On the Duality of Unsteady MHD Al$_2$O$_3$-Cu/Water Hybrid Nanofluid Flow over a Stretching/Shrinking Curved Surface with Newtonian Heating

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ABSTRACT

The notable progress in contemporary engineering technology has prompted a greater emphasis on curved surfaces, due to their wide-ranging utilization in transportation, industrial domains, and electronics. However, additional research is necessary to broaden the scope of applications involving curved surfaces. This study explores the unsteady magnetohydrodynamic (MHD) Copper-Alumina/water hybrid nanofluid flow through a permeable curved stretching/shrinking surface with Newtonian heating applied. Due to the curved nature of the geometry, the present problem is modeled using curvilinear coordinates. The addition of Newtonian heating is due to its vital role in the cooling and heating process for industrial purposes. The partial differential equations (PDEs) of the fluid flow will be reduced through a similarity transformation to ordinary differential equations (ODEs). A numerical solution is obtained by resolving the equations of continuity, momentum, and energy using the bvp4c solver in MATLAB. Furthermore, a comprehensive graphical analysis is conducted to examine the impacts of various physical parameters on the velocity and temperature profiles as well as Local Nusselt number and skin friction. These include the parameters on suction, magnetic, Newtonian heating, nanoparticle volume fraction, and stretch/shrink parameters. By systematically varying these parameters, a dual solution was noticed on the graphs while observing their influence on the flow and heat transfer characteristics. The results show that the range of solutions has expanded with an increase in copper volume fraction and magnetic parameters. A shrinking sheet exhibits greater skin friction when the value of copper and magnetic parameters is increased. In the meantime, the stretching sheet portrayed an opposite trend. The local Nusselt number is enhanced with the strengthened magnetic values and Newtonian heating parameters. Besides, the presence of suction is also responsible for a noteworthy decrease in the rate of heat transfer.

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1. Introduction

Globally, heat transfer analysis is becoming increasingly important due to the implication of the heating/cooling process in a wide range of industries and consumer products. To address the limitations of conventional working fluids, nanofluids are formulated by dispersing nano-solid particles possessing superior thermal conductivity within the fluid. This will result in enhancing the thermal conductivity of the nanofluid, as demonstrated notionally by Choi *et al.,* [1]. Subsequently, Suresh *et al.,* [2] and Momin [3] have published new experimental work on increasing the thermal conductivity of the base fluid. It was discovered that dispersing two distinct categories of nanoparticles in a base fluid, which is known as a hybrid nanofluid (HN) boosts the heat capacity of the traditional fluid. The investigation on HN based on micropolar fluid theory through exponentially stretched surfaces was performed by Subhani and Nadeem [4]. Meanwhile, Elattar *et al.,* [5] investigated the HN flow of slender stretching surfaces with the influence of hall current and chemical reaction. Yasir and Khan [6] conducted comparative research on the dynamic viscosity and thermal conductivity of blood-based HN. Other related studies were done by Aladdin *et al.,* [7] and Yasir *et al.,* [8] using carbon nanotubes and TiO_2-SiO_2 as nanoparticles, respectively. A series of literature done by previous studies [9-11] listed the most recent noteworthy articles on a HN by considering a multitude of factors and scenarios.

The study of boundary layer flow on stretching/shrinking surfaces has gained significant research interest in recent times, primarily due to its potential applications in various fields, such as manufacturing, industrial processes, and polymer technology [12]. In contrast with the findings of the stretching sheet, Bachok *et al.,* [13] observed that the solutions for the shrinking sheet are non-unique. To date, Zainal *et al.,* [14] explore the flow of HN over a stretch/shrink surface considering Arrhenius kinetics and radiation effect. Notably, their findings revealed the existence of dual solutions. A few more research on the boundary layer flow of stretching/shrinking surface is accessible in Ref. [15-17]. The study of flat surfaces has been extensively conducted, but exposure to curved surfaces was scarce. In the modern engineering sector, the flow through curved surfaces has gained extensive attention due to its diverse applications in transportation, industry, and electronics. Sajid *et al.,* [18] conducted a study that extended the previous work of Crane [19] by considering a curved stretching sheet. Afterward, Abbas *et al.,* [20] expanded upon the analysis conducted by Sajid *et al.,* [18] by investigating the heat transfer characteristics with the prominent of magnetic field for two distinct heating processes, namely prescribed surface temperature (PST) and prescribed heat flux (PHF). Few researchers [21-23] have addressed the fluid flow problems passing the stretching/shrinking curved surface.

The presence of conditions on the surface of a plate has a significant impact on flow behaviors in heat transfer. By imposing suitable restrictions at the boundary surface, the flow characteristics can be altered. Interesting physical challenges arise when there is a controlled transfer of heat from the boundary surface to the fluid flow field. Conventional thermal boundary requirements, such as constant temperature or constant heat flux, may not be applicable in certain cases where additional factors come into play. In such situations, the consideration of Newtonian heating conditions becomes necessary. Newtonian heating (or conjugate convective flows) refers to the scenario in which heat is transferred to the convective fluid through a bounding surface with limited heat capacity. The inclusion of Newtonian heating conditions in heat transfer analysis can be particularly relevant in high-temperature applications, such as those involving extremely high heat fluxes or intense energy generation. Merkin [24] was the pioneer in studying the Newtonian heating conditions in free convective boundary layer flow over an infinite vertical surface. Hamza *et al.,* [25] proposed an analysis on the unsteady MHD free convection flow of an exothermic Arrhenius kinetic

fluid with Newtonian heating. The significance of such applications has therefore caused numerous researchers [26-28] to incorporate the Newtonian heating condition into their investigations of convective heat transfer.

In all the studies cited above, it is discovered that the behaviour of unsteady boundary layer flow of a hybrid nanofluid over a curved surface has yet to be investigated. Therefore, in this study, we extend the work of Roşca and Pop [29] by choosing the hybrid nanofluids as the nanoparticles together with the effect of Newtonian heating. The outcomes of the current study on the flow behavior of hybrid nanofluid in curved surfaces are anticipated to provide valuable insights for academics, engineers, and researchers. This knowledge will enable them to better understand the characteristics of this fluid and make predictions about its properties, thereby exploring potential applications in diverse industrial and engineering processes.

2. Mathematical Formulation

The investigation focuses on the unsteady boundary laminar flow of an incompressible fluid over a curved surface that undergoes simultaneous stretching and shrinking, forming a coiled shape with a radius of R around the curvilinear coordinates r and s, as depicted in Figure 1(a). The larger value of R corresponds to a sheet that is slightly curvy. Meanwhile, the geometry of the stretched and shrunk surface with no stream velocity appeared as in Figures 1(b) and 1(c). Therefore, the surface was assumed to be stretched or shrunk along the direction with the velocity $u_w(s) = as$, where a is a positive constant and $v_w(t)$ is the mass flux velocity. Meanwhile t is defined as the time.
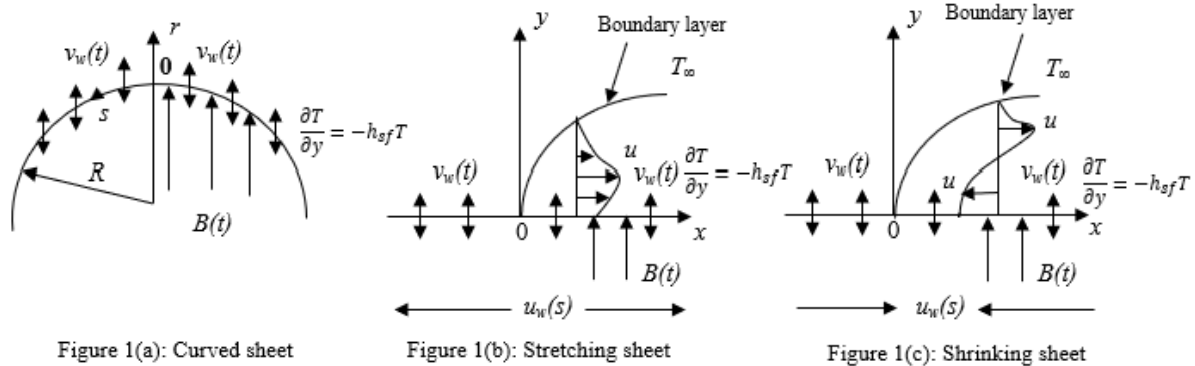

Fig. 1. Schematic diagram of the flow for a curved surface

The governing equations are given as below (see Abbas *et al.,* [20], Roşca and Pop [28]):

$$\frac{\partial}{\partial r}\left\{(r+R)v\right\} + R\frac{\partial u}{\partial s} = 0 \tag{1}$$

$$\frac{u^2}{r+R} = \frac{1}{\rho_{hnf}}\frac{\partial p}{\partial r} \tag{2}$$

$$\frac{\partial u}{\partial t} + v\frac{\partial T}{\partial r} + \frac{uR}{r+R}\frac{\partial u}{\partial s} + \frac{uv}{r+R} = -\frac{1}{\rho_{hnf}}\frac{R}{r+R}\frac{\partial p}{\partial s} + \frac{\mu_{hnf}}{\rho_{hnf}}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r+R}\frac{\partial u}{\partial r} - \frac{u}{(r+R)^2}\right) - \frac{\sigma_{hnf}}{\rho_{hnf}}B^2 u \tag{3}$$

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial r} + \frac{uR}{r+R}\frac{\partial T}{\partial s} = \frac{k_{hnf}}{(\rho Cp)_{hnf}}\left(\frac{1}{r+R}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) \tag{4}$$

Along with

$$t < 0 : u = v = 0 \text{ for any } r \text{ and } s,$$

$$t \geq 0 : v = v_w(t), \quad u = \frac{u_w(s)}{1-\alpha t}\lambda, \quad \frac{\partial T}{\partial r} = -h_{sf}T \qquad \text{at} \quad r = 0 \tag{5}$$

$$u \to 0, \frac{\partial u}{\partial r} \to 0, T \to T_\infty \qquad \qquad \text{as} \quad r \to \infty$$

Here, u and v represent the velocity components aligned with r and s directions, respectively. Then, p, ρ and μ signify as pressure, density, and dynamic viscosity. The electrical conductivity is denoted by σ. Next, λ characterize the stretching or shrinking parameter, i.e. $\lambda > 0$ for the stretching surface, $\lambda < 0$ for the shrinking surface, and $\lambda = 0$ for the static surface. A transverse magnetic field of strength $B(t) = B_0\sqrt{1-\alpha t}$ is applied in the r direction (Naveed *et al.,* [29]). Here, B_0 and α are constant where $\alpha > 0$ is for accelerated sheet and $\alpha < 0$ is for decelerated sheet. T is the temperature, k is thermal conductivity, C_p is heat capacitance and h_{sf} is the heat transfer coefficient. Note that the subscript of "hnf" denotes hybrid nanofluid.

Table 1 tabulates the thermophysical properties of the hybrid nanofluids from the correlation by Takabi and Salehi [30] where $\phi_{hnf} = \phi_1 + \phi_2$. Hence, the subscript of '1', '2' and 'f' denote the Alumina nanoparticle, Copper nanoparticle and base fluid, respectively. Table 2 illustrates the values for the correlation of the thermophysical properties for nanoparticles and base fluid (see Oztop and Abu Nada [31] and Waini *et al.,* [32]).

Table 1
Correlation on thermophysical properties of hybrid nanofluid

Properties	Hybrid nanofluid
Dynamic viscosity (μ)	$\dfrac{\mu_{hnf}}{\mu_f} = \dfrac{1}{(1-\phi_{hnf})^{2.5}}$
Density (ρ)	$\rho_{hnf} = (1-\phi_{hnf})\rho_f + \phi_1\rho_1 + \phi_2\rho_2$
Heat capacity (ρC_p)	$(\rho C_p)_{hnf} = (1-\phi_{hnf})(\rho C_p)_f + \phi_1(\rho C_p)_1 + \phi_2(\rho C_p)_2$
Thermal conductivity (k)	$\dfrac{k_{hnf}}{k_f} = \left[\dfrac{\left(\dfrac{\phi_1 k_1 + \phi_2 k_2}{\phi_{hnf}}\right) + 2k_f + 2(\phi_1 k_1 + \phi_2 k_2) - 2\phi_{hnf}k_f}{\left(\dfrac{\phi_1 k_1 + \phi_2 k_2}{\phi_{hnf}}\right) + 2k_f - (\phi_1 k_1 + \phi_2 k_2) + \phi_{hnf}k_f}\right]$
Electrical conductivity (σ)	$\dfrac{\sigma_{hnf}}{\sigma_f} = \left[\dfrac{\left(\dfrac{\phi_1\sigma_1 + \phi_2\sigma_2}{\phi_{hnf}}\right) + 2\sigma_f + 2(\phi_1\sigma_1 + \phi_2\sigma_2) - 2\phi_{hnf}\sigma_f}{\left(\dfrac{\phi_1\sigma_1 + \phi_2\sigma_2}{\phi_{hnf}}\right) + 2\sigma_f - (\phi_1\sigma_1 + \phi_2\sigma_2) + \phi_{hnf}\sigma_f}\right]$

Table 2
Values of thermophysical properties for nanoparticles and base fluid

Physical properties	Base fluid water	Copper (Cu)	Alumina (Al₂O₃)
C_p (J/(kgK))	4179	385	765
ρ (kg/m³)	997.1	8933	3970
k (W/(mK))	0.613	400	40
σ (S/m)	0.05	5.96×10^7	3.69×10^7

A set of similarity variables is introduced which assures that Eq. (1) is satisfied are as below:

$$u = \frac{as}{1-\alpha t} f'(\eta), \quad v = -\frac{R}{r+R}\sqrt{\frac{a\upsilon_f}{1-\alpha t}} f(\eta), \quad \eta = \sqrt{\frac{a}{\upsilon_f(1-\alpha t)}} r,$$

$$p = \frac{\rho_f a^2 s^2}{(1-\alpha t)^2} P(\eta), \quad \theta(\eta) = \frac{T-T_\infty}{T_\infty}.$$

(6)

Using the Eq. (6), Eq. (2) – Eq. (4) together with the boundary conditions Eq. (5) are transformed as below:

$$\frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f}(f^{iv} + \frac{2}{\eta+K}f''' - \frac{1}{(\eta+K)^2}f'' + \frac{1}{(\eta+K)^3}f') + \frac{K}{\eta+K}(ff''' - ff'')$$

$$+\frac{K}{(\eta+K)^2}(ff'' - f'^2) - \frac{K}{(\eta+K)^3}ff' - \frac{\beta}{2}(\eta f''' + 3f'') - \frac{\beta}{\eta+K}(f' + \frac{\eta}{2}f'')$$

$$-\frac{\sigma_{hnf}/\sigma_f}{\rho_{hnf}/\rho_f}M(f'' + \frac{1}{\eta+K}f') = 0.$$

(7)

$$\frac{k_{hnf}/k_f}{(\rho Cp)_{hnf}/(\rho Cp)_f}\frac{1}{Pr}\left(\frac{1}{\eta+K}\theta' + \theta''\right) - \frac{\eta}{2}\beta\theta' + \frac{K}{\eta+K}f\theta' = 0.$$

(8)

Along with the boundary conditions:

$$f(0) = S, \quad f'(0) = \lambda, \quad \theta'(0) = -N(\theta(\eta)+1)$$
$$f'(\eta) \to 0, \quad f''(\eta) \to 0, \quad \theta(\eta) \to 0 \qquad \text{as} \qquad \eta \to \infty$$

(9)

Where $N = h_{sf}\sqrt{\frac{v_f(1-\alpha t)}{a}}$ is the Newtonian heating parameter, $K = \sqrt{\frac{a}{v_f(1-\alpha t)}}R$ is the curvature, $\beta = \frac{\alpha}{a}$ is the unsteadiness parameter, $M = \frac{\sigma_f}{\rho_f}B_0^2$ is the magnetic parameter, and Pr is the Prandtl number.

The attentiveness study for this problem is skin friction C_f and Nusselt number Nu_s. Both can be represented as

$$C_f = \frac{\tau_s}{\rho_f u_w(s)^2}, \quad Nu_s = \frac{sq_s}{k_f(T_w - T_\infty)}$$

(10)

With τ_s is the surface shear stress and q_s is the heat flux of the wall. The expression is given as follows:

$$\tau_s = \mu_{hnf} \left(\frac{\partial u}{\partial r} - \frac{1}{r+R} u \right)_{r=0} , \quad q_s = -k_{hnf} \left(\frac{\partial T}{\partial r} \right)_{r=0} . \tag{11}$$

By using Eq. (6) and Eq. (11), the Eq. (10) are transformed into:

$$C_{fr} = \frac{\mu_{hnf}}{\mu_f} \left(f''(0) - \frac{\lambda}{K} \right), \quad Nu_{sr} = -\frac{k_{hnf}}{k_f} \left(\frac{1}{\theta(0)} + 1 \right) \tag{12}$$

Where $Re_s = \frac{u_w(s)s}{v_f}$ is the Reynolds number, $C_{fr} = C_f \, Re_s^{\frac{1}{2}} (1-\alpha t)^{\frac{3}{2}}$ is the reduced skin friction coefficient and $Nu_{sr} = Nu_s \left(\frac{Re_s}{1-\alpha t} \right)^{-\frac{1}{2}}$ is the local Nusselt number.

3. Result and Discussion

3.1 Validation of Results

The computational strategy to elucidate the boundary value problem for Eq. (7) – Eq. (9) can be conducted by solving them numerically in MATLAB using the bvp4c package. A comparison from the prior study published by Abbas *et al.,* [20] and Roşca and Pop [28] for the possible solutions to the current flow problem has been furnished in Table 3 for certain limiting cases to verify that our model and numerical computation are credible. Therefore, we are confident in presenting our results in the next section.

Table 3
Comparison values of $f''(0)$ with $\phi_1 = \phi_2 = S = \beta = M = N = 0$, $\lambda = 1$ for several values of curvature parameter, K

| K	Abbas *et al.,* [20]	Roşca and Pop [28]	Present results
5	1.15763	1.15076	1.15763
10	1.07349	1.07172	1.07349
20	1.03561	1.03501	1.03561
30	1.02353	1.02315	1.02353
40	1.01759	1.01729	1.01759
50	1.01405	1.01380	1.01405
100	1.00704	1.00687	1.00704
200	1.00356	1.00342	1.00356
1000	1.00079	1.00068	1.00079
∞	1.00000	1.00000	1.00000

3.2 Effect of Emerging Parameters

The discussion on the results of physical interest towards the governing parameters examined for this study is elaborated in this section. The sequel to the graphical results is obtained by computing Eq. (7) – Eq. (9) into the bvp4c, MATLAB. The emphasizing on the dual solutions can be discerned

through Figures 2 – 7. Meanwhile, profiles of the hybrid nanofluid with affected limitations, including suction (S), magnetic (M), and Newtonian heating (N) parameters, were illustrated in Figures 8 - 12.

The graphical illustration for skin friction coefficient C_{fr} and local Nusselt number Nu_{sr} are presented in Figure 2 and Figure 3 for three distinct volume fractions of copper (ϕ_2) over changes of stretching/shrinking parameter λ. As we observed, two potential solutions are obtainable in this case, possibly as a result of the reversal of flow from the shrinking condition. Consequently, a critical point, λ_c for the detachment of the boundary layer appeared to be noticeable in the figures. Here, the critical point as $\phi_2 = 0$ is $\lambda_c = -0.948184$, $\phi_2 = 0.01$ is $\lambda_c = -0.988699$ and $\phi_2 = 0.02$ is $\lambda_c = -1.026986$. It can be clearly noted that the increment of the volume fraction of copper from 0.01 to 0.02 impeded the detachment of the boundary layer from occurring rapidly. Interestingly, the value of C_{fr} is affected with the elevation of ϕ_2 at the stretching region, while contradictory behavior of flow was observed at the shrinking region.

According to Figure 3, the local Nusselt number Nu_{sr} decreases with the increment of ϕ_2 from 0.01 to 0.02 for either stretching or shrinking region. It also worth noting that the value of Nu_{sr} for the first solution is greater compared to the second solution. To the author's concern, a small amount of ϕ_2 is said to be sufficient to intensify the heat transfer rate. This happen due to the systems' cooling rate with the increases in the value of copper volume fraction, ϕ_2 which has reduced the cooling capacity as the viscous fluid is transformed into the hybrid nanofluid.
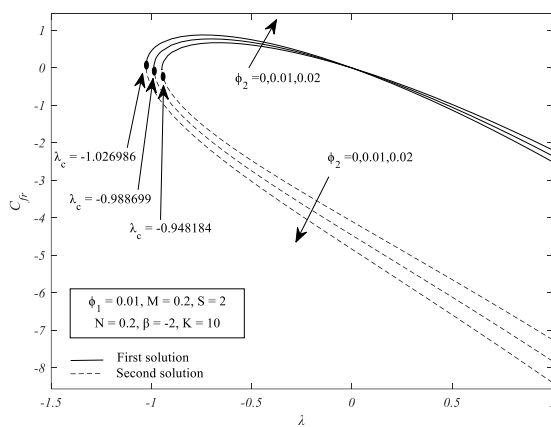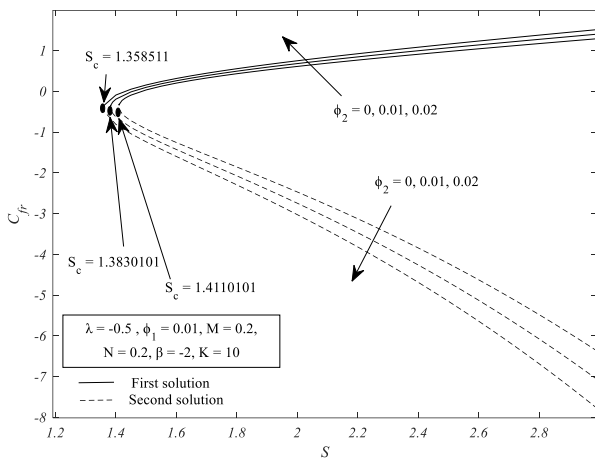

Fig. 2. Skin friction, C_{fr} against λ for distinct values of copper volume fraction, ϕ_2
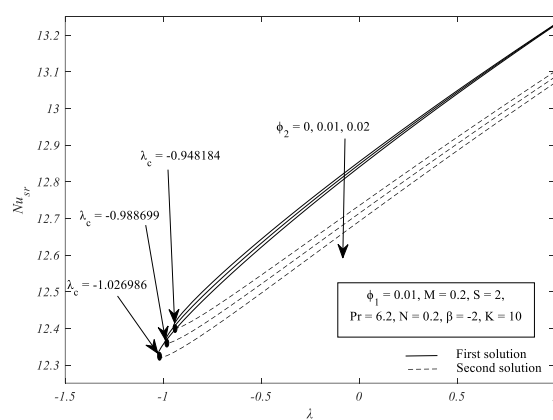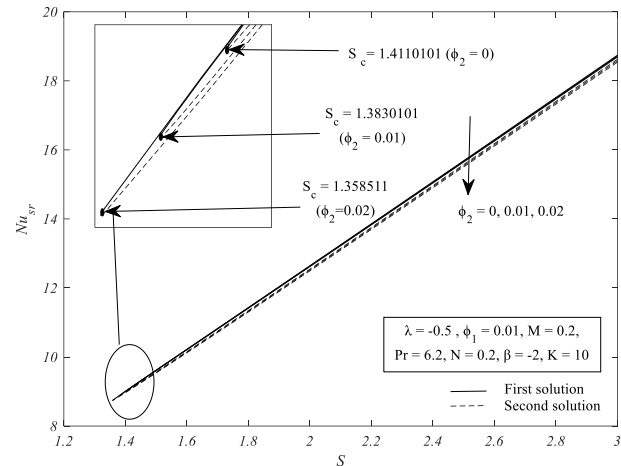
Fig. 3. Local Nusselt number, Nu_{sr} against λ, for distinct values of copper volume fraction, ϕ_2

Figure 4 and Figure 5 depict the sequel of the physical interest of this study towards suction S with three distinctive values of ϕ_2. In Figure 4, the slightly changes in viscosity of nanoparticles exhibits strong suction within the shrinking region and generate the emergence of the dual solution i.e.: $\phi_2 = 0 \left(S_c = 1.4110101\right)$, $\phi_2 = 0.01 \left(S_c = 1.3830101\right)$ and $\phi_2 = 0.02 \left(S_c = 1.358511\right)$. This explains that the concentration of nanoparticles increases, causing substantial obstructions to the fluid's ability to flow freely, resulting in a higher level of suction required. However, the distribution of Nu_{sr} shows an opposed trend when ϕ_2 is added for both first and second solution regardless of the direction of the sheet. This result seems deviate with the findings from Sarkar *et al.,* [33] but

concurred with the findings from Khashi'ie *et al.,* [34] on the role of suction with the increment of ϕ_2 in related to the heat transfer process. The addendum of ϕ_2 from 1% to 2% shorten the magnitude of S_c from 1.3830101 to 1.358511 which manifested that small concentration of ϕ_2 $\left(<1\%\right)$ is sufficient in suspending the process of boundary layer separation.
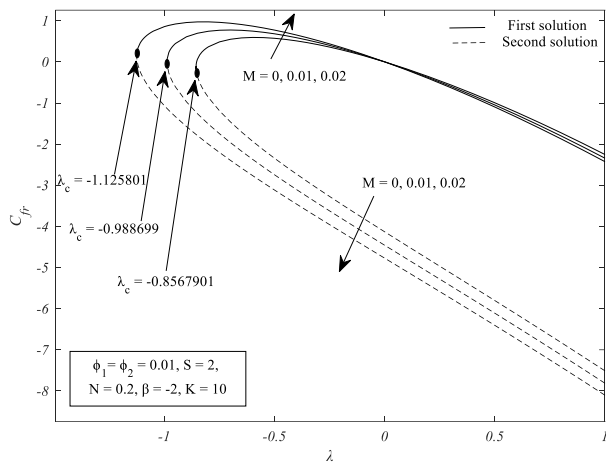

**Fig. 4.** Skin friction, C_{fr} against S for distinct values of copper volume fraction, ϕ_2

Fig. 5. Local Nusselt number, Nu_{sr} against S, for distinct values of copper volume fraction, ϕ_2

Meanwhile, the variation of skin friction coefficient C_{fr} and local Nusselt number Nu_{sr} against stretching/shrinking parameter λ for three distinct values of magnetic field M are deliberated in Figure 6 and Figure 7. Obviously, the strong M widen the range of the solution and also delaying the boundary layer separation process to occur in the shrinking region, ie: $M = 0\left(\lambda_c = -0.8567901\right)$, $M = 0.01\left(\lambda_c = -0.988699\right)$ and $M = 0.02\left(\lambda_c = -1.125801\right)$. Besides, both C_{fr} and Nu_{sr} increased as M is added to the flow. This means that the interaction between the magnetic field and hybrid nanofluid will trigger a resistance force which is called the Lorentz force. The presence of this force impedes the separation of the boundary layer by repelling the fluid motion. Apart from that, the constant governing parameter of S added together with the Lorentz force circulates simultaneously which activates the process of heat transfer.

Fig. 6. Skin friction, C_{fr} against λ for distinct values of magnetic parameter, M
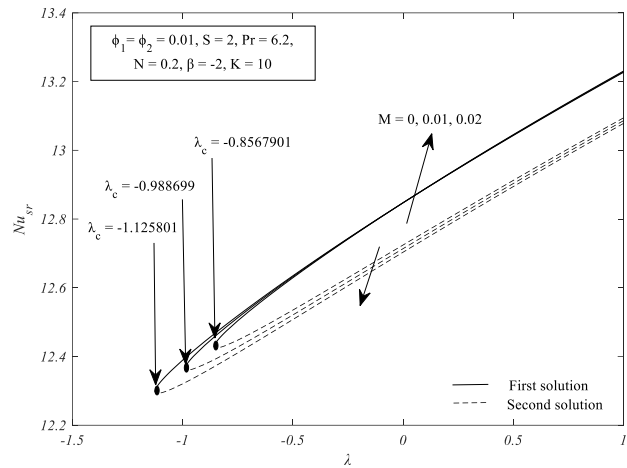

Fig. 7. Local Nusselt number, Nu_{sr} against λ, for distinct values of magnetic parameter, M

Figures 8-11 exhibit the velocity and temperature profiles corresponding to various values of ϕ_2 for the hybrid nanofluid, as well as the suction parameter S. In certain circumstances for the velocity profile, the increasing of ϕ_2 and S resulted in an upsurge for the first solution, however, a contradicting behavior was noticed for the second solution. The selected value of $\lambda \approx 0.5$ which is close to the critical value for the justification of the resulting dual solutions obtained from the figures. On a curved surface, fluid particles experience varying flow velocities due to curvature effects. The increased shear stress near the surface due to higher viscosity may lead to a reduced momentum boundary layer thickness, particularly on the convex side of the curved surface. Simultaneously, the enhanced thermal conductivity from nanoparticles can result in a broader thermal boundary layer, particularly on the concave side of the curved surface.
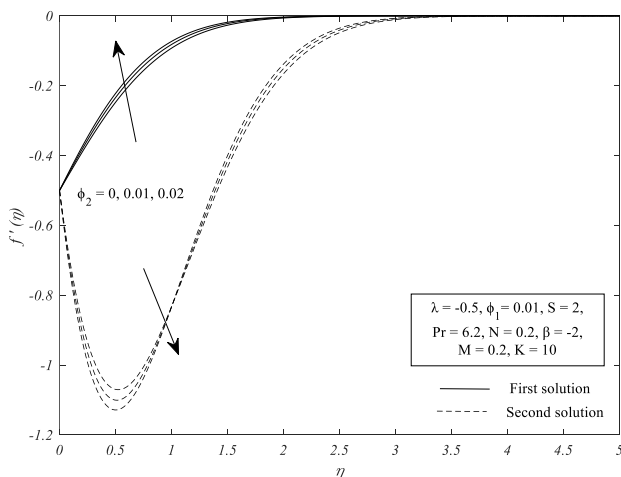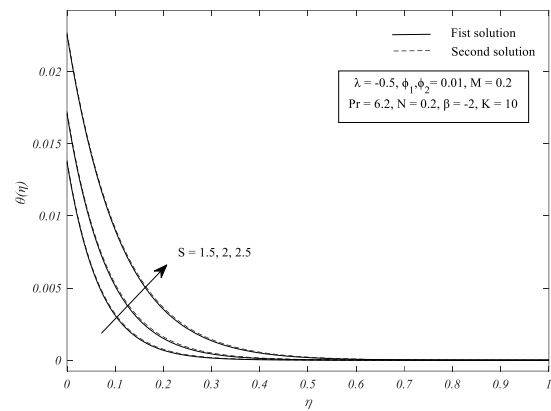

Fig. 8. Velocity profile, $f'(\eta)$ for distinct values copper volume fraction, ϕ_2
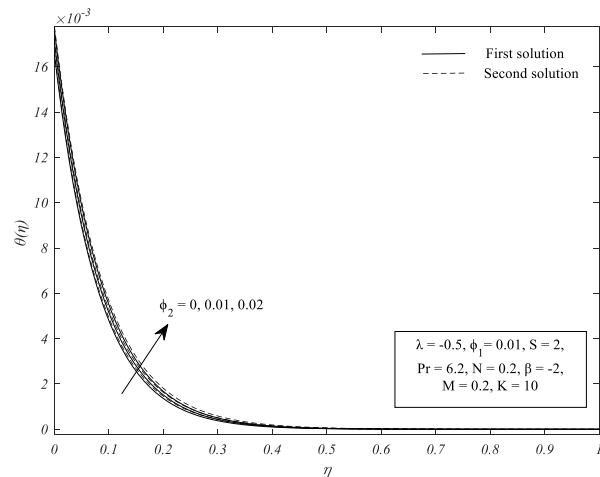
Fig. 9. Temperature profile, $\theta(\eta)$ for distinct values copper volume fraction, ϕ_2
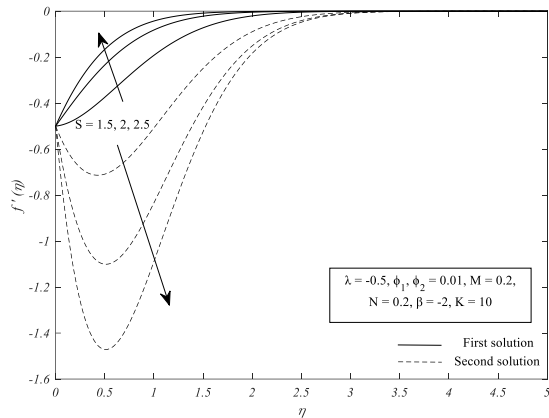
Fig. 10. Velocity profile, $f'(\eta)$ for distinct values of suction, S

Fig. 11. Temperature profile, $\theta(\eta)$ for distinct values of suction, S

Figure 12 displays the temperature profile for the effect of N. By observation, adding the Newtonian heating parameter increases the temperature of the fluid. This may cause a reduction in the fluids' density and amplify the surface temperature. Additionally, the fluid will have more resistance and will produce more heat. As more heat is transferred into the fluid, the temperature profile within the fluid changes. This can lead to a wider thermal boundary layer. All the figures asymptotically fulfilled the boundary condition Eq. (9).
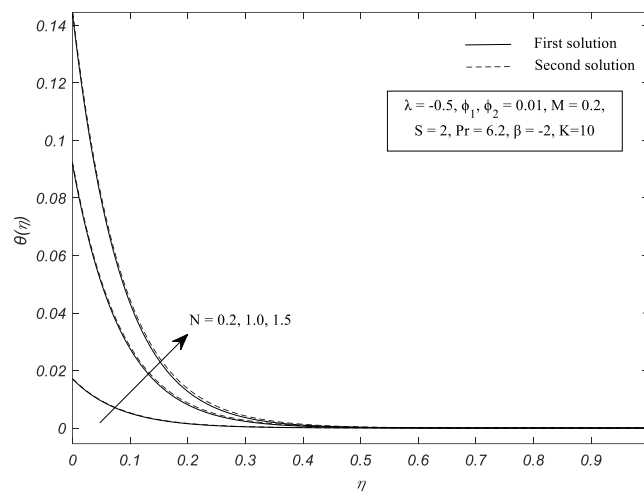
Fig. 12. Temperature profile, $\theta(\eta)$ for different values of newtonian heating parameter, N

Result for the Nusselt number, Nu_{sr} with three distinct values of Newtonian heating parameter N and several values of stretching/shrinking parameter λ was tabularized in Table 4. It explicates that the increasing value of λ has increase the value of the Nusselt number, Nu_{sr} for both first and second solution. It is worth mentioning that Table 4 indicates the analysis done from Figure 12.

Table 4

Result for Nusselt number, Nu_{sr} with three distinct values of the newtonian heating parameter, N through stretching/shrinking parameter, λ

N	λ	Nusselt number, Nu_{sr}	
		First solution	**Second solution**
0.2	1	13.22888015	13.08637066
	0.5	13.04441517	12.90351932
	0	12.84752271	12.71397837
	-0.5	12.63239237	12.52104346
1.0	1	13.22888022	13.08637072
	0.5	13.04441520	12.90351939
	0	12.84752276	12.71397841
	-0.5	12.63239237	12.52104348
1.5	1	13.22888023	13.08637073
	0.5	13.04441520	12.90351940
	0	12.84752277	12.71397844
	-0.5	12.63239237	12.52104348

4. Conclusions

This innovative study presents a numerical investigation into the flow characteristics of unsteady MHD hybrid nanofluid on a curved surface that is subject to permeable stretching or shrinking, incorporating the effects of Newtonian heating. The use of magnetohydrodynamics is to examine how electrically conducting fluids behave in a magnetic field. Meanwhile, Newtonian heating is a cooling process whereby the internal resistance is significantly less than surface resistance during the heating process. For better understanding, the summarized findings are as below:

i. The dual solution exists at both stretching/shrinking region.
ii. The utilization of hybrid nanofluid $\left(\phi_{hnf}\right)$ and M enhance the range of achievable solutions and mitigates the occurrence of boundary layer separation, thereby decelerating the process.
iii. The skin friction and Nusselt number exhibit a positive enhancement in magnitude with the inclusion of M .
iv. The Newtonian heating parameter N elevates the temperature of the hybrid nanofluid flow.

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