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Newtonian Heating in Magnetohydrodynamic (MHD) Hybrid Nanofluid Flow Near the Stagnation Point over Nonlinear Stretching and Shrinking Sheet

Nurwardah Mohd Puzi[1], Mashitah Aziz[1], Nur Syazana Anuar[1,*], Norfifah Bachok[2,3], Ioan Pop[4]

[1] School of Mathematical Sciences, College of Computing, Informatics and Mathematics, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia
[2] Department of Mathematics & Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia
[3] Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia
[4] Department of Mathematics, Babes-Bolyai University, 400084 Cluj-Napoca, Romania

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ABSTRACT

Hybrid nanofluids have demonstrated superior heat transfer performance in numerous applications. However, there remains a need for further research to broaden the scope of their potential applications. The unique behavior of hybrid nanofluids, driven by their potential for improved thermal efficiency, continues to be a focal point of investigation and exploration. This study focuses on the effects of Newtonian heating in MHD hybrid nanofluid near the stagnation point over a nonlinear stretching/shrinking sheet. The Tiwari and Das model, which is a single-phase model, was used to develop the mathematical model. The base fluid and the nanoparticles are assumed to be in thermal equilibrium; hence there is no thermal slip between them. The combination of metal (Cu) and metal oxide (Al_2O_3) nanoparticles with water (H_2O) as the base fluid is used for the analysis. Furthermore, the governing equations are transformed using a similarity transformation technique into similarity equations, which are then solved numerically using a bvp4c function in MATLAB software. Numerical comparison with the published literature is conducted to validate the numerical results, and excellent agreement is found. The impact of physical parameters on the velocity, temperature, skin friction, and local Nusselt number is graphically deliberated. The outcomes suggest that non-unique solutions are found in a specific range of the shrinking parameter. It is also observed that increasing Cu (copper) nanoparticle volume fractions cause an increase in the skin friction coefficient and the local Nusselt number. The presence of magnetic and nonlinear parameters widens the range of solutions to exist while different observation is noticed with an increase in the volume fraction of Cu. Other than that, it has been shown that the Nusselt number increases as the magnetic parameter increases. Lastly, the rise of Newtonian heating contributes to an increase in the temperature profile. This investigation is crucial for understanding the thermal behavior of Cu-Al_2O_3/ H_2O under the influence of physical factors like a magnetic field and Newtonian heating.

* *Corresponding author.*
E-mail address: nursyazana931@uitm.edu.my (Nur Syazana Anuar)

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1. Introduction

The performance of natural convective heat transfer may be impacted by the magnetic field's presence in several industries and has become the basis for various industrial, scientific, and technical applications, including the electronics sector, cooling of nuclear reactors by liquid sodium, and induction flow meter. Furthermore, magnetohydrodynamic (MHD) controls the metallurgical procedures and boundary layer flow. Based on all these significant benefits, analysts and researchers monitor MHD flows continuously. Pavlov [1] explored the MHD boundary layer flow of an electrically conducting fluid caused by a stretched sheet and obtained an exact similarity solution. Ishak *et al.,* [2] considered the flow due to an extensible moving sheet in an electrically conducting fluid. Merkin and Kumaran [3] studied the unsteady MHD boundary-layer flow on a shrinking sheet, and they concluded that the nature of the solution depends on a dimensionless magnetic parameter. Mabood *et al.,* [4] investigated MHD flow and heat transfer of nanofluid over a nonlinearly stretching sheet using the Runge–Kutta–Fehlberg fourth, fifth order method. The magnetic field and chemical reaction effect on power-law fluid over an axisymmetric stretched sheet was studied by Jamalabadi [5]. Subsequently, a few recent investigators [6 - 8] have been in this direction.

Convective heat transfer is the most critical mechanism for heat transfer in a fluid flow. Convection can be improved by changing the boundary conditions, the flow geometry or by increasing the thermal conductivity. Therefore, nanometer-sized metallic particles were introduced by Choi and Eastman [9] to industrial fluids, known as "Nanofluids," to improve their heat transfer properties. Khanafer *et al.,* [10] appeared to be the first to investigate the heat transfer performance of nanofluids inside an enclosure while considering solid particle dispersion. Since then, various researchers have examined different fluid models in the presence of nanoparticles, including Buongiorno [11], Tiwari and Das [12], and Nield and Kuznetsov [13]. Furthermore, a different type of nanofluid, called a hybrid nanofluid, is being studied to boost the heat transfer performance even more. Suresh *et al.,* [14] and Momin [15] have published new experimental work on improving the thermal conductivity of the base fluid. Afterward, Devi and Devi [16] proposed thermophysical property correlations for hybrid nanofluids while studying a boundary layer flow problem. In the relevant literature, Nadeem *et al.,* [17] investigated the characteristics of hybrid nanofluids in three-dimensional stagnation point flow. They determined that hybrid nanofluids' rate and thermal transformation are considerably more than simple nanofluids. As a result, the researcher puts in much work and is very interested in hybrid nanofluids. Since then, there has been an excellent analysis of hybrid nanofluid (see [18 – 21])

The most recent heating phenomenon, known as Newtonian heating (NH), was first described by Merkin [22]. In this phenomenon, the heat transfer rate from a bounding surface with a finite heat capacity is proportional to the local surface temperature and is commonly referred to as conjugate convective flow. Due to their practical applications in a variety of engineering devices, such as the design of heat exchangers, conjugate heat transfer around fins, and convection flow setup where bounding surfaces absorb heat from solar radiations, these effects have been used by numerous researchers (Hayat *et al.,* [23]). The impact of Newtonian heating on a radiating hydromagnetic flow past an impulsively moving infinite vertical plate in the presence of a Hall current was investigated by Reddy [24]. Recently, Gangadhar *et al.,* [25] observed that Newtonian heating causes an increase in temperature in their work. Other recent investigations reported by authors which relevant to this topic can be found in the references [26 – 28].

Due to its significance in practical circumstances such as polymer sheet extractors, microfluidics, space acoustics, glass blowing, and avionics, the flow produced by stretching/shrinking a sheet is a fundamental element in fluid mechanics. Most of the literature focuses on analyzing boundary layer flow with a linear surface; however, stretching/shrinking is not always linear, and only a few researchers have examined nonlinear stretching/shrinking sheets. In view of this, Rana and Bhargava [29] studied a nanofluid's flow and heat transfer over a nonlinearly stretching sheet, while Mabood and Ismail [30] extended their work by considering the effect of MHD. Later, Anuar *at et*. [31] analyzed the nonlinear stretching/shrinking sheet in a hybrid nanofluid and concluded that the nonlinear parameter delays the flow separation, while a non-unique solution exists for the shrinking case. However, only a few attempts have been made in this direction [32 – 34].

Stagnation point flow describes the flow near a stagnation streamline, formed due to a flow impinging on a surface orthogonally. The study on the stagnation point flow has increased significantly as the stagnation point flow involves the interaction of quite a lot of physical problems of polymer industrial applications. The idea of stagnation point flow was initially proposed by Hiemenz [35] in 1911. Furthermore, Rawat and Kumar [36] investigated the Cu–water nanofluid stagnation point flow past a stretching/shrinking sheet using Cattaneo–Christov model in presence of heat generation/absorption. Following that, Negi *et al.,* [37] gave insights into the heat transfer rate of nanofluid in a porous medium near the stagnation point over a stretching sheet. Yaseen et al, [38] studied the unsteady buoyancy and stagnation point flow incorporated by Ag-TiO_2 hybrid nanoparticles towards a spinning disk with Hall effects.

The literature mentioned above states that hybrid nanofluid holds significant scientific potential across various domains, including technology and industry, thanks to its superior heat transfer rate compared to conventional fluids. Previous studies have considered the flow of hybrid nanofluid over the nonlinearly stretching and shrinking surface. However, to enhance the practical applicability of the current model, the authors have incorporated the MHD effect and Newtonian heating at the surface. The nonlinear partial differential equations (PDEs) system defines these flow models well. The similarity variable is applied to the partial differential equations, which are converted into ordinary differential equations. MATLAB software was used to carry out the calculations. Graphs were also employed to show how emerging parameters affected fluid temperature and velocity. In addition, the critical point of boundary layer separation is recorded for each control parameter, serving as a guideline to control flow transition, and the appropriate ranges of parameter configurations are also presented to achieve dual solutions successfully. These findings serve as a significant milestone in expanding the understanding of hybrid nanofluid and set a benchmark for future research in this area.

2. Problem Formulation

We considered a two-dimensional and steady flow of a hybrid nanofluid past a nonlinear stretching and shrinking surface with the velocity $u_w\left(x\right)=ax^n$ (where $a>0$ and n is the nonlinear parameter) in the stagnation point region. The coordinate system (x,y) is chosen as x-axis measures along the direction of the horizontal surface while y-axis is perpendicular to it. The dispersion of copper (Cu) and alumina (Al_2O_3) nanoparticles into the base fluid water is intended to improve the energy transport mechanism. The physical structure of the fluid flow is displayed in Figure 1. It is assumed that the surface is subjected to Newtonian heating as proposed by Merkin [22], and the fluid is electrically conducted with magnetic field strength in the form $\beta\left(x\right)=b_0x^{\frac{n-1}{2}}$, where b_0 denote the constant magnetic field. The free velocity of the hybrid nanofluid at the free stream is $u_\infty\left(x\right)=bx^n$ with $b>0$ denoting the stagnation point's strength, while the temperature outside the boundary layer is considered as T_∞.

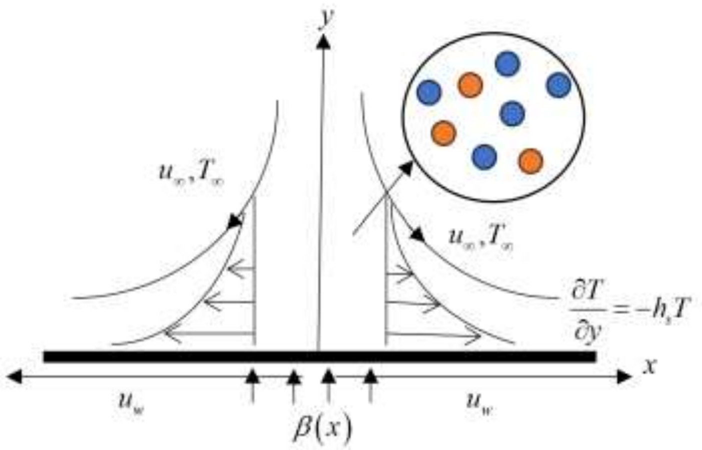

Fig. 1. Schematic illustration of the flow for stretching sheet

The governing boundary layer equations, which regulate the fluid flow and energy transport mechanism, can be defined as follows (see Bachok *et al.,* [39], Anuar *et al.,* [31]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_\infty \frac{\partial u_\infty}{\partial x} + \frac{\mu_{hnf}}{\rho_{hnf}}\frac{\partial^2 u}{\partial y^2} + \frac{\sigma_{hnf}\beta^2(x)}{\rho_{hnf}}(u_\infty - u) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{hnf}\frac{\partial^2 T}{\partial y^2} \tag{3}$$

The restriction at the boundary is executed as follows (see Merkin [22]):

$$\begin{aligned} &u = u_w(x), \quad v = 0, \quad \frac{\partial T}{\partial y} = -h_s T \quad \text{at} \quad y = 0 \\ &\quad u \to u_\infty(x), \quad T \to T_\infty \quad \text{as} \quad y \to \infty \end{aligned} \tag{4}$$

u and v indicate the corresponding component velocities in the respective x- and y- directions, respectively. Now, wherein, the involved mathematical symbols or letters in the equations above such as T, μ, ρ, σ, and α are respectively the temperature, dynamic viscosity, density, electrical conductivity, and thermal diffusivity, where the subscript 'hnf' denotes the hybrid nanofluid. Here, $h_s = h_0 x^{\frac{n-1}{2}}$ is the heat transfer coefficient where h_0 is the constant.

The heat transfer process in fluids is expected to be improved by the current mode. Therefore, the thermophysical properties of hybrid nanofluid's are thus shown in Table 1 (see Devi and Devi [16]).

Table 1
Thermophysical properties of hybrid nanofluid.

Properties	Hybrid nanofluid
Thermal diffusivity	$\alpha_{hnf} = \dfrac{k_{hnf}}{\left(\rho C_p\right)_{hnf}}$
Density	$\rho_{hnf} = \varphi_2 \rho_{s_2} + \left(1-\varphi_2\right)\left[\left(1-\varphi_1\right)\rho_f + \varphi_1 \rho_{s1}\right]$
Dynamic viscosity	$\mu_{hnf} = \dfrac{\mu_f}{\left(1-\varphi_1\right)^{2.5}\left(1-\varphi_2\right)^{2.5}}$
Heat capacity	$\left(\rho C_p\right)_{hnf} = \varphi_2\left(\rho C_p\right)_{s_2} + \left(1-\varphi_2\right)\left[\left(1-\varphi_1\right)\left(\rho C_p\right)_f + \varphi_1\left(\rho C_p\right)_{s1}\right]$
Thermal conductivity	$\dfrac{k_{hnf}}{k_{bf}} = \dfrac{k_{s2} + 2k_{bf} - 2\varphi_2\left(k_{bf} - k_{s2}\right)}{k_{s2} + 2k_{bf} + \varphi_2\left(k_{bf} - k_{s2}\right)}$ where $\dfrac{k_{bf}}{k_f} = \dfrac{k_{s1} + 2k_f - 2\varphi_1\left(k_f - k_{s1}\right)}{k_{s1} + 2k_f + \varphi_1\left(k_f - k_{s1}\right)}$
Electrical conductivity	$\dfrac{\sigma_{hnf}}{\sigma_{bf}} = \dfrac{\sigma_{s2}\left(1+2\varphi_2\right) + 2\sigma_{bf}\left(1-\varphi_2\right)}{\sigma_{s2}\left(1-\varphi_2\right) + \sigma_{bf}\left(2+\varphi_2\right)}$ where $\dfrac{\sigma_{bf}}{\sigma_f} = \dfrac{\sigma_{s1}\left(1+2\varphi_1\right) + 2\sigma_f\left(1-\varphi_1\right)}{\sigma_{s1}\left(1-\varphi_1\right) + \sigma_f\left(2+\varphi_1\right)}$

where k signifies the thermal conductivity, ρC_p is the heat capacity and φ refers to the nanoparticle volume fraction. While the subscripts bf , $s1$ and $s2$ denote the base fluid, Al_2O_3 and Cu nanoparticle. The purpose of this study was to examine the steady boundary layer equations for hybrid nanofluids, which were formed by mixing Cu nanoparticles $\left(\varphi_2\right)$ with 0.01 volume of $A1_2O_3$ nanoparticle $\left(\varphi_1\right)$ in a base fluid water. In this research, the thermophysical properties of Devi and Devi [16] were used since it has been demonstrated that their results are in excellent agreement with the experimental findings of Suresh *et al.,* [14]. Consequently, we predicted that these results would provide useful guidance and understanding for increasing the heat transfer rate. The valuable thermo-physical characteristics of nanoparticles and base fluid are presented in Table 2 (see Oztop and Abu-Nada [40] and Khanafer *et al.,* [10]).

Table 2
Thermophysical properties of nanoparticles and base fluid.

Physical Properties	Water	Cu	Al_2O_3
$C_p\left(J/kgK\right)$	4179	385	765
$\rho(kg/m^3)$	997.1	8933	3970
$k\left(W/mK\right)$	0.613	400	40
$\sigma\left(s/m\right)$	0.05	5.96 x 10^7	3.69 x 10^7

To get the similarity solution, we utilize the following similarity transformation (see Malvandi *et al.,* [41]):

$$\eta = \left(\frac{(n+1)b}{2\nu_f}\right)^{\frac{1}{2}} y x^{\frac{n-1}{2}}, \ \psi = \left(\frac{2b\nu_f}{n+1}\right)^{\frac{1}{2}} x^{\frac{n+1}{2}} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_\infty} \tag{5}$$

where η is the similarity variable, ψ is the stream function and ν_f is the kinematic viscosity, while f and θ are the dimensionless velocity and temperature function. A stream function $\psi(x, y)$ is introduced as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$.

Furthermore, using the similarity transformation (5) in the aforementioned governing equations, the first equation is automatically satisfied while the remaining equations (2)-(4) respectively become

$$\frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f} f''' + ff'' - \left(\frac{2n}{n+1}\right)\left(f'^2 - 1\right) + \frac{\sigma_{hnf}/\sigma_f}{\rho_{hnf}/\rho_f}\left(\frac{2M}{n+1}\right)(1 - f') = 0 \tag{6}$$

$$\frac{1}{\Pr}\frac{k_{hnf}/k_f}{\left(\rho C_p\right)_{hnf}/\left(\rho C_p\right)_f}\theta'' + f\theta' = 0 \tag{7}$$

and the boundary conditions are:

$$\begin{gathered} f(0) = 0, \quad f'(0) = \lambda, \quad \theta'(0) = -\gamma\left(1 + \theta(0)\right), \\ f'(\infty) \to 1, \quad \theta(\infty) \to 0 \quad \text{as} \quad \eta \to \infty \end{gathered} \tag{8}$$

In the above equations, the involved physical parameters such as magnetic parameter M, stretching/shrinking parameter λ and Newtonian heating γ are expressed as:

$$M = \frac{b_0^2\sigma_f}{b\rho_f}, \quad \lambda = \frac{a}{b}, \quad \gamma = h_0\left(\frac{2\nu_f}{(n+1)b}\right)^{\frac{1}{2}} \tag{9}$$

3. Physical Quantities

The physical quantities of interest are skin friction coefficient, C_f and Nusselt number Nu_x. The mathematical expressions for the drag force and heat transfer analysis are defined as:

$$C_f = \frac{\tau_w}{\rho_f u_\infty^2} \quad \text{and} \quad Nu_x = \frac{xq_w}{k_f\left(T - T_\infty\right)} \tag{10}$$

where $\tau_w = \mu_{hnf}\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is the shear stress and $q_w = -k_{hnf}\left(\frac{\partial T}{\partial y}\right)_{y=0}$ is the heat flux at the surface. Utilizing the predefined similarity relations, expressions (10) become

$$C_f\left(\mathrm{Re}_x\right)^{\frac{1}{2}} = \frac{\mu_{hnf}}{\mu_f}\sqrt{\frac{n+1}{2}}f''(0) \quad \text{and} \quad Nu_x\left(\mathrm{Re}_x\right)^{-\frac{1}{2}} = \frac{k_{hnf}}{k_f}\gamma\left(1 + \frac{1}{\theta(0)}\right)\sqrt{\frac{n+1}{2}} \tag{11}$$

Here, $\mathrm{Re}_x = u_\infty x/\nu_f$ is the local Reynold number.

4. Results and Discussion

4.1 Validation of Results

Bvp4c solver in Matlab software is used to perform the solutions for the ODEs (6) and (7) together with boundary conditions (8). This bvp4c function is designed using the 3-stage Lobatto IIIa formula under the finite difference scheme with fourth order accuracy (see Shampine *et al.,* [42]). Before further computations are made with the MATLAB solver, this study was validated by comparing them to previously published papers. This step was taken to ensure the accuracy and reliability of the methods used in the research. The numerical comparison values of the skin friction $f''(0)$ are tabulated in Table 3. The comparative analysis is made with Wang [43], Bachok *et al.,* [39], and Anuar *et al.,* [31] for different stretching/shrinking parameters when $n=1$, $M=0$ and $\gamma=0$ in a regular fluid $(\varphi_1=\varphi_2=0)$. From table 3, a good agreement has been obtained, where the values in brackets signify the second solution, and therefore the current numerical scheme is reliable and can be used for subsequent calculations.

Table 3
Values of $f''(0)$ for different λ when $n=1$ and $M=0, \varphi_1=\varphi_2=0$.

λ	Wang [43]	Bachok *et al.,* [39]	Anuar *et al.,* [31]	Present Result
2	-1.88731	-1.887307	-1.887307	-1.887307
1	0	0	0	0
0.5	0.71330	0.713295	0.713295	0.713295
0	1.23588	1.232588	1.232588	1.232588
-0.5	1.49567	1.495670	1.495670	1.495670
-1	1.32882 [0]	1.328817 [0]	1.328817	1.328817 [0]
-1.15	1.08223 [0.116702]	1.082231 [0.116702]	1.082231 [0.116702]	1.082231 [0.116702]
-1.2		0.932473 [0.233650]	0.932473 [0.233650]	0.932473 [0.233650]
-1.2465	0.55430	0.584281 [0.554297]	0.584282 [0.554296]	0.584281 [0.554296]

"[ ]" second solution

4.2 Analysis of Results

The local skin friction coefficient $C_f \mathrm{Re}_x^{1/2}$ and the local Nusselt number $Nu_x \mathrm{Re}_x^{-1/2}$ with stretching/shrinking parameter λ for selected value of Copper nanoparticle φ_2 when $\varphi_1=0.01, \gamma=0.2, M=0.1, n=2$ are plotted in Figure 2 and 3. The Prandtl number Pr is set to be 6.2 (Khanafer *et al.,* [10] and Oztop *et al.,* [40]) which represents water as a base fluid. The solid lines in these figures correspond to the first solution, while the dotted lines correspond to the second solution. The first solution shows a significant difference, whereas the second solution shows only a minor difference. It is noted that the solutions are not unique for a certain stretching/shrinking parameter. The dual solution range for Al_2O_3/water nanofluid $(\varphi_1=0.01, \varphi_2=0)$ is $-1.397715<\lambda<-1$, Cu-Al_2O_3/water hybrid nanofluid $(\varphi_1=\varphi_2=0.01)$ is $-1.395637<\lambda<-1$, Cu-Al_2O_3/water hybrid nanofluid $(\varphi_1=0.01, \varphi_2=0.02)$ is $-1.393861<\lambda<-1$. Hence a larger range of dual solutions is observed with Al_2O_3/water nanofluid. In addition, due to the separation of the boundary layer from the surface, no solutions are obtained when $\lambda < \lambda_c$; λ_c is the critical point where the first and second solution is connected. It has been observed that the solution domain expands with the decrease of φ_2, where the critical values λ_c tend to move slightly to the right. The figures also show that when $\lambda \geq -1$, a unique solution exists. From Figures 2 and 3, it is noted that values of skin friction coefficient and local Nusselt number increases as the value of Copper nanoparticle increases. Next, Figures 4 and 5 are plotted to describe the effect of φ_2 towards the velocity $f'(\eta)$ and temperature $\theta(\eta)$ profiles. The result shows that by the increment of Copper nanoparticle from 0 to 0.01, the momentum and thermal boundary layer thickness reduce for the shrinking case $(\lambda = -1.3)$, therefore the velocity and temperature of the fluid declined for both solutions. The fluid motion is slowed down due to the reason of the boost up of the viscosity in the fluid when the fluid is concentrated with the nanoparticle. However, increasing the Copper nanoparticle concentration causes the magnetic hybrid nanofluid's temperature to descend for the shrinking case. The increment values of skin friction can be attributed to the increased presence of solid particles. These particles create more points of interaction with the fluid flow, causing resistance and energy loss. The irregularities on the nonlinear surface, combined with the presence of nanoparticles, intensify the frictional forces exerted on the fluid flow, leading to an overall rise in skin friction rate. Simultaneously, the rise in the local Nusselt number signifies enhanced heat transfer near the surface. The nanoparticles' accumulation can enhance thermal conductivity and convective heat transfer, promoting the exchange of heat between the fluid and the surface. In addition, these profiles satisfy the boundary condition (8) and converge asymptotically. It is also observed that the first solution has a thinner boundary layer than the second solution.

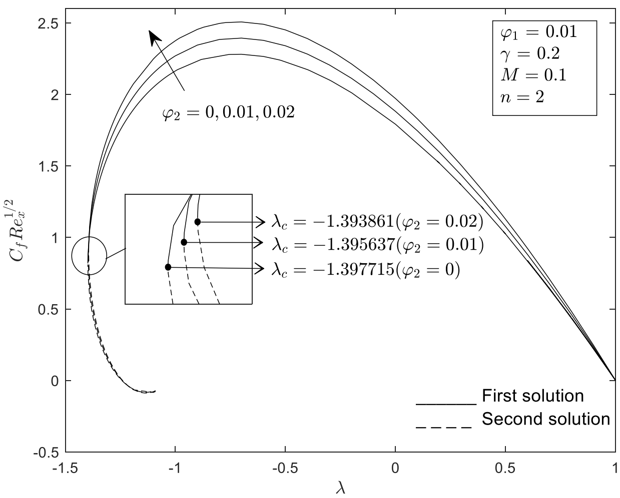

Fig. 2. Local skin friction $C_f \mathrm{Re}_x^{1/2}$ with λ for selected values of φ_2

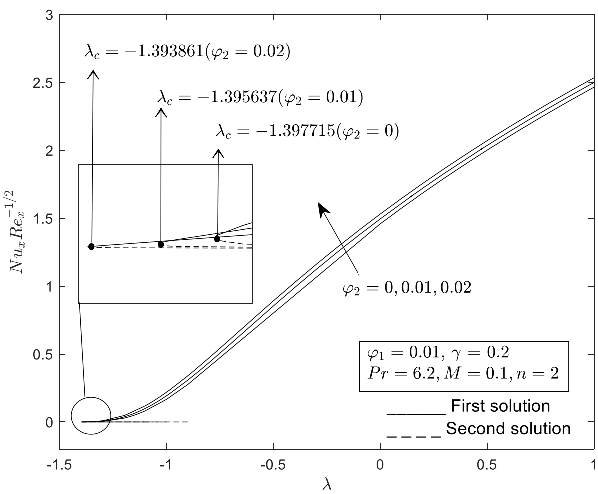

Fig. 3. Local Nusselt number $Nu_x \mathrm{Re}_x^{-1/2}$ with λ for selected values of φ_2

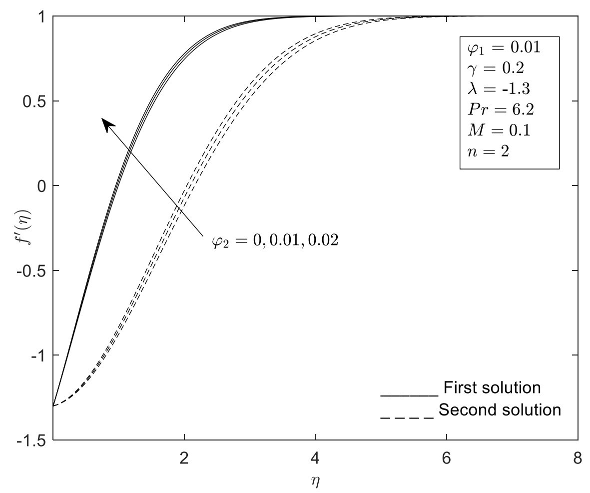

Fig. 4. Velocity profile $f'(\eta)$ for selected values of φ_2

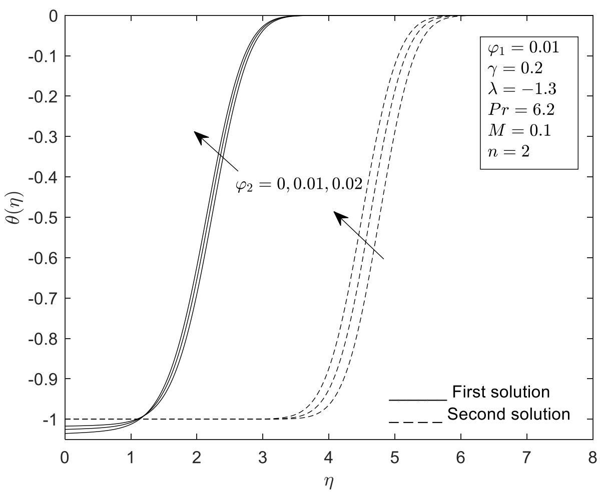

Fig. 5. Temperature profile $\theta(\eta)$ for selected values of φ_2

Figures 6 - 9 unveil the impact of magnetic parameter M on the local skin friction $C_f \mathrm{Re}_x^{1/2}$ and Nusselt number $Nu_x \mathrm{Re}_x^{-1/2}$ against stretching/shrinking parameter λ, velocity $f'(\eta)$ and temperature $\theta(\eta)$ for Cu-Al_2O_3/water hybrid nanofluid $(\varphi_1 = \varphi_2 = 0.01)$ when $\gamma = 0.2$ and $n = 2$. The same observation is noticed for the existence of dual solution. Here, the critical values increase by enlarging the value of magnetic parameter which consequently delays the separation of boundary layer to happen. Therefore, the critical values of λ when $M = 0, 0.1$ and 0.2 are $-1.349699, -1.394999$ and -1.441583. In addition, both skin friction coefficient and heat transfer rate increase with the accretion of magnetic parameter, M (see in Figures 6 and 7). The Lorenz force, arising from the interaction between the magnetic field and moving charged particles in the fluid, becomes more pronounced as the magnetic parameter increases. The increase of the retarding force upsurge in the hybrid nanofluid is associated with the increase of M, which magnify the motion of the hybrid nanoparticles and results in a thinning momentum boundary layer thickness (see in Figure 8). Further, Figure 9 shows the thermal boundary layer becomes thinner as the transverse magnetic field value rises. Interestingly, heat is actively produced by the generation of Lorentz force, which raises the temperature of the hybrid nanofluid while circulating an active heat transfer process.

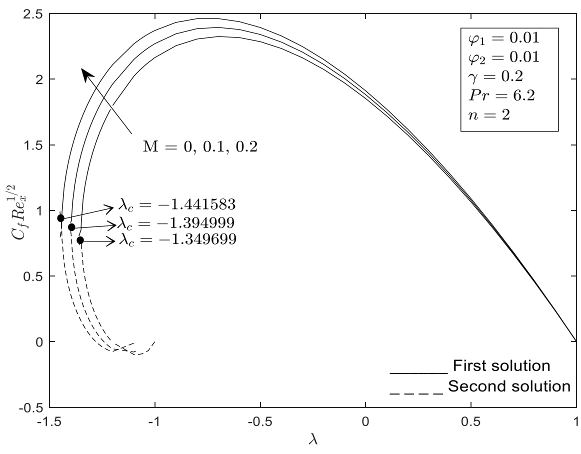

Fig. 6. Local skin friction $C_f \mathrm{Re}_x^{1/2}$ with λ for selected values of M

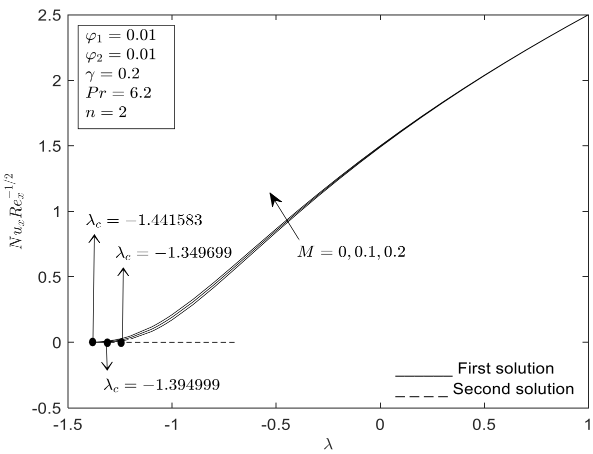

Fig. 7. Local Nusselt number $Nu_x \mathrm{Re}_x^{-1/2}$ with λ for selected values of M

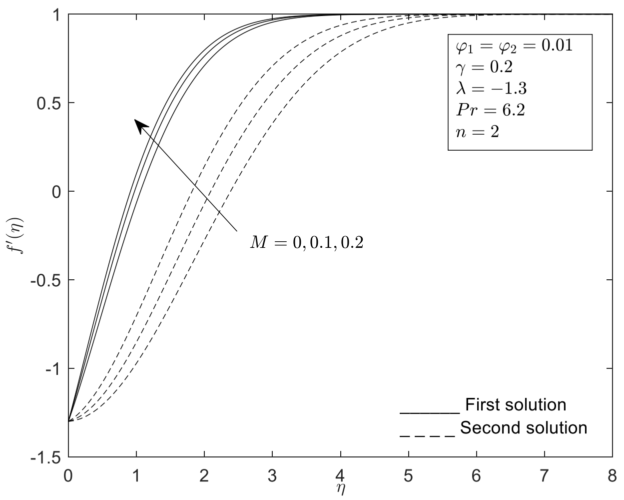

Fig. 8. Velocity profile $f'(\eta)$ for selected values of M

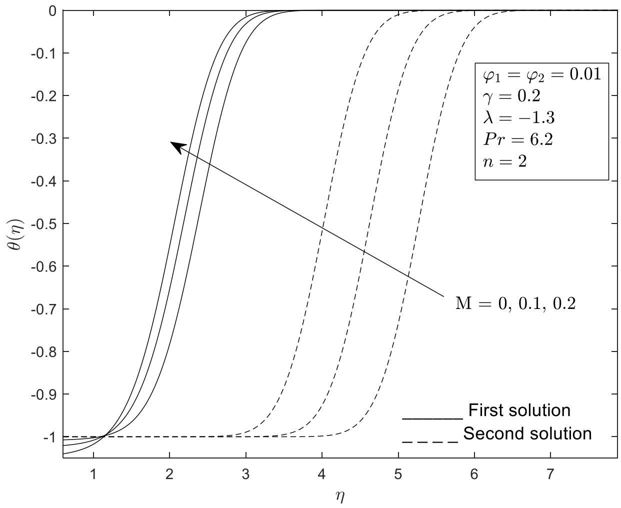

Fig. 9. Temperature profile $\theta(\eta)$ for selected values of M

Aspect of nonlinear parameter n on the local skin friction $C_f \operatorname{Re}_x^{1/2}$, local Nusselt number $Nu_x \operatorname{Re}_x^{-1/2}$, velocity $f'(\eta)$ and temperature $\theta(\eta)$ profiles are described in Figures 10 – 13 for magnetic hybrid nanofluid when $\gamma = 0.2$. It is observed that the local skin friction $C_f \operatorname{Re}_x^{1/2}$ and local Nusselt number $Nu_x \operatorname{Re}_x^{-1/2}$ increase with increasing value of nonlinear parameter n. It is important to note that $n=1$ denotes the linear sheet while $n=2$ and 4 represent the nonlinear sheet. Obviously, the range of the similarity solution is widening as shown in the Figures 10 and 11. For the varying value of the nonlinear parameter $(n=1, 2, 4)$, we have found the following critical values such as $-1.331122, -1.392999$ and -1.435999. It is observed that the velocity and temperature profiles of the magnetic hybrid nanofluid is insignificantly reduces with increasing values of n. The irregularities caused by the nonlinear parameter provoke changes in fluid momentum, intensifying shear stresses at the fluid-solid interface. This leads to an increase in local skin friction. Meanwhile, the changing surface conditions caused by the nonlinear parameter perturb the fluid flow. This perturbation enhances the convective heat transfer rate, leading to a higher local Nusselt number.

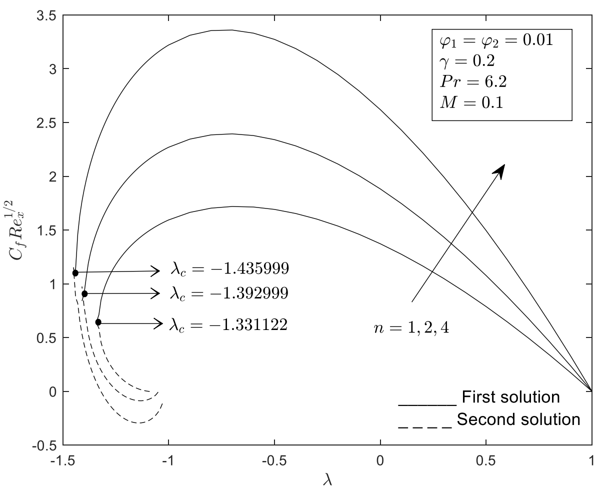

Fig. 10. Local skin friction $C_f \operatorname{Re}_x^{1/2}$ with λ for selected values of n

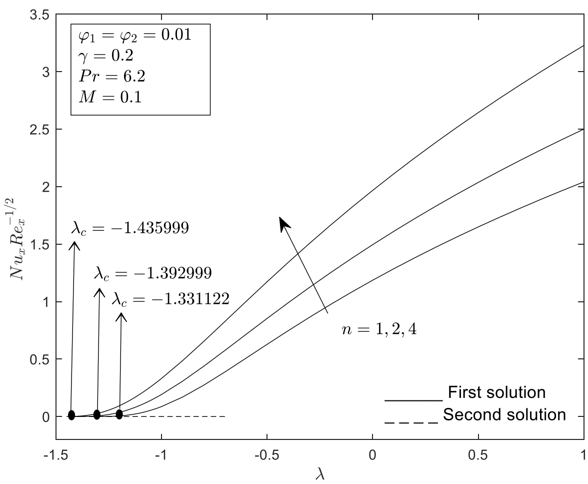

Fig. 11. Local Nusselt number $Nu_x \operatorname{Re}_x^{-1/2}$ with λ for selected values of n

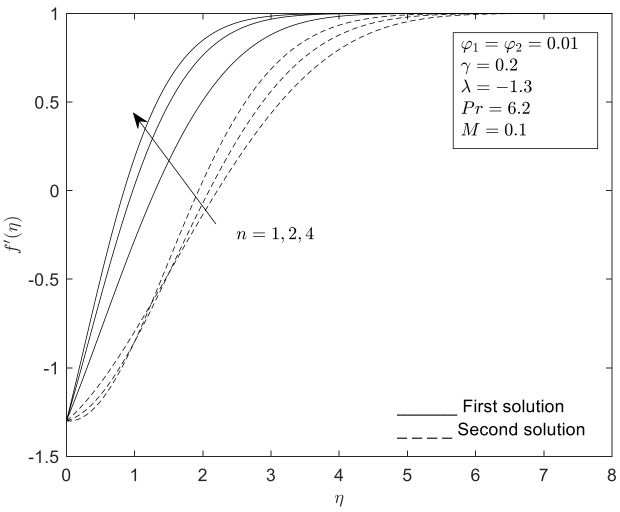

Fig. 12. Velocity profile $f'(\eta)$ for selected values of n

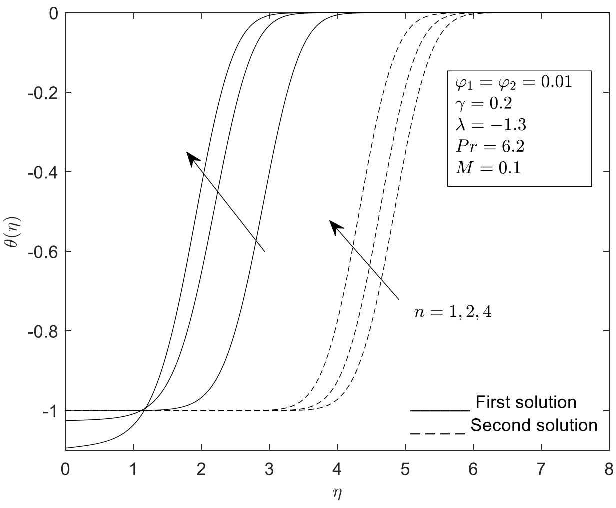

Fig. 13. Temperature profile $\theta(\eta)$ for selected values of n

Figure 14 shows temperature profile $\theta(\eta)$ for selected values of Newtonian heating γ for magnetic hybrid nanofluid. It is discovered that the rise of γ contribute to decrease of thermal boundary layer, however no large variation is seen in the second solution. The Newtonian heating parameter has no effects on the velocity profile and local skin friction as the momentum equation (6) is independent of this parameter. As $\gamma \to \infty$, the Newtonian heating condition becomes prescribed wall temperature case, and the fluid temperature tends to zero. Table 4 gives the values of $Nu_x \mathrm{Re}_x^{-1/2}$ at the different values of stretching/shrinking λ and Newtonian heating γ parameters. The Nusselt number increase with stretching/shrinking parameter λ whereas it decreases with increasing Newtonian heating parameter γ. It is obvious that stronger convective heating allows the thermal effect to penetrate deeper into the fluid. The Newtonian heating parameter is proportional to the conjugate heat transfer coefficient. When we increase the Newtonian heating parameter, the temperature rises.

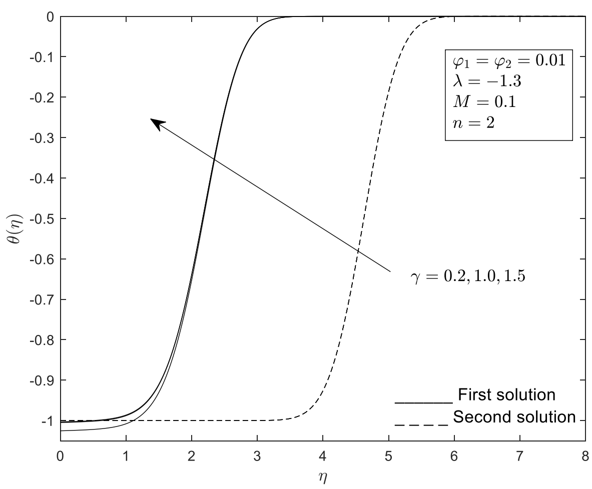

Fig. 14. Temperature profile $\theta(\eta)$ for selected values of γ

Table 4
Values of $Nu_x \mathrm{Re}_x^{-1/2}$ for different γ when $n=2, M=0.1$ for magnetic hybrid nanofluid $(\varphi_1=\varphi_2=0.01)$

γ	λ	$Nu_x \mathrm{Re}_x^{-1/2}$
0.2	1	2.499562565
	0.5	2.036265487
	-0.5	0.853422172
	-1.0	0.192212482
	-1.2	0.036500917
1.0	1	2.499562558
	0.5	2.03626548
	-0.5	0.853422173
	-1.0	0.192212477
	-1.2	0.036500916
1.5	1	2.499562556
	0.5	2.036265476
	-0.5	0.85342218
	-1.0	0.192212477
	-1.2	0.036500916

5. Concluding Remarks

In this paper, the flow and heat transfer over a nonlinear stretching/shrinking sheet in a MHD hybrid nanofluid with Newtonian heating is studied. The numerical technique is used to solve the mathematical model. Hence, the findings of this study can be summarized as follows:

i. Magnetic hybrid nanofluid enhances the heat transfer rate compared with the regular nanofluid.
ii. Increasing of Copper nanoparticle φ_2, magnetic parameter M and nonlinear parameter n enhance the values of local skin friction and local Nusselt number.
iii. Dual solutions exist for a specific limit of the shrinking sheet parameter $\lambda < -1$, whereas unique solution is obtained when $\lambda \geq -1$.
iv. The range of λ for which the dual solutions exist increase as magnetic M and nonlinear n parameters increase while it is decrease as copper nanoparticle volume fraction φ_2 increase.
v. The temperature of fluid exerts a direct relationship towards Newtonian heating and magnetic field. Therefore, the local Nusselt number increases for large values of Newtonian heating and magnetic field parameters.

Based on the findings of this study, it is suggested that the cooling/heating industries can improve heat transfer operation by including a small scale of magnetic field and Newtonian heating effects.

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