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Relationship between Viscosity and Surface Tension as a Solution to Fractional Differential Equations

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ABSTRACT

Viscosity, also known as thickness, is the measure of a fluid's resistance to the stress it experiences, caused by cohesive forces between fluid particles. Surface tension is the force or downward pull caused by unbalanced attractive forces at the liquid interface, leading to the contraction of the liquid surface. Numerous studies have been conducted, particularly on determining the relationship between these two phenomena. However, very few studies are associated with fractional differential equations. This paper presents research results based on empirical data obtained from laboratory testing on viscosity and surface tension and explores their interrelationship. Through scatter plot data and regression of logarithmic functions, the obtained function is claimed as a solution to a first-order linear differential equation, with the graph of its solution matching the actual data. Subsequently, this differential equation will be generalized into a fractional differential equation with guaranteed existence and uniqueness of solutions. The method used to find solutions is the Adomian-Laplace Decomposition, and the result is that the graph of the solution function coincides with the graph of the exact solution. This indicates that the relationship between viscosity and surface tension can be described using a solution derived from a fractional differential equation model.

1. Introduction

Mathematics, particularly in the field of calculus, has experienced rapid development, notably following the introduction of fractional derivatives. This advancement represents an extension of ordinary derivatives, traditionally defined for natural numbers, to include rational orders. Beyond the foundational concepts of fractional integrals and derivatives, substantial progress has been observed in the formulation and application of fractional derivative equations over the past two decades. For example, Mu'lla [1] offers a thorough exploration of about Fractional Calculus, Fractional Differential Equations and Applications. Previously, fractional differential equations themselves had been extensively discussed by Podlubny [2] in "Fractional Differential Equations" (1999). The next

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development has included studies on systems of fractional differential equations as exemplified by Asgari [3] who presented numerical solutions. In terms of concept or definition of fractional derivatives, beyond the well-known Riemann-Liouville, Grundwald-Letnikov, and Caputo approaches, Khalil *et al.,* [4] introduced a novel perspective with the conformable fractional derivative.

To ensure the existence and uniqueness of solutions to fractional differential equations, significant research has yielded several theorems, as documented in previous studies [5-7]. These results encompass both linear and nonlinear forms of fractional differential equations. Once existence and uniqueness are established, the next step is finding the solution of the fractional differential equations. Numerous methods have been developed for this purpose. Assessing the accuracy of numerical methods can be achieved by comparing their results with exact solutions, although exact solutions are often challenging to obtain for certain types of fractional differential equations.

In general, to solve nonlinear fractional differential equations, Arafa *et al.,* [8] have employed the Mittag-Lefler function method, while Elzaki and Chamekh [9] utilized the New Decomposition Method. On the other hand, Johansyah *et al.,* [10,11] have addressed fractional differential equations using the Combined Adomian Decomposition Method with Kamal Integral Transformation. To tackle the Economic Growth Acceleration Model with Memory Effects, the Combined theorem of Adomian Decomposition Methods and Kashuri–Fundo Transformation Methods has been applied. Other methods include the Adomian Decomposition Method by Sumiati *et al.,* [12], Homotopy Asymptotic method by Hamarsheh *et al.,* [13], Laplace Transformation, and so on. A special form of nonlinear fractional differential equations is the Riccati form, whose solutions can be sought using various methods such as the Homotopy Asymptotic Method, Variational Homotopy Perturbation Method, and the Variational Iteration Method [14-18]. The relationship between Bivariate q-Generalized Extreme Value Distribution (BqGEV) and surface tension or viscosity may be observed in models of fluid behavior under extreme conditions or changes in fluidic properties that affect the distribution of observed random variables [19].

The applications of fractional differential equations have significantly expanded across various fields, including economics, as seen in the Black-Scholes equation, mathematical biology, physics, and others [20]. Particularly in the field of physics, extensive research has been conducted on the relationship between viscosity and surface tension. This knowledge helps in predicting fluid flow behavior under various conditions, including turbulent and laminar flow, and in accurate mathematical modeling. Additionally, for scientific research, the relationship between viscosity and surface tension aids in understanding flow phenomena from micro to macro scales, and in building better mathematical models for numerical simulations. Pelofsky [21], in his paper "Surface Tension-Viscosity Relation for Liquids", declared that this relationship can be expressed as

$$\gamma = A \exp\left(-\frac{B}{\eta}\right) \tag{1}$$

where *A* and *B* are constants, γ is the surface tension, and η is viscosity. Earlier research has explored the use of vegetable-based lubricant oils for knee cartilage replacement, along with low surface tension liquids like distilled water and 3% butanol [22,23]. In addition, Ghatee *et al.,* [24] and Ahmari and Amiri [25] discuss the correlation between viscosity and surface tension, while Zheng *et al.,* [26] introduced a novel relationship between viscosity and surface tension as

$$\ln \eta = A + \frac{B}{\gamma^n + C} \tag{2}$$

However, previous studies have not utilized fractional derivatives. Fractional models that describe the relationship between viscosity and surface tension began emerging after 2012, as evidenced by Rusyaman *et al.,* [27,28], Servadei and Valdinoci [29], and Pandey and Holm [30], along with other publications [31–33]. Viscosity pertains to the thickness of liquid substances, while elasticity is commonly used for solids. Viscoelasticity, a combination of both, represents an advancement concept from viscosity alone. Only a limited amount of research has explored this specific area [34–36]. This paper contributes to existing literature by introducing a differential equation model that clarifies the relationship between surface tension in lubricating oil and its viscosity.

2. Materials and Methods

2.1 Fractional Derivative

In this section, we present several basic theories that support the main problem, including fractional derivatives, fractional differential equations, and their unique theorem.

Here are two of several versions of the definition of fractional derivatives.

Definition 1. α-order fractional derivative of $f(t)$ according to Riemann-Liouville is given by

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{-n+\alpha+1}}ds\,, \alpha > 0\,, t > 0. \tag{4}$$

Definition 2. α-order fractional derivative of $f(t)$ according to Caputo is given by

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{-n+\alpha+1}}ds. \tag{5}$$

From the two versions of the definition above, it can be concluded that the fractional derivative of $f(t) = t^p$ with order α and $p \neq 0$ is

$$D^{\alpha}x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\, x^{p-\alpha}\,. \tag{6}$$

The difference between the two definitions is that when $p = 0$ or $f(t) = 1$ (a constant function). From Eq. (5), $D^{\alpha}f(t) = 0$ according to Caputo, but not zero according to Riemann-Liouville in Eq. (6), that is:

$$D^{\alpha}f(t) = D^{\alpha}1 = \frac{1}{\Gamma(1-\alpha)}\, x^{-\alpha}.$$

2.2 Fractional Differential Equation

Fractional differential equations are a generalization of ordinary differential equations into equations of non-integer order. The general form of a fractional differential equation is

$$D^{\alpha}y(t) = u\big(t\,, y(t)\big). \tag{7}$$

Theorem 1 [6]. If $f(t) \in L_1(0,T)$, and $Q(t)$ continuous function in the closed and bounded interval $[0,T]$, then the fractional differential equation

$$D^{\alpha}y + Q\, y = P(t)\text{ , with } u(0) = y_0 \tag{8}$$

has a unique solution $y(t) \in L_1(0,T)$.

Note:

$$L_1(0,T) = \left\{ f \,:\, \int_0^T |f(t)|\, dt \,< +\infty \right\} \tag{9}$$

is set of integrable functions on $[0,T]$.

3. Viscosity and Surface Tension on Differential Equation

Laboratory tests were performed to measure the viscosity and surface tension of multiple types of lubricating oils found in Indonesia, using the methodology described by Rusyaman *et al.,* [27]. A total of 25 different samples from various brands were chosen based on their availability in the Indonesian market. Viscosity was determined using an Ostwald viscometer at the Pharmacy Laboratory, Bandung Institute of Technology, while surface tension was assessed using surface tensiometers at the Physics Laboratory, Padjadjaran University.

Assumptions made in the context of this study, including: i) fluid properties are homogenous throughout experiment; ii.) fluid is under steady-state and behaves ideally without significant external influences, e.g., temperature fluctuations or contaminations. Data collected under controlled laboratory conditions to minimize external variables. Potential limitation of the method used including generalizability of the method to all types of fluids or in different temperature conditions.

The results are presented in Figure 1 and Figure 2, where viscosity is measured in Pascal seconds (Pa.s = N.s/m^2) and surface tension in Newtons per meter (N/m).

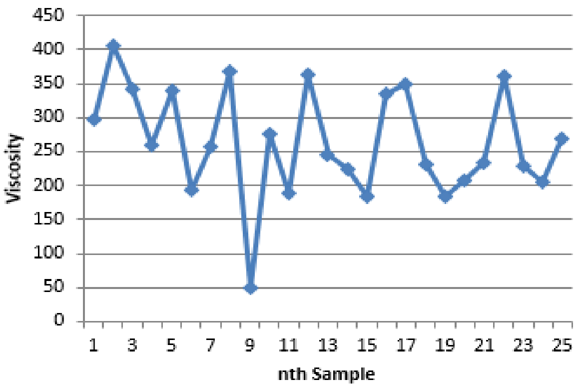

Fig. 1. Viscosity (η)

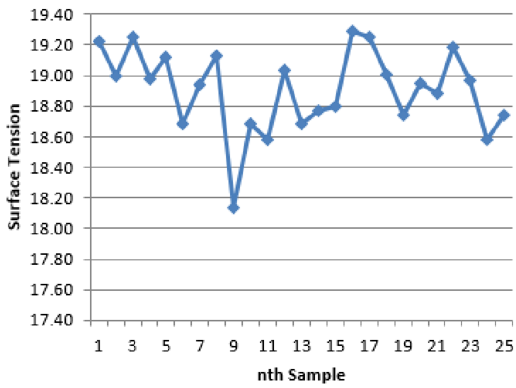

Fig. 2. Surface tension (γ)

By visually examining the patterns in the two sets of data above, it is evident that the correlation between the viscosity and surface tension of lubricating oil is quite strong, at 0.82. In general, as the viscosity value increases, so does the surface tension value. This relationship is visually depicted in the scatter plot shown in Figure 3 below, where the horizontal axis represents viscosity and the vertical axis represents surface tension.

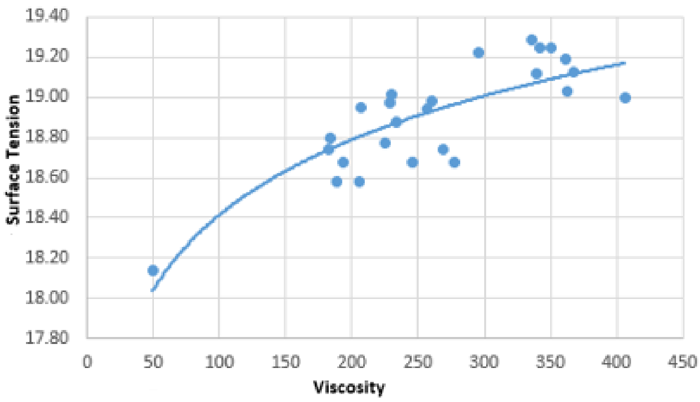

Fig. 3. Scatter plot of lubricating oil data

Based on Figure 3, in this research will consider γ as a function of η with its logarithmic regression result being

$$\gamma = 0.5393 \ln \eta + 15.932 . \tag{10}$$

Eq. (10) can be transformed into exponential form as follows

$$\eta = \exp(1.8543\, \gamma - 29.5420) .$$

To generalize, the symbol η will henceforth be replaced by y to denote viscosity, while the symbol γ will be replaced by t for surface tension. Therefore, Eq. (10) transforms to

$$y = \exp(1.8543\, t - 29.5420) . \tag{11}$$

Based on the regression relationship in (11) and the scatter plot in Figure 3, let's first examine the exact solution of the first-order ordinary differential equation through the following statement.

Theorem 2.
Function (11) is a solution of an ordinary derivative equation

$$y'(t) + Q\, y(t) = P(t) \tag{12}$$

with $(t) = k\, .\, e^{at+b}$, $Q = 2$ and initial condition $y(18) = 46$.

Proof:
The solution obtained is as follows

$$y(t) = e^{-\int Q\, dt} \left[\int e^{\int Q\, dt} . k\, e^{at+b}\, dt\ + c\right]$$
$$= \ e^{-2t}\, [\int k\, e^{2t}\, e^{at+b}\, dt\ + c]$$
$$= e^{-2t}\, \left[\frac{k}{2+a}\, e^{(2+a)t+b} + c\right],$$

where for $k = 3.854$, $a = 1.854$ and $b = -29.542$ the resulting solution function is

$$y(t) = \exp(1.8543\, t - 29.5420),$$

that identical to Figure 4 or Figure 5 with the following graph

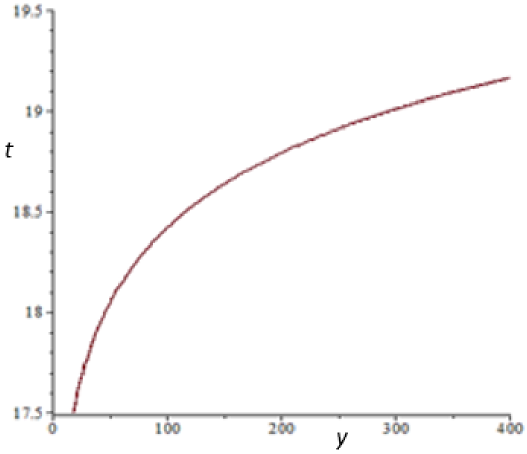

Fig. 4. $t(y) = 0.5393 \ln y + 15.932$

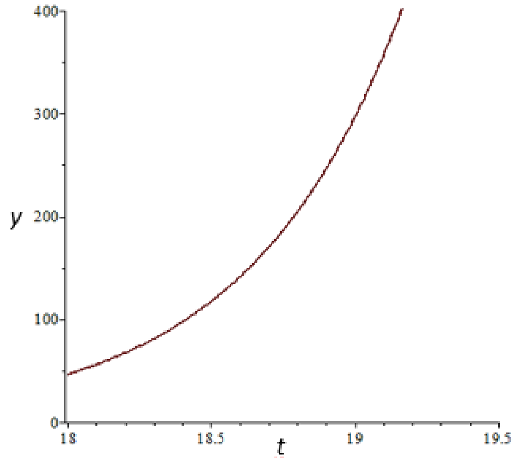

Fig. 5. $y(t) = \exp(1.8543\, t - 29.5420)$

Subsequently, the ordinary differential equation in (12) was transformed into a fractional-order differential equation. The reason for fractionalizing the derivative order is that fractional differential equations can provide a more accurate model for processes involving fractional characteristics, where properties such as viscosity or other phenomena can be better described using fractional derivatives rather than ordinary derivatives. Another reason is that fractional differential equations allow for the recognition of long-term memory effects and nonlocal properties in physical processes. This is useful in modeling phenomena involving responses to stimuli that occur not only at the current time but also at earlier times. By replacing the original derivative order of 1 with a fractional number α, thus the general form of the fractional differential equation (12) becomes:

$$D^{\alpha} y(t) + Q\, y(t) = P(t)\,;\ 0 < \alpha \leq 1. \quad (13)$$

where y is viscosity, t is surface tension, $P(t)$ is an exponential function, α is the fractional order, and Q is the chosen real constant which is 2.

4. Solution of Fractional Differential Equation by Adomian-Laplace Decomposition

The Adomian-Laplace Decomposition Method is an analytic technique used to solve ordinary differential equations or fractional differential equations. The essence of this method lies in decomposing the unknown solution function $y(t)$ to $\sum_{n=1}^{\infty} y_n(t)$ which consists of terms (decomposition) based on the mathematical properties of the given equation. Subsequently, differential equations for each $y_n(t)$ are sequentially solved from $y_0(t)$ hingga $y_n(t)$. The overall solution for $y(t)$ is obtained by summing all the derived terms. By combining the Adomian decomposition technique with Laplace transformation to separate the time variable, solutions can be obtained in a more systematic manner. This method is often utilized in applied mathematics and physics.

Next, the fractional differential equation in Eq. (13) will be solved using the Adomian-Laplace Decomposition Method.

Given a special form of fractional differential equation in Eq. (13)

$$D^{\alpha} y(t) + 2y(t) = k.e^{at+b}\,;\ y(18) = y_0. \quad (14)$$

With a guarantee of the existence and uniqueness of the solution, here is a technique for finding fractional differential equation solutions using the Adomian-Laplace Decomposition Method.

Let $x = t - 18$, then $t = x + 18$ and Eq. (13) becomes:

$$D^{\alpha} y(x+18) + 2\, y(x+18) = k.e^{\,a(x+18)+b}. \tag{15}$$

By applying the Laplace transform to both sides, the following is obtained:

$$\mathcal{L}[D^{\alpha} y(x+18)] + 2\mathcal{L}[y(x+18)] = k.e^{b}\mathcal{L}\left[e^{a(x+18)}\right]$$
$$s^{\alpha}\mathcal{L}[y(x+18)] = s^{\alpha-1}y(18) + \frac{k.e^{b+18a}}{s-a} - 2\mathcal{L}[y(x+18)]$$
$$\mathcal{L}[y(x+18)] = \frac{y(18)}{s} + \frac{k.e^{b+18a}}{s^{\alpha}(s-a)} - 2\frac{\mathcal{L}[y(x+18)]}{s^{\alpha}}$$
$$y(x+18) = y(18)\mathcal{L}^{-1}\left[\frac{1}{s}\right] + k.e^{b+18a}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}(s-a)}\right] - 2\mathcal{L}^{-1}\left[\frac{\mathcal{L}[y(x+18)]}{s^{\alpha}}\right].$$

Assume that $y(x+18) = \sum_{n=0}^{\infty} y_n(x+18)$, then

$$\sum_{n=0}^{\infty} y_n(x+18) = y(18)\,\mathcal{L}^{-1}\left[\frac{1}{s}\right] + k\,.\,e^{b+18a}\,\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}\,(s-a)}\right] - 2\,\mathcal{L}^{-1}\left[\frac{\mathcal{L}[\sum_{n=0}^{\infty} y_n(x+18)]}{s^{\alpha}}\right].$$

Thus, the recursive relation for the solution is obtained as follows:

$$y_0(x+18) = y(18)\mathcal{L}^{-1}\left[\frac{1}{s}\right] + k\,.\,e^{b+18a}\,L^{-1}\left[\frac{1}{s^{\alpha}\,(s-a)}\right]$$
$$= y(18) + \frac{k}{a^{\alpha}}e^{ax+b+18a}\left\{1 - \frac{\Gamma(\alpha,a.x)}{\Gamma(\alpha)}\right\}$$
$$y_n(x+18) = -2\mathcal{L}^{-1}\left[\frac{\mathcal{L}[y_{n-1}(x+18)]}{s^{\alpha}}\right].$$

The solution function can be written as

$$y(x+18) = y_0(x+18) + y_1(x+18) + y_2(x+18) + \cdots$$

The following are the calculations for $y_1(x+18)$ and $y_2(x+18)$:

$$\mathcal{L}[y_0(x+18)] = y(18)\,\mathcal{L}\left[\mathcal{L}^{-1}\left[\frac{1}{s}\right]\right] + k\,.\,e^{b+18a}\,\mathcal{L}\left[\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}\,(s-a)}\right]\right]$$
$$= \frac{y(18)}{s} + k\,.\,e^{b+18a}\,\frac{1}{s^{\alpha}\,(s-a)}.$$

$$y_1(x+18) = -2\,\mathcal{L}^{-1}\left[\frac{\mathcal{L}[y_0(x+18)]}{s^{\alpha}}\right]$$
$$= -2\,\mathcal{L}^{-1}\left[\frac{y(18)}{s^{\alpha+1}} + k\,.\,e^{b+18a}\,\frac{1}{s^{2\alpha}\,(s-a)}\right]$$
$$= -2\,\left\{y(18)\,\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] + k\,.\,e^{b+18a}\,\mathcal{L}^{-1}\left[\frac{1}{s^{2\alpha}\,(s-a)}\right]\right\}$$
$$= -\frac{2\,y(18)}{\Gamma(\alpha+1)}\,x^{\alpha} - \frac{2\,k}{a^{2\alpha}}\,e^{ax+b+18a}\,\left\{1 - \frac{\Gamma(2\alpha,a\,.x)}{\Gamma(2\alpha)}\right\}$$

$$\mathcal{L}[y_1(x+18)] = -2\,\mathcal{L}\left[\mathcal{L}^{-1}\left[\frac{y(18)}{s^{\alpha+1}} + k\,.\,e^{b+18a}\,\frac{1}{s^{2\alpha}\,(s-a)}\right]\right]$$
$$= -\frac{2\,y(18)}{s^{\alpha+1}} - 2\,k\,.\,e^{b+18a}\,\frac{1}{s^{2\alpha}\,(s-a)}$$
$$y_2(x+18) = -2\,\mathcal{L}^{-1}\left[\frac{\mathcal{L}[y_1(x+18)]}{s^{\alpha}}\right]$$
$$= -2\,\mathcal{L}^{-1}\left[-\frac{2\,y(18)}{s^{2\alpha+1}} - 2\,k\,.\,e^{b+18a}\,\frac{1}{s^{3\alpha}\,(s-a)}\right]$$
$$= -2\,\left\{-2\,y(18)\,\mathcal{L}^{-1}\left[\frac{1}{s^{2\alpha+1}}\right] - 2\,k\,.\,e^{b+18a}\,\mathcal{L}^{-1}\left[\frac{1}{s^{3\alpha}\,(s-a)}\right]\right\}$$
$$= \frac{4\,y(18)}{\Gamma(2\alpha+1)}\,x^{2\alpha} + \frac{4\,k}{a^{3\alpha}}\,e^{ax+b+18a}\,\left\{1 - \frac{\Gamma(3\alpha,a\,.x)}{\Gamma(3\alpha)}\right\}.$$

Based on the above calculations, the solution function up to $n = 6$ is obtained as follows:

$$y(x+18) = y(18) + \frac{k}{a^{\alpha}}\,e^{ax+b+18a}\,\left\{1 - \frac{\Gamma(\alpha,a\,.x)}{\Gamma(\alpha)}\right\} - \frac{2\,y(18)}{\Gamma(\alpha+1)}\,x^{\alpha} - \frac{2\,k}{a^{2\alpha}}\,e^{ax+b+18a}\,\left\{1 - \frac{\Gamma(2\alpha,a\,.x)}{\Gamma(2\alpha)}\right\} + \frac{4\,y(18)}{\Gamma(2\alpha+1)}\,x^{2\alpha} + \frac{4\,k}{a^{3\alpha}}\,e^{ax+b+18a}\,\left\{1 - \frac{\Gamma(3\alpha,a\,.x)}{\Gamma(3\alpha)}\right\} - \frac{8\,y(18)}{\Gamma(3\alpha+1)}\,x^{3\alpha} - \frac{8\,k}{a^{4\alpha}}\,e^{ax+b+18a}\,\left\{1 - \frac{\Gamma(4\alpha,a\,.x)}{\Gamma(4\alpha)}\right\} + \frac{16\,y(18)}{\Gamma(4\alpha+1)}\,x^{4\alpha} + \frac{16\,k}{a^{5\alpha}}\,e^{ax+b+18a}\,\left\{1 - \frac{\Gamma(5\alpha,a\,.x)}{\Gamma(5\alpha)}\right\} - \frac{32\,y(18)}{\Gamma(5\alpha+1)}\,x^{5\alpha} - \frac{32\,k}{a^{6\alpha}}\,e^{ax+b+18a}\,\left\{1 - \frac{\Gamma(6\alpha,a\,.x)}{\Gamma(6\alpha)}\right\} + \frac{64\,y(18)}{\Gamma(6\alpha+1)}\,x^{6\alpha} + \frac{64\,k}{a^{7\alpha}}\,e^{ax+b+18a}\,\left\{1 - \frac{\Gamma(7\alpha,a\,.x)}{\Gamma(7\alpha)}\right\}.$$

By substituting $x = t - 18$, the solution function in terms of t is obtained as follows:

$$y(t) = y(18) + \frac{k}{a^{\alpha}}\,e^{at+b}\,\left\{1 - \frac{\Gamma(\alpha,a\,.t-18\,a)}{\Gamma(\alpha)}\right\} - \frac{2\,y(18)}{\Gamma(\alpha+1)}\,(t-18)^{\alpha} - \frac{2\,k}{a^{2\alpha}}\,e^{at+b}\,\left\{1 - \frac{\Gamma(2\alpha,a\,.t-18\,a)}{\Gamma(2\alpha)}\right\} + \frac{4\,y(18)}{\Gamma(2\alpha+1)}\,(t-18)^{2\alpha} + \frac{4\,k}{a^{3\alpha}}\,e^{at+b}\,\left\{1 - \frac{\Gamma(3\alpha,a\,.t-18\,a)}{\Gamma(3\alpha)}\right\} - \frac{8\,y(18)}{\Gamma(3\alpha+1)}\,(t-18)^{3\alpha} - \frac{8\,k}{a^{4\alpha}}\,e^{at+b}\,\left\{1 - \frac{\Gamma(4\alpha,a\,.t-18\,a)}{\Gamma(4\alpha)}\right\} + \frac{16\,y(18)}{\Gamma(4\alpha+1)}\,(t-18)^{4\alpha} + \frac{16\,k}{a^{5\alpha}}\,e^{at+b}\,\left\{1 - \frac{\Gamma(5\alpha,a\,.t-18\,a)}{\Gamma(5\alpha)}\right\} - \frac{32\,y(18)}{\Gamma(5\alpha+1)}\,(t-18)^{5\alpha} - \frac{32\,k}{a^{6\alpha}}\,e^{at+b}\,\left\{1 - \frac{\Gamma(6\alpha,a\,.t-18\,a)}{\Gamma(6\alpha)}\right\} + \frac{64\,y(18)}{\Gamma(6\alpha+1)}\,(t-18)^{6\alpha} + \frac{64\,k}{a^{7\alpha}}\,e^{at+b}\,\left\{1 - \frac{\Gamma(7\alpha,a\,.t-18\,a)}{\Gamma(7\alpha)}\right\}. \tag{16}$$

The following are four example cases for the solution function in Eq. (16) above.

Case 1: $\alpha = 0.2,\; k = 3.854,\; a = 1.854,\; b = -29.542,\; y(18) = 46$

$$y(t) = 25{,}009.96 + 67.609\,e^{1.854\,t-29.542} - 0.743\,e^{1.854\,t-29.542}\,\Gamma(0.2\,,1.854\,t - 33.372) + 2.715\,e^{1.854\,t-29.542}\,\Gamma(0.4\,,1.854\,t - 33.372) - 7.153\,e^{1.854\,t-29.542}\,\Gamma(0.6\,,1.854\,t - 33.372) + 16.162\,e^{1.854\,t-29.542}\,\Gamma(0.8\,,1.854\,t - 33.372) + 64.027\,e^{1.854\,t-29.542}\,\Gamma(1.2\,,1.854\,t - 33.372) - 117.128\,e^{1.854\,t-29.542}\,\Gamma(1.4\,,1.854\,t - 33.372) - 100.199\,(t-18)^{0.2} + 207.379\,(t-18)^{0.4} - 411.856\,(t-18)^{0.6} + 790.222\,(t-18)^{0.8} - 1{,}472\,t + 2{,}671.985\,(t-18)^{1.2}.$$

Case 2: $\alpha = 0.8,\ k = 3.854,\ a = 1.854, b = -29.542,\ y(18) = 46$

$y(t) = -5.072681332 \cdot 10^6 + 5.333\, e^{1.854\, t-29.542} - 2.02\, e^{1.854\, t-29.542}\, \Gamma(0.8\,, 1.854\, t - 33.372) + 3.213\, e^{1.854\, t-29.542}\, \Gamma(1.6\,, 1.854\, t - 33.372) - 2.821\, e^{1.854\, t-29.542}\, \Gamma(2.4\,, 1.854\, t - 33.372) + 1.766\, e^{1.854\, t-29.542}\, \Gamma(3.2\,, 1.854\, t - 33.372) + 0.357\, e^{1.854\, t-29.542}\, \Gamma(4.8\,, 1.854\, t - 33.372) - 0.126\, e^{1.854\, t-29.542}\, \Gamma(5.6\,, 1.854\, t - 33.372) - 98.778\, (t-18)^{0.8} + 128.705\, (t-18)^{1.6} - 123.44\, (t-18)^{2.4} + 94.886\, (t-18)^{3.2} + 1.196549789 \cdot 10^6\, t - 1.05829591 \cdot 10^5\, t^2 + 4{,}160.129\, t^3 - 61.333\, t^4 + 34.384\, (t-18)^{4.8}.$

Case 3: $\alpha = 0.925,\ k = 3.854,\ a = 1.854, b = -29.542, y(18) = 46$

$y(t) = 46 + 3.426\, e^{1.854\, t-29.542} - 2.075\, e^{1.854\, t-29.542}\, \Gamma(0.925\,, 1.854\, t - 33.372) + 2.603\, e^{1.854\, t-29.542}\, \Gamma(1.85\,, 1.854\, t - 33.372) - 1.693\, e^{1.854\, t-29.542}\, \Gamma(2.775\,, 1.854\, t - 33.372) + 0.754\, e^{1.854\, t-29.542}\, \Gamma(3.7\,, 1.854\, t - 33.372) - 0.256\, e^{1.854\, t-29.542}\, \Gamma(4.625\,, 1.854\, t - 33.372) + 0.0706\, e^{1.854\, t-29.542}\, \Gamma(5.55\,, 1.854\, t - 33.372) - 0.0164\, e^{1.854\, t-29.542}\, \Gamma(6.475\,, 1.854t - 33.372) - 94.789\, (t-18)^{0.925} + 105.18\, (t-18)^{1.85} - 80.77\, (t-18)^{2.775} + 47.695\, (t-18)^{3.7} - 22.957\, (t-18)^{4.625} + 9.348\, (t-18)^{5.55}.$

Case 4: $\alpha = 1,\ k = 3.854,\ a = 1.854, b = -29.542,\ y(18) = 46$

$y(t) = -1.272424594 \cdot 10^8 + 2.7\, e^{1.854\, t-29.542} + 4.353632318 \cdot 10^7\, t - 6.21135946 \cdot 10^6\, t^2 + 4.73005392 \cdot 10^5\, t^3 - 20{,}278.46946\, t^4 + 464.080967\, t^5 - 4.429535502\, t^6.$

Figure 6 is a combined graph of the exact solution function in Eq. (13) and the solution functions for the four cases above obtained using the Adomian-Laplace Decomposition Method.

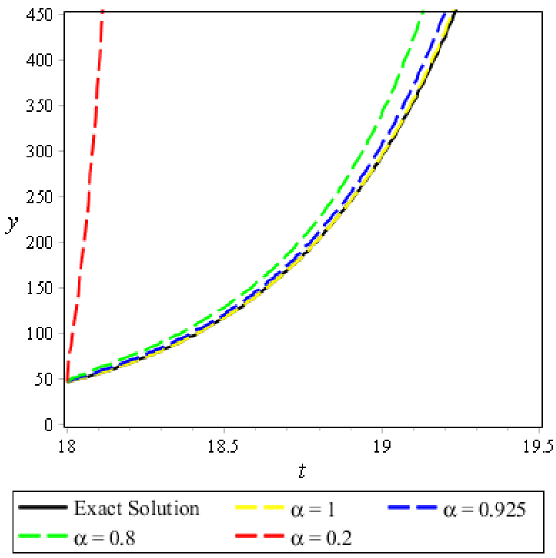

Fig. 6. Combined graph of exact solution and Adomian-Laplace Decomposition solutions

If the sequence of orders (α_n) converges to a number α, then the sequence of solution functions will converge to the solution function of the fractional differential equation of order α. In this case, for fractional order α which goes to 1, the graph of the solution function also goes to the graph of the solution function for order $\alpha = 1$ which coincides with the graph of the exact solution (depicted in yellow and black graph). The pattern of convergence of the sequence of solution functions to its exact solution demonstrates the correctness of this approach. This shows that empirically the solution of the fractional differential Eq. (11) truly reflects the relationship between surface tension and viscosity.

Our findings have important implications. Understanding the relationship between surface tension and viscosity through a fractional differential equation may guide the development of more effective and efficient materials and processes for manufacturing. The use of the Adomian-Laplace Decomposition method to solve fractional differential equations contributes to the methodology in applied mathematics and engineering.

5. Conclusions

The viscosity and surface tension of lubricating oil demonstrated a clear relationship that can be mathematically described by either a logarithmic or exponential function. This function represents the solution to a first-order linear differential equation, suggesting that this equation effectively models the connection between viscosity and surface tension. When this differential equation is extended to a fractional form, it yields a highly accurate result where the solution function closely matches the exact solution's graph. This underscores the applicability of a fractional differential equation model in describing the viscosity-surface tension relationship. Furthermore, employing the fractional model reveals insights into the memory effect associated with it. Overall, knowledge of the relationship between viscosity and surface tension allows the development of more advanced technologies and a deeper understanding of fluid behavior in various contexts.

As a suggestion for further research, the order of the fractional differential equation can be increased to non-linear, especially the Riccati form. In addition, further studies may be needed to extend the model to different types of fluids, investigating additional properties beyond viscosity and surface tension, or applying similar methods to other scientific disciplines where fractional differential equations could be relevant.

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