem [7]. We used appropriate discretization techniques for the direct method to ensure accurate comparisons, such as Euler and Runge-Kutta method [1]. For the indirect method, we leveraged numerical solvers for the TPBVP, such as the hybrid Newton-Brent shooting method.

We analyzed the convergence behaviour of both methods for different problem settings. This analysis includes examining the convergence rate and stability of the algorithms. We compared the accuracy of the optimal solutions obtained from both methods and evaluated the closeness of the approximated performance index to the actual optimal value.

Let us consider a dynamic system governed by the state variable $y(t)$ and the control variable $u(t)$, where t represents time. The objective is to determine the optimal performance index J at the final time t_f as Eq. (5);

$$\text{Maximize } J\left[u(t)\right] = \int_{t_i}^{t_f} L \, dt \tag{5}$$

The L function is defined as follows Eq. (6).

$$L = 0.75\sqrt{u} - Q(y)\left(0.95 + uzy\sin(0.1\pi t)\right) \tag{6}$$

subject to time t, control u, state variable y and $Q(y)$ is denoted as a two-stage piecewise function. The dynamics of the system are described by the ODE in Eq. (7);

$$\dot{y}(t) = u(t) \tag{7}$$

The objective function is subject to an initial state that is equal to zero, and the piecewise function can be defined as Eq. **Error! Reference source not found.**. The proposed problem will use the following $Q(y)$ system: equivalent to the two-stage piecewise constant function.

$$Q(y) = \begin{cases} 1.5 & \text{for} \quad y \le 0.5z \\ 2 & \text{for} \quad y > 0.5z \end{cases} \tag{8}$$

Zinober and Kaivanto [26] focused on the optimization of production processes while considering piecewise continuous royalty payment obligations. This research investigated the challenges of royalty payment obligations exhibiting piecewise continuous characteristics. The authors arranged the piecewise royalty payment into a matrix formulation. However, the problem cannot be solved due to the difficulty related to the non-differentiable function when the royalty payment level changes.

Figure 1 illustrates the plot for the two-stage piecewise function. This function cannot be differentiated at certain periods when the level of piecewise increases. Therefore, a continuous hyperbolic tangent (tanh) approach was applied to overcome the difficulty. Hyperbolic tangent (tanh) modelling, also known as the hyperbolic tangent (tanh) function modelling, is a mathematical approach used to overcome non-differentiability and discontinuities in certain functions. The hyperbolic tangent function, denoted as $\tanh(x)$, is defined as the ratio of the hyperbolic sine (sinh) to the hyperbolic cosine (cosh) of a real number x. It has a sigmoidal shape and maps input values from the real number line to a range between negative and positive ones.

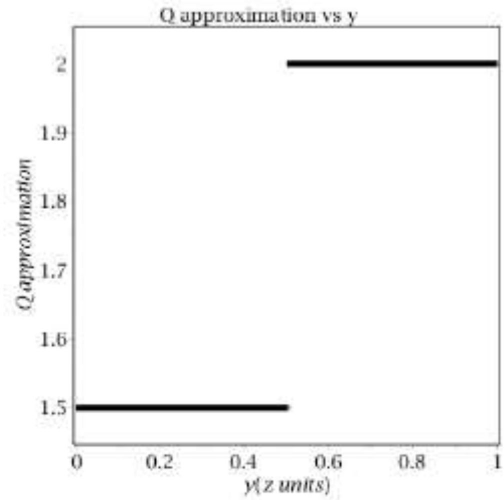


Fig. 1. Two-stage piecewise function

Hyperbolic modelling was applied worldwide to solve real-world problems. Md Ali *et al.,* [15] presented a hyperbolic tangent (tanh) fluid model for the stagnation flow of a hybrid nanofluid over a stretching sheet. The study explored the behaviour of the nanofluid under various conditions, particularly its fluid flow characteristics and thermal properties using the hyperbolic tangent (tanh) function. Shafiq *et al.,* [20] contributed to statistical modelling for bioconvective hyperbolic tangent (tanh) nanofluid near a stretching surface under zero mass flux. The study employs statistical modelling techniques to investigate the intricate interplay of fluid dynamics, heat transfer and bioconvective effects. Kawashima *et al.,* [12] addressed the initial value problem (IVP) by investigating the behaviour of the hyperbolic-dispersive system in terms of its initial values.

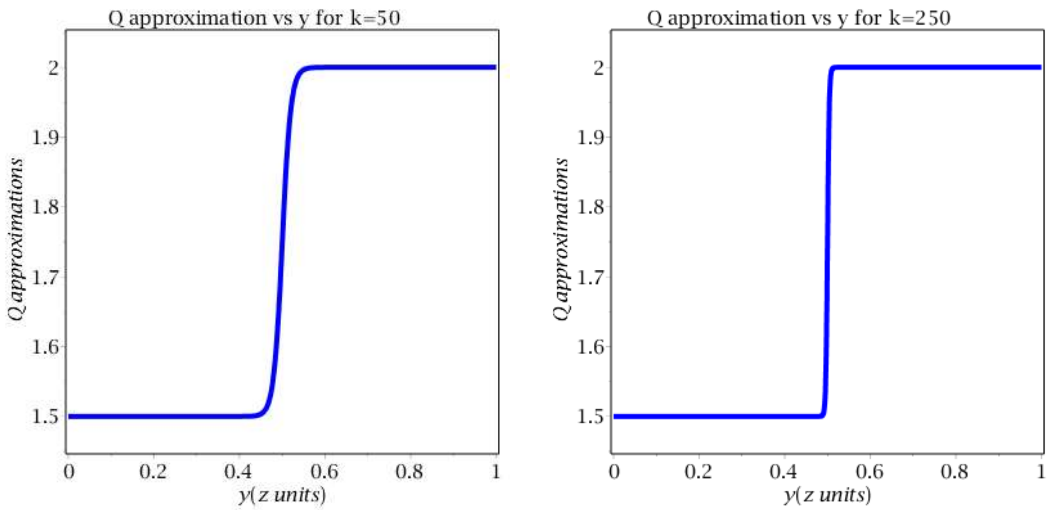
In the context of non-standard OC problems, hyperbolic tangent (tanh) modelling is employed to handle piecewise functions that are not differentiable at certain time frames. Such functions can pose challenges in optimization algorithms that rely on derivatives for convergence. The critical advantage of hyperbolic tangent (tanh) modelling lies in its continuous and smooth nature. The overall function becomes differentiable throughout the entire domain by approximating non-differentiable piecewise functions using hyperbolic tangent (tanh) functions. This enables the use of derivative-based optimization techniques, making the problem more amenable to solution approaches like the indirect method.

Thus, $Q(y)$ in Eq. **Error! Reference source not found.** can be converted into the hyperbolic tangent (tanh) function as Eq. (9);

$$Q(y)=1.75+0.25\tanh\left(k\left(y-0.5z\right)\right) \quad (9)$$

The proposed settings are $t_i = 0$ and $t_f = 10$ while the initial known state value is $y(0)=0$ and the final state value $y(t_f)$ is free. In this study, the smoothing value k is equal to 50 and 250. These smoothing values were applied to determine the step size during iteration.

Figure 2 depicts the plot representing the two-stage piecewise function, which adopts a continuous hyperbolic tangent (tanh) approximation to achieve smoothness for values of k set at 50 and 250. Several essential conditions need to be met in this context, including the state equation, the costate equation, and the stationary condition. Additionally, the initial state condition of $y(0)=0$ is specified, and an initial costate value for $p(0)$ is assumed. The integral's boundary condition must also hold true at the final time t_f. Furthermore, the iterated value of z employed in the state equation must ensure that it corresponds to the value of $y(t_f)$ at the final time, ensuring the scalar function approaches zero. This requirement is fulfilled within our algorithm only when the costate equation converges. Consequently, the optimal solution is then obtained.



(a) (b)

Fig. 2. Continuous approximation of hyperbolic tangent (tanh) function (a) for smoothing value k equal to 50 (b) for smoothing value k equal to 250

The outcomes acquired from the discretization and the shooting strategy are ideal solution which is exceedingly precise. The optimal value of the costate at the final time is determined to be 0.057015 when k is set at 50. The optimal solution is presented in Table 1 below, corresponding to the chosen smoothing value of k at 50.

Table 1
Optimal results of the shooting and discretization methods with k equal to 50

Methods	Final state value	Initial costate value	Objective function
Shooting	0.351150	-0.031924	0.643248
Euler	0.349611	-0.032073	0.646771

Runge-Kutta	0.351759	-0.031720	0.646803

Referring to Table 1, the final state values, initial costate values, along with performance index exhibit remarkable disparities between the shooting method and the discretization techniques. In terms of final state approximation, the Euler method closely matches up to one decimal point, whereas the Runge-Kutta method demonstrates convergence up to three decimal places when compared to the shooting technique. Simultaneously, the shooting, Euler, and Runge-Kutta methods provide reasonably congruent answers up to two decimal places.

When considering the initial time $t_i = 0$, the costate value produced by the Runge-Kutta approach offers an appropriately corresponding solution, maintaining similarity up to three decimal places compared to the shooting method. Meanwhile, the Euler method delivers an initial costate outcome comparable up to two decimal places when contrasted with the shooting method.

The objective function values at the final time $t_f = 10$ yield solutions that are akin up to two decimal places for all the methods.

Displayed in Figure 3 is the optimal curve illustrating the objective function, and the optimal curve depicting the state values exhibit similarities between the shooting technique and the discretization methods. Simultaneously, when examining the plots for the costate and control values, slight discrepancies emerge between the discretization outcomes and the shooting results.

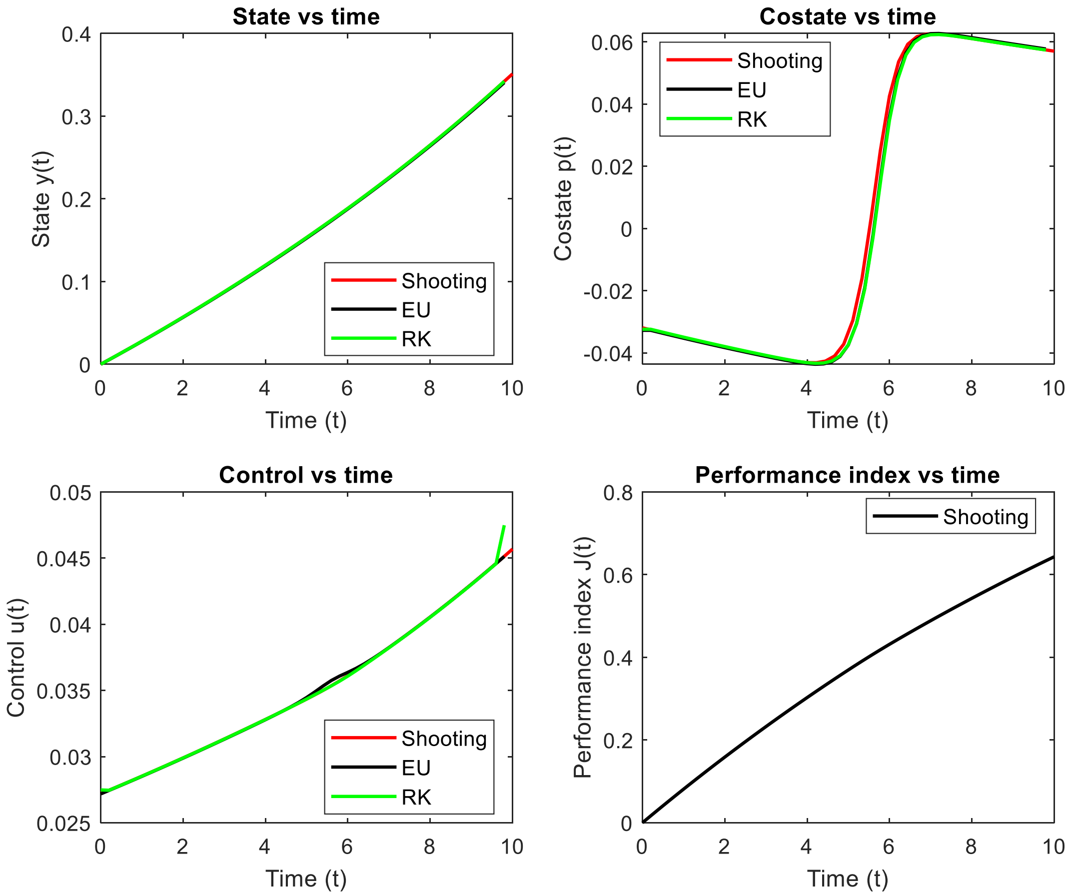


Fig. 3. The plots for the optimal state, costate, control and objective function when the smoothing value is equal to 50 (EU=Euler; RK=Runge-Kutta)

With a smoothing value of k set at 250, the optimal final costate value is calculated to be 0.056910. These optimal outcomes for the final state, initial costate, and objective function are presented in Table 2 for the specific case of k being 250.

Table 2
Optimal results of the shooting and discretization methods with k equal to 250

Methods	Final state value	Initial costate value	Objective function
Shooting	0.351080	-0.031822	0.643299
Euler	0.349742	-0.032759	0.646844
Runge-Kutta	0.351821	-0.032116	0.646865

Referring to Table 2, the optimal final state values display congruence up to three decimal places for the Runge-Kutta method, whereas the Euler results align up to only one decimal place when compared to the shooting results. In a broader context, the optimal values for the costate at the initial time, as obtained from both the shooting and direct methods, demonstrate similarity up to two decimal places. Similarly, the optimal objective function values show a parallel outcome, with results akin up to two decimal places across the shooting, Euler, and Runge-Kutta methods.

Examining Figure 4, it becomes apparent that there are discernible distinctions in the plots between the Euler and Runge-Kutta methods, particularly in relation to the optimal costate and control variables. Notably, when the value of k surpasses 50, the discretization methods tend to generate inaccurate values for both the costate and control variables. This inaccuracy could potentially stem from the discretization error incurred during the process, as discussed in prior works [24][25].

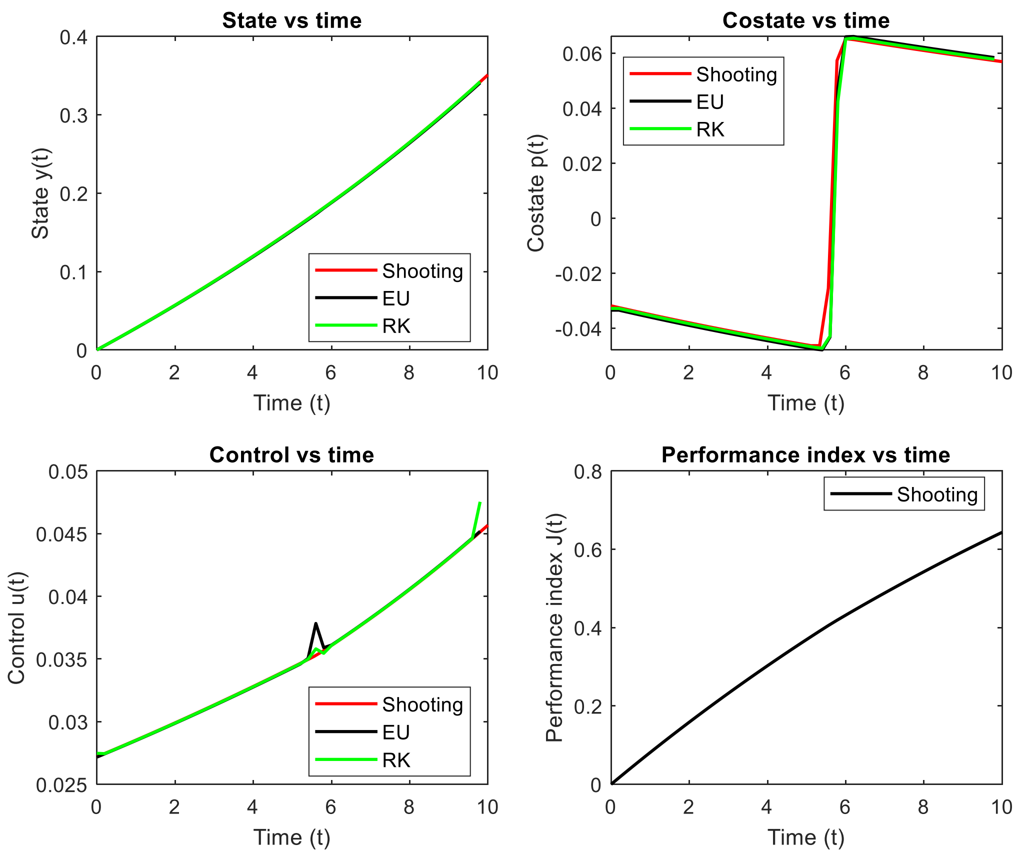


Fig. 4. The plots for the optimal state, costate, control and objective function when the smoothing value is equal to 250 (EU=Euler; RK=Runge-Kutta)

Despite this, the curve representing the shooting values appears notably smoother compared to the curves derived from the discretized values. Consequently, it can be observed that the higher the value of k becomes, the more refined the plot becomes in the case of shooting calculations. This outcome is conducive to achieving higher accuracy in the plot, as it assists in minimizing deviations.

7. Contribution, Implication and Future Direction

This paper thoroughly investigated the application of direct and indirect methods to address non-standard OC problems, focusing on the two-stage piecewise function problem as a case study. Our research compared the effectiveness of these solution approaches in handling challenges posed by non-standard scenarios involving piecewise function and unknown final state. Through numerical experiments and comparative analyses, we have gained valuable insights into the strengths and limitations of the direct and indirect methods. The key findings of the study can be summarized as follows:

i. Convergence: Both direct and indirect methods exhibited satisfactory convergence in most test cases. However, the direct method encountered challenges in cases with abrupt changes in the system dynamics, where the problem became non-smooth.
ii. Computational efficiency: The direct method demonstrated higher computational efficiency for problems with a small number of state and control variables. Conversely, the indirect method proved more efficient for larger-scale problems with complex dynamics.
iii. Accuracy and optimality: In general, both methods provided accurate approximations of the optimal solution. However, the direct method occasionally struggled to locate the global optimal solution in multi-modal cost landscapes.
iv. Robustness: The indirect method displayed greater robustness in handling uncertainties and perturbations in system parameters, making it more suitable for scenarios with uncertain dynamics.

The findings of this research contribute significantly to the understanding of direct and indirect methods in the context of non-standard OC problems. By examining the piecewise function problem, we have demonstrated the applicability of these approaches to real-world applications where non-standard elements are prevalent. The direct method offers simplicity and efficiency for problems with relatively smooth dynamics, making it suitable for applications with moderate complexities. On the other hand, the indirect method provides robustness and global optimality guarantees, making it well-suited for problems with discontinuities, piecewise function and unknown final state.

The insights gained from this study pave the way for future research directions in the field of non-standard OC:

i. Hybrid approaches: Investigating hybrid approaches that combine direct and indirect methods may provide the best of both worlds, offering efficiency and robustness in handling non-standard scenarios.
ii. Adaptivity and sensitivity analysis: Developing adaptive methods that can switch between direct and indirect techniques based on the problem's characteristics could enhance the overall performance.
iii. Applications in other domains: Extending the study to other real-world applications with non-standard features, such as economic systems or environmental models, can demonstrate the versatility of the solution approaches.

iv. OC with uncertainties: Exploring the integration of uncertainty quantification and optimization to address non-standard problems with uncertainties remains an open area for research.

By understanding the strengths and limitations of each method, researchers and practitioners can make informed choices when addressing complex optimization challenges in diverse domains. Pursuing further advancements and innovative techniques in the field will undoubtedly lead to more effective and robust solutions for a wide range of real-world applications.

8. Conclusion

In conclusion, this paper investigated the direct and indirect methods as solution approaches for non-standard OC problems, focusing on the two-stage piecewise function problem as a case study. The numerical experiments and comparative analysis shed light on the effectiveness of each method in handling non-standard scenarios, contributing to the understanding of their applicability in real-world applications. The insights gained from this study can aid researchers and practitioners in effectively addressing complex optimization challenges encountered in various domains, especially in scenarios involving piecewise functions. The insights gained from this research can lead to advancements in solving non-standard OC problems and offer opportunities for future developments in optimization techniques.

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