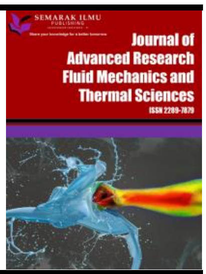
Journal of Advanced Research in Fluid Mechanics and Thermal Sciences

Journal homepage:
https://semarakilmu.com.my/journals/index.php/fluid_mechanics_thermal_sciences/index
ISSN: 2289-7879



A Mathematical Model for the Velocity of Thin Film Flow of a Third Grade Fluid Down in an Inclined Plane

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ARTICLE INFO

Article history:
Received 30 July 2022
Received in revised form 8 December 2022
Accepted 20 December 2022
Available online 6 January 2023

Keywords:
Motion equation; homotopy perturbation method; Pade´ approximation; thin film flow; third-grade fluid

ABSTRACT

In this paper, a new technique to solve a mathematical model of thin film flow of third-grade fluid in an inclined plane is presented. This technique is based on combining the homotopy perturbation method with the Laplace transform and Pade´ approximation approach. The results we obtained showed that the proposed technique has high accuracy than other classical methods to solve this model. The measurements of error are tabulated. the validity and usefulness of the new method are derived. Moreover, the influence of parameters on the velocity profile is discussed graphically.

1. Introduction

In the past few years, the study of non-Newtonian fluids has become very important due to its technological, engineering, and industrial applications, especially with the emergence of the polymer industry, the petroleum industry, plastics, and others. Therefore, there is no definitive model for non-Newtonian fluids. Subcategories of these fluids are constructed based on a physical concept that enables them to be effectively modeled. One of these fluid subclasses is a third-grade fluid [1]. Many researchers have made great efforts to discuss solutions to the problems of flowing Newtonian and non-Newtonian fluids numerically and analytically, such as Mahmood *et al.,* [2], who solved the problem of the effect of thermal buoyancy force on the flow of an unstable third-order magneto hydrodynamic (MHD) fluid with convective cooling by using the finite element method for calculating fluid velocity and temperature. Mohammed *et al.,* [3] applied the finite element method to the equation of continuity and energy conservation for the flow of a compressible Newtonian fluid in a two-dimensional channel. Chinyoka and Makinde [4] suggested the finite difference method to MHD for a third-order fluid flow with ohm heating. Abbas *et al.,* [5] studied the effects of MHD on the phenomenon of heat and mass transfer of a third-grade fluid on an exponentially extended plate

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https://doi.org/10.37934/arfmts.102.1.140152

under the influence of Darcy-Furchheimer's law. The partial differential equations were converted by similarity procedure to the ordinary differential equations and solved using Matlab built-in numerical solver bvp4c, where they analyzed the effect of physical properties, such as Nusselt number, temperature, velocity field, mass concentration and friction coefficient in the graphs. Keimanesha *et al.,* [6] studied the third-order non-Newtonian fluid flow problem between two parallel plates and it was treated by the developed differential transform method and compared the results with the symmetric turbulence differential transformation method and the fourth-order Range-Kutta method. Gul and Ullah [7] solved the momentum and energy equations using the Adomian Decomposition Method (ADM). The results they obtained are illustrated by graphs of the velocity, average velocity, and volumetric flow rate of the flow. Alam *et al.,* [8] discussed the application of ADM to the momentum and energy equations for the flow of a thin non-Newtonian fluid on the outer surface of the vertical cylinder of Johnson-Segalman fluid, and a comparison was made with the numerical solution. Okedayo *et al.,* [9] developed the ADM method to solve the problem of flow and heat transfer of a third-order fluid, discussed the results graphically, and concluded that increasing the magnetic parameter leads to a decrease in flow velocity and temperature. Gul *et al.,* [10] derived the solution to the temperature-dependent viscosity effect of a third-grade dynamic hydraulic fluid flow perpendicular to the momentum-energy equations. Jasim [11] presented a solution to a magnetic-hydrodynamic Jeffrey Hummel fluid flow (JHF) problem with nanoparticles. After converting the equations of this problem into an ordinary differential equation and solving the latter using the new analytical method (NAM) and the fourth-order Runge-Kutta method (RK 4), Siddiqui *et al.,* [12] tried to solve the problem of thin-film flow of Eyring-Powell fluid on a vertically moving belt by the two methods of variable frequency (ADM) and variation iteration method (VIM), and the numerical results proved an excellent agreement between the solutions of the two methods. Ali *et al.,* [13] highlighted the study of the motion of a third-order fluid through an impermeable cylinder with a heat source analyzed using a homotopy analytical method (HAM). However, some researchers who have tried to tackle the current problem in different ways, such as Kumaran *et al.,* [14] and Hayat *et al.,* [15], found the exact solution to the problem according to certain assumptions. Shah *et al.,* [16] were able to solve the problem of thin film flow of a third-grade fluid in an inclined plane using the spline collection method, and it was compared with the optimal homotopy perturbation method (OHPM), numerical solution, and homotopy perturbation method (HPM). Mahmood and Khan [17] solved the same problem using the HPM method and the perturbation method (PM). The results include finding the velocity and the rate of flow volume. Manshoor *et al.,* [18] handled the problem of the flow of a third-order thin fluid with an inclined plane using the variation parameter method (VPM), which graphically explained the effect of physical parameters on the fluid flow velocity. Sadighi *et al.,* [19] used PM and ADM methods to solve the flow of a third-order thin fluid problem and found the results were completely identical for these methods. Sadighi *et al.,* [20] and Azimi and Azimi [21] treated the same problem of flow using HPM. The results for this method gave a smooth solution. Siddique *et al.,* [22] adopted the fuzzy perturbation method to compute the solution of the flow problem of thin films at an inclined plane in a fuzzy environment. Sarwani *et al.,* [23] specified a formula for calculating the biodiesel's specific energy emission rate (g/km) based on the volumetric emissions percentage. Asghar and Ying [24] examined the expansion/contraction plate's influence on a three-dimensional hybrid nanofluid's boundary layer flow characteristics and heat transfer rate. Sharafatmandjoor and Azwadi [25] researchers looked at how external factors affected the velocity time series data of microorganisms. Abidin *et al.,* [26] an anisotropic porous layer saturated with a viscoelastic double diffusive binary fluid is examined numerically for the start of Darcy-Rayleigh convection. Akaje and Olajuwon [27] they studied the effect of nonlinear thermal radiation on the heat transfer of a MHD Casson nanofluid with a stagnation point associated with Thompson and

Troian slip boundary conditions in the center of Dacry-Forchheimme. Budiana *et al.,* [28] they conducted a numerical simulation to develop the thermal treatment of inductive asphalt by two methods of electrolysis using ANSYS Electronics Desktop software and thermal analysis using ANSYS Transient Thermal Analysis. Despite the accuracy and efficiency of the analytical methods (mentioned in literature) that are used by the researchers to solve the flow problems and, as well as the current problem, some of them require great time and effort in their numerical calculations and others need a small parameter like the homotopy method is depends on the small perturbation parameter. At the same time, there are analytical methods used to solve differential equations without these limitations, such as the Laplace transformation, which was discovered by Simon (1942), and the Pade´ approximation method (1890) discovered by Henry Pade´. Many researchers have invested time and effort in studying HPM to solve nonlinear problems and used it along with the Pade´ approximation (HPM-Pade´), Laplace Transform (LP-HPM) and the technique of parameter expansion, recently, Filobello-Nino proposed a modification to the HPM method and Li-He relied on the modification and combined it with the parameter expansion method to solve the micromechanical system problem [29-33]. According to our simple information, we note that these two methods are not used to find an approximate analytical solution to the current problem. This is an incentive for us to use these analytical methods to find the solution to the studied problem. Beyond that, it encourages us to improve them by combining the HPM, Laplace transforms, and Pade´ approximate method (HPLPM) and then using them to solve the current problem. Thus, the goal of this work is to present a new analytical technique used to find an approximate analytical solution to the third-order non-Newtonian fluid flow problem with an inclined plane. The results proved the effectiveness and sufficient ability of our method to obtain the optimal analytical solution by comparing it with the HPM-Pade´ and HPM. It is further observed here that the results of the studies by Sadighi *et al.,* [19] and Siddiqui *et al.,* [20] match the results obtained by HPLP. Furthermore, it has fewer errors than other methods. The remaining sections of this research are as follows: In Section 2, the mathematical model of the problem of flow is defined. The analytical methods used are presented in Section 3. In Section 4, we introduce the algorithm of the new method and its application to the current problem, and the discussion of results is given in Section 5. and in the last section, some conclusions are written.

2. Mathematical Model of Flow

Governing equations of the motion of an incompressible fluid, neglecting the thermal effects, are [14]

$$\nabla . V = 0 \tag{1}$$

$$\rho \frac{DV}{Dt} = -\nabla P + \rho f + div\tau , \tag{2}$$

where ρ is the constant density, V is the velocity vector, P is the pressure, τ is the stress tensor, and D/Dt denotes the material derivative.

The stress tensor defined by a third-grade fluid is given by

$$\tau = \sum_{i=1}^{3} S_i , \tag{3}$$

where,

$$S_1 = \mu A_1 \tag{4}$$

$$S_2 = \alpha_1 A_1 + \alpha_2 A_1^2 \tag{5}$$

$$S_3 = \beta_1 A_3 + \beta_2(A_1 A_2 + A_2 A_1) + \beta_3(trA_2)A_1 \tag{6}$$

Here, μ is the coefficient of viscosity, α_1 , α_2 , β_1 , β_2 and β_3, are material constants, and $A_n's$ are defined by $A_0 = I$, the identity tensor, and

$$A_n = \frac{DA_{n-1}}{Dt} + A_{n-1}(\Delta V) + (\Delta V)^T A_{n-1} \, , n \geq 1 \tag{7}$$

Consider a thin film of an incompressible fluid of a third-grade flow down an inclined plane, with the ambient air assumed stationary so that the flow is due to gravity alone. Assume that the surface tension of the fluid is negligible and the film is of uniform δ . Seek a velocity field of the form

$$v = (u(y), 0,0) \tag{8}$$

Substituting for v and τ in Eq. (1) and Eq. (2) and assuming the absence of pressure gradient, we obtain

$$\mu\frac{d^2u}{dy^2} + 6(\beta_2 + \beta_3)(\frac{du}{dy})^2\frac{d^2u}{dy^2} + \rho g \sin\alpha = 0 \tag{9}$$

Subject to the boundary conditions of

$$u(y) = 0 \, , at \; y = 0 \tag{10}$$

$$\frac{du}{dy} = 0 \, , at \; y = \delta \tag{11}$$

Let $\beta = \frac{\beta^*}{\mu}$, $where$ $\beta^* = \beta_2 + \beta_3$, $\frac{\rho g \, sin\alpha}{\mu} = m$. Then Eq. (9) becomes

$$\frac{d^2u}{dy^2} + 6\beta(\frac{du}{dy})^2\frac{d^2u}{dy^2} + m = 0 \tag{12}$$

3. Some Analytical Methods

Make a quick review of the homotopy perturbation method and Pade´ approximation,

3.1 Basic Idea of the Homotopy Perturbation Method

Consider the nonlinear differential equation [30]

$$\mathrm{A}(u) - f(z) = 0 \; z \in \Omega \, , \tag{13}$$

the boundary condition is

$$B\left(u, \frac{du}{dn}\right) = 0, z \in \Gamma\ , \tag{14}$$

where A , B , $f(z)$ and Γ are general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω , respectively

The A operator can be divided into a linear part L and a nonlinear part N. Eq. (13) may therefore be written as.

$$L(u) + N(u) - f(z) = 0, f(z) = 0 \tag{15}$$

Which satisfies by the homotopy technique, we construct a homotopy $v(z,p): \Omega\ \times [0,1] \rightarrow \mathrm{R}$

$$H(v,q) = (1-q)[L(v) - L(v_0)] + q[L(v) + N(v)] = 0 \tag{16}$$

or

$$H(v,q) = L(v) - L(v_0) + qL(v_0) + q[N(v)] = 0 \tag{17}$$

where $q \in [0,1]$ is an embedding parameter, while v_0 is an initial approximation of Eq. (13) which satisfies the boundary condition, obviously, from Eq. (16) and Eq. (17), we have

$$\begin{aligned} H(v,0) &= L(v) - L(v_0) = 0\ , \\ H(v,1) &= L(v) + N(v) = 0 \end{aligned} \tag{18}$$

In topology this is called deformation, while $L(v) - L(v_0)$ and $L(v) + N(v)$ are called homotopy. The parameter q as a small parameter, applying the classical perturbation technique, we can assume that the solution of Eq. (16) and Eq. (17) can be written as power series in q as follows

$$v = \sum_{i=0}^{\infty} q^i v_i \tag{19}$$

Putting $q = 1$ in Eq. (19), we have $u = \lim_{q \rightarrow 1} v\ = \sum_{i=0}^{\infty} v_i$.

3.2 Pade´-approximation

Definition: Pade´ approximation is a ratio of two polynomials that come from the Tayler series expansion of a function u(y) and define as [34]

$\mathrm{u(y)} = \mathrm{P_r^p} = \frac{\sum_{n=0}^{p} a_n y^n}{\sum_{n=0}^{r} b_n y^n}$, where $b_0 = 1$, the function u(y) writing by

$\mathrm{u(y)} = \sum_{n=0}^{\infty} c_n y^n$

Also, $\mathrm{u(y)} - \mathrm{P_r^p} = O(y^{p+r+1})$

thus,

$$\sum_{n=0}^{\infty} c_n y^n = \frac{\sum_{n=0}^{p} a_n y^n}{\sum_{n=0}^{r} b_n y^n} \tag{20}$$

from Eq. (20), obtain the following system equations

$$a_0 = c_0$$
$$a_1 = c_1 + c_0 b_1$$
$$a_2 = c_2 + c_1 b_1 + c_0 b_2$$
.
.
.
By given c_n

To solve the above system of equations for a_n and b_n , we take the numerator degree p and the denominator degree r, that is mean, we want to solve the following system

$$a_0 = c_0$$
$$a_1 = c_1 + c_0 b_1$$
$$a_2 = c_2 + c_1 b_1 + c_0 b_2$$
.
.
.

$$a_p = c_p + c_{p-1} b_1 + c_{l-2} b_2 + \cdots + c_0 b_l$$
$$0 = c_{p+1} + c_p b_1 + c_{p-1} b_2 + \cdots c_{p-r+1} b_r$$
$$0 = c_{p+2} + c_{p+1} b_1 + c_p b_2 + \cdots + c_{p-r+2} b_r$$
.
.
.
$$0 = c_{p+r} + c_{p+r-1} b_1 + c_{p+r-2} b_2 + \cdots + c_p b_r \, .$$

For more details to solve such as the above system by examples refer Khatib [35].

4. The HPLP Algorithm

In this section, we summarize the basic idea of HPLP depending on the Homotopy perturbation method with Laplace transform and Pade´ approximation, in the following steps

Step 1: From of the homotopy perturbation method, we get the solution

$$u(y) = \lim_{p \to 1} v = \sum_{i=0}^{n} v_i$$

Step 2: Take Laplace transform (l) of u, we have

$$l(u(y)) = l\{\sum_{i=0}^{n} v_i \, , y, s\}$$

Step 3: Applying Pade´- approximate on Step 2, gives

$$\mathrm{P}_r^P\ (l(u(s)) = \mathrm{Pade}'\{l\left(u\left(\frac{1}{s}\right)\right), [\mathrm{p}, r]\}\ , 1 \leq r, \mathrm{p} \leq r + \mathrm{p}$$

Step 4: Take the inverse Laplace transform of Step 3.

$$u = l^{-1}\left[P_r^q\left(l(u\left(\frac{1}{s}\right))\right), s, y\right]$$

such that u approximate solution.

4.1 Application HPLP on a Mathematical Model of Fluid

In this section, we solve the mathematical model for thin film flow of a third-grade fluid down in an inclined plane by a new method.

Apply the HPM to Eq. (12), we obtain

$$L(u) - L(u_0) + qL(u_0) + q\left(6\beta\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} + m\right) = 0, \tag{21}$$

where $L = \frac{d^2}{dy^2} + m$ is the linear operator, $u_0(y)$ the initial guess approximation.

By rearranging the Eq. (21), we obtain

$$L(u) = L(u_0) - q\left(6\beta\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2}\right) \tag{22}$$

substituting Eq. (16) into Eq. (22), we have

$$L\left(\sum_{i=0}^{\infty} q^i u_i\right) = m - q\left(6\beta\left(\frac{d\left(\sum_{i=0}^{\infty} q^i u_i\right)}{dy}\right)^2 \frac{d^2\left(\sum_{i=0}^{\infty} q^i u_i\right)}{dy^2}\right) \tag{23}$$

with the boundary condition

$$u_i(y) = 0\ at\ y = 0\ , \frac{du_i}{dy} = 0\ at\ y = 1\ , for\ i = 0,1,2, \dots \tag{24}$$

Solving Eq. (22) with the corresponding boundary conditions then

$$u_0 = m\left(y - \frac{y^2}{2}\right) \tag{25}$$

$$u_1 = 6\beta m^3\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2} - \frac{y}{3}\right) \tag{26}$$

$$u_2 = 36\beta^2 m^5\left(\frac{y^5}{3} - \frac{y^6}{18} - \frac{5y^4}{6} + \frac{10y^3}{9} - \frac{5y^2}{6} + \frac{y}{3}\right) \tag{27}$$

$$u_3 = 216\beta^3 m^7 \left(\frac{y^8}{18} - \frac{4y^7}{9} + \frac{14y^6}{9} - \frac{56y^5}{18} + \frac{70y^4}{18} - \frac{56y^3}{18} + \frac{28y^2}{18} - \frac{8y}{18}\right) \quad (28)$$

The solutions of this model are formulation as;

$U = \sum_{i=0}^{3} u_i$

Thus, the solutions are

$$U(y) = m\left(y - \frac{y^2}{2}\right) + 6\beta m^3 \left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2} - \frac{y}{3}\right) + 36\beta^2 m^5 \left(\frac{y^5}{3} - \frac{y^6}{18} - \frac{5y^4}{6} + \frac{10y^3}{9} - \frac{5y^2}{6} + \frac{y}{3}\right) + 216\beta^3 m^7 \left(\frac{y^8}{18} - \frac{4y^7}{9} + \frac{14y^6}{9} - \frac{56y^5}{18} + \frac{70y^4}{18} - \frac{56y^3}{18} + \frac{28y^2}{18} - \frac{8y}{18}\right) \quad (29)$$

Now, take Laplace transform of both sides of the Eq. (29) with respect to y, we have

$$l(U) = m\left(\frac{1}{s^2} - \frac{1}{s^3}\right) + 6\beta m^3 \left(\frac{4}{3s^5} - \frac{2}{s^4} + \frac{1}{s^3} - \frac{1}{3s^2}\right) + 36\beta^2 m^5 \left(\frac{40}{s^6} - \frac{40}{s^7} - \frac{20}{s^5} + \frac{20}{3s^4} - \frac{5}{3s^3} + \frac{1}{3s^2}\right) + 216\beta^3 m^7 \left(\frac{2240}{s^9} - \frac{2240}{s^8} + \frac{1120}{s^7} - \frac{1120}{3s^6} + \frac{280}{3s^5} - \frac{56}{3s^4} + \frac{28}{9s^3} - \frac{4}{9s^2}\right) \quad (30)$$

Let $s = \frac{1}{y}$, substituting in Eq. (30) become

$$l(U) = m(y^2 - y^3) + 6\beta m^3 \left(\frac{4}{3}y^5 - 2y^4 + y^3 - \frac{1}{3}y^2\right) + 36\beta^2 m^5 \left(40y^6 - 40y^7 - 20y^5 + 20y^4 - 5y^3 + \frac{1}{3}y^2\right) + 216\beta^3 m^7 \left(2240y^9 - 2240y^8 + 1120y^7 - 1120y^6 + \frac{280}{3}y^5 - \frac{56}{3}y^4 + \frac{28}{9}y^3 - \frac{4}{9}y^2\right) \quad (31)$$

The Pade´ approximation P_r^p of Eq. (32) where $m = 0.25, \beta = 1.4$ is

$$P_2^7 = \frac{-0.0931y\text{^}2 - .4884y\text{^}3 - 1.5445y\text{^}4 - .3917y\text{^}5 - 4.8538y\text{^}6 + 9.6410*y\text{^}7}{1 + 3.9452y + 6.3204y^2} \quad (32)$$

substituting $y = \frac{1}{s}$ and taking inverse Laplace transform of the Eq. (32)

$$U = l^{-1}\left(\frac{-0.0931\left(\frac{1}{s}\right)^2 - .4884\left(\frac{1}{s}\right)^3 - 1.5445\left(\frac{1}{s}\right)^4 - .3917\left(\frac{1}{s}\right)^5 - 4.8538\left(\frac{1}{s}\right)^6 + 9.6410\left(\frac{1}{s}\right)^7}{1 + 3.9452\frac{1}{s} + 6.3204\left(\frac{1}{s}\right)^2}\right)$$

thus,

$$. \ U(y) = 0.8366e - 1 + 0.6356e - 1 * y\text{^}4 - .2867 * y\text{^}3 + .3852 * y\text{^}2 - .1469 * y + 1.3374 * 10\text{^}(-63) * exp(-1.9726 * y) * (9.3526 * 10\text{^}61 * sin(1.5586 * y) - 6.2554 * 10\text{^}61 * cos(1.5586 * y)$$

Define the errors L_2 , L_∞ by

$$\|E\|_{L_2} = \sqrt{h^2 \sum_{i=0}^{n} |U_3 - U_2|^2} \,,$$

$$\|E\|_{L_\infty} = max_{i=0..n}(|U_3 - U_2|) \,.$$

5. Results and Discussion

A new analytical approximate method was successfully implemented to solve the thin-film flow problem of a third-order fluid with an inclined plane. The results of this method show the effect of the elastic parameter β and the parameter m on the fluid flow velocity graphically. In Figure 1, other different values were taken, m = 0.2, 0.3, 0.4, and the value of β = 0.75 was fixed. In Figure 2, the value of β = 1.4 1.8, 2, m = 0.25 and m = 0.5, β = 0.1, 0.3, 0.5. It is clear that the flow velocity of the fluid decreases as the values of β are increased and it increases with the increase in the parameter m from 1 to 10. In Figure 3, the value of β = 0.2 and m = 0.7 and in Figure 4, β = 12 and m = 0.1 These two figures represent the comparison between the new technology HPLP and HPM, HPM- Pade´, the convergence between them is good. In Table 1 and Table 2, the error between the proposed method and HPM- Pade´ and HPM method is calculated. We can say that these results show an agreement with results in the studies by Manshoor *et al.,* [18], Sadighi *et al.,* [19], and Siddiqui *et al.,* [20]. Through this comparison, it is clear that the errors for the new technique are small, and the solutions are accurate.

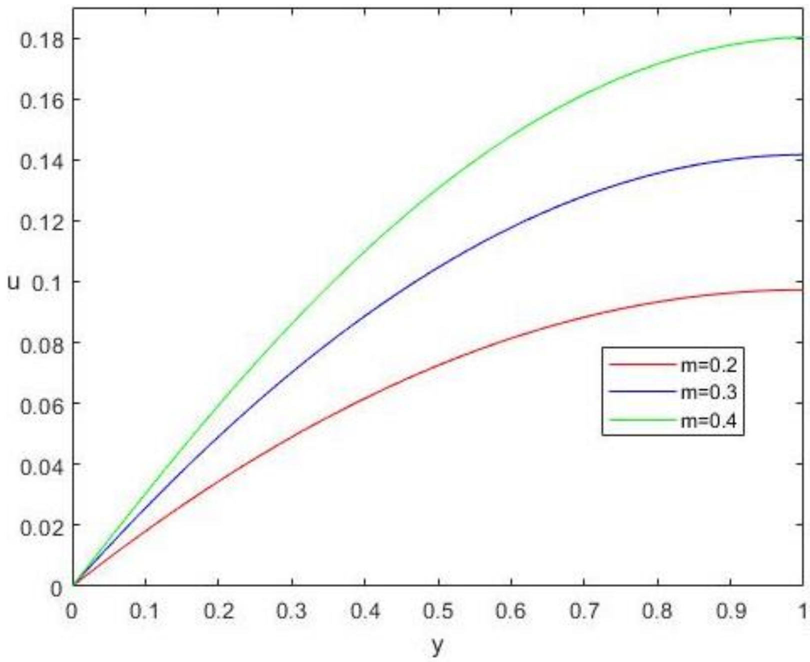

Fig. 1. Analytical approximate solution by HPLP where β=0.75

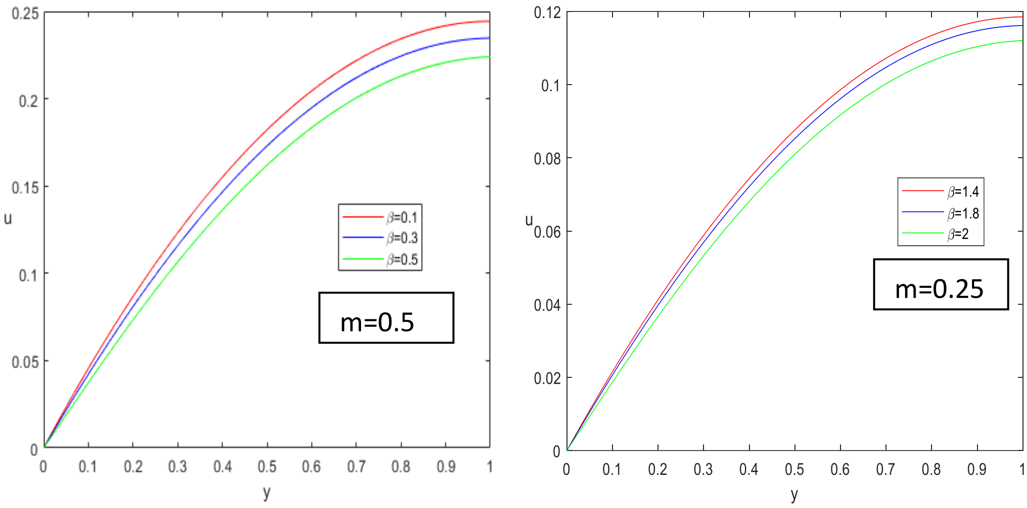

Fig. 2. Analytical approximate solution by HPLP

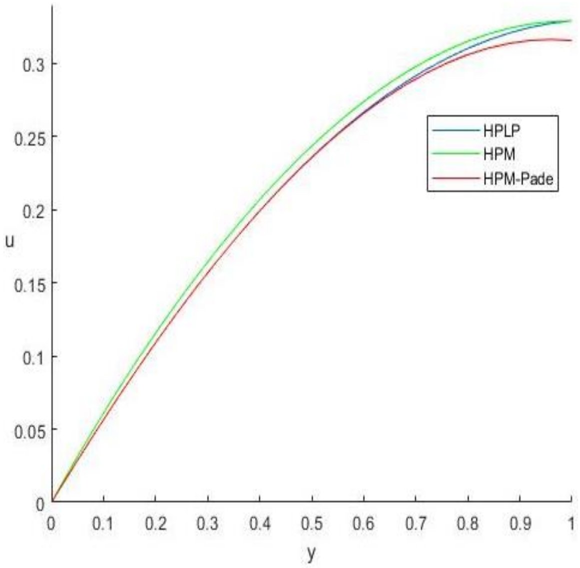

Fig. 3. A comparison of the solutions between HPLP, HPM and PM- Pade´`where m=0.7, $\beta = 0.2$

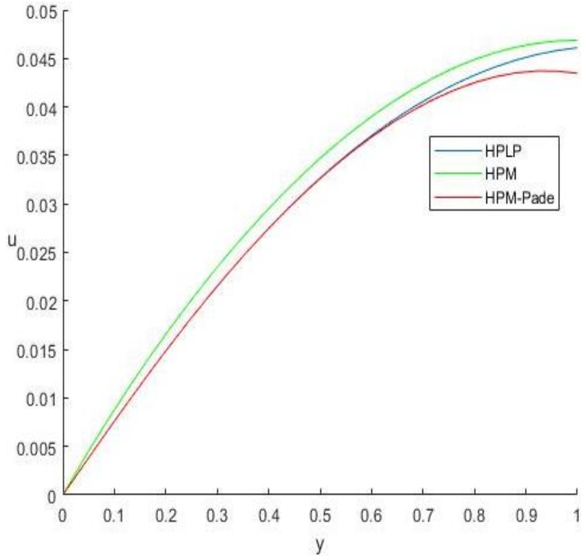

Fig. 4. A comparison of the solutions between HPLP, HPM and HPM-Pade´where m=0.1, $\beta = 12$

Table 1
Error of the methods HPLP, PM, HPM-Pade`, HPM. When $\beta = 1.4$

m	$Error\ measurement$	HPLP	HPM	PM	HPM-Pade
0.2	L_2	3.9×10^{-4}	1.1×10^{-1}	1.1×10^{-1}	1.1×10^{-3}
	L_∞	4.2×10^{-4}	1.2×10^{-1}	1.2×10^{-1}	1.4×10^{-3}
0.1	L_2	2.6×10^{-6}	3.5×10^{-5}	3.5×10^{-5}	2.1×10^{-4}
	L_∞	4.4×10^{-6}	3.9×10^{-5}	3.9×10^{-5}	5.2×10^{-4}
0.01	L_2	5.8×10^{-14}	3.5×10^{-10}	3.5×10^{-10}	4.8×10^{-10}
	L_∞	7.4×10^{-14}	3.9×10^{-10}	3.9×10^{-10}	4.5×10^{-10}

Table 2
Error of the methods HPLP, PM, HPM-Pade`, HPM. When $m = 0.04$

β	$Error\ measurement$	HPLP	HPM [20]	PM [19]	HPM-Pade
0.3	L_2	4.4×10^{-11}	1.7×10^{-8}	1.7×10^{-8}	1.8×10^{-8}
	L_∞	4.9×10^{-11}	1.8×10^{-8}	1.8×10^{-8}	2.3×10^{-8}
0.5	L_2	2.3×10^{-10}	4.6×10^{-8}	4.6×10^{-8}	4.9×10^{-8}
	L_∞	2.4×10^{-10}	5.1×10^{-8}	5.1×10^{-8}	6.4×10^{-8}
1	L_2	1.8×10^{-9}	1.9×10^{-7}	1.9×10^{-7}	2.0×10^{-7}
	L_∞	2.0×10^{-9}	2.1×10^{-7}	2.1×10^{-7}	2.6×10^{-7}

6. Conclusion

In this work, I suggest a newly developed technique, HPLP, for solving problems of thin-film flow in an inclined plane. The results we obtained from solving the problem with the new method HPLP, which is effective and has high accuracy, are for finding the approximate analytical solution to this problem. By comparing the current method with HPM- Pade´, HPM noted that the new technique is accurate and its results are in good agreement with the methods VPM, HPM, PM, and ADM, and we conclude that the new method is highly efficient for finding the approximate analytical solution to the third-order fluid flow problem.

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