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Effect of Lorentz Force and Magnetic Viscosity on Shliomis Model Ferro Flow due to Rotating Disk

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ABSTRACT

The present work involves the study of ferrofluid flow over a rotating disk analysed through Shliomis model in the presence of alternating magnetic field applied in the radial direction. Shliomos model is the appropriate model to analyse ferrofluid flows as it provides prominence to the particle's movement in the fluid. Here the flow alters due to the viscosity dependent on applied magnetic field in the presence of Lorentz force. The governing equations leading to nonlinear partial differential equations are reduced to nonlinear ordinary differential equations using similarity transformation and solved numerically using the shooting method with the error tolerance of 10-6. The results obtained are represented in the form of graphs and the effects of parameters are analysed on the velocity profiles and pressure. The magnetic parameter declines whereas the magnetic field dependent viscosity enhances the velocity of the flow by reducing the pressure, which finds its application in many hardware devices. Also, the comparative study provides a quantitative agreement for the limiting case of the present work.

1. Introduction

The magnetic field can be introduced into the boundary layer flow through two mechanisms. The first, where the magnetic force acts on the fluid within the boundary layer directly. Secondly, the effect of magnetic field on free stream flow, alters the flow variables in turn reflects on the change in pressure gradient which imposes changes in boundary layer flow. The second mechanism requires approximate assumption to analyse the system, further there exists certain flows where the magnetic field in the free stream vanishes. Thus it is more suitable to analyse the system based on the direct act of the magnetic field as mentioned by Sparrow and Cess [1]. Further the introduction of magnetic field leads to two situations, the one in which the particles magnetic moment and vorticity are collinear, there exists no change in the viscosity of the fluid. On the other hand if the particle's

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magnetic moment and vorticity are non - collinear, the fluid is induced with additional viscosity which in turn exhibits resistance to the flow as explained in previous studies [2,3]. The presence of magnetic flux in an electrically conducting medium exerts a force perpendicular to the applied medium represented as $\vec{J} \times \vec{B}$. This force is called as Lorentz force, which is a body force acting on the medium. If the electric field induced exists, the Lorentz force takes the form $\vec{J} = \sigma\left(\vec{E} + \vec{q} \times \vec{B}\right)$ and in the case of large conductivity $\vec{J} = \nabla \times \vec{H}$ [4]. Thus the concept of analysing the fluid flow with Lorentz force and magnetic field dependent viscosity has been a keen interest for researchers from many decades on the fluid being Newtonian, non - Newtonian, nanofluids and in the recent years on ferrofluids.

The suspension of colloidal ferric particles which are of the size 3 – 10 nm in the polar or non-polar base fluid is termed as ferrofluid. Ferrofluids are the synthesized fluid and are controlled by an applied external magnetic field. This property of ferrofluid has multidisciplinary strategies which leads researchers to focus on different aspects of ferrofluid. In particular, the synthesis of nano particles with its stability are focused by chemists. Physicists concentrate on physical properties of ferrofluid. Engineers create required model with ferrofluid for particular purposes. Biomedical characteristics attracts biologists and physicians and so on. Thus aforementioned purposes provides usage of ferrofluid in dampers, sealants, lubrication, bearing, drug delivery for hypothermia, contrast agent, to remove tumour cells, enhancement of heat conductivity dependent on shape of ferric particles, pool boiling and so on as studied by Nargund and Asha [5] and Asha and Achala [6].

Ferrofluid flows are analysed through three different models. In Neuringer - Rosensweig model the magnetization and magnetic field are parallel with the stress tensor is symmetric. Jenkins model involves the co-rotational derivative of magnetization and the magnetic field is parallel to the magnetization. In Shliomis model the magnetization and applied magnetic field are perpendicular and the stress tensor is asymmetric. In addition, the prominence is provided to the particle velocity and is considered to be the perfect model to demonstrate the flow system of ferrofluid [7]. The Shliomis model is used to find load carrying capacity and pressure distribution in lubrication, in loud speakers to dampen vibrations, hard disk drives, contrast agent for MRI and to study thermal effects in bearings [8,9].

Hayat *et al.,* [10] in their study of flow of magneto Williamson fluid over a variable thickness stretching sheet explored that the magnetic parameter decays the velocity profile of nanoliquid. Ullah *et al.,* [11] specified that the velocity enhances against Hall current in the presence of ion-slip for a Phan-Thein-Tanner fluid passed through the channel. Ellahi [12] provides an analytical solution through which he analyse the effects of magnetohydrodynamic and viscosity dependent temperature on the conducting nanofluid flow problem in a pipe by two different models - Reynolds' model and Vogel's model of viscosity. Sheikholeslami and Ganji [13] in the study on ferrofluid flow and heat transfer analysed the effect of nano particles volume fraction, magnetic parameter generated by ferrohydrodynamics and Hartmann number generated by magnetohydrodynamics. In which they specify that fluid flow enhances with increase of nano particles volume fraction whereas the heat transfer rate enhances and diminishes with increasing values of Hartmann number and magnetic parameter respectively.

Khan *et al.,* [14] in the study of double stratified micropolar fluid flow over a permeable sheet have analysed the effect of various parameters. In which they specify that in the presence of magnetic parameter an increase in the velocity profile, whereas decreases temperature and concentration are observed for an increasing heat source parameter. Jalili *et al.,* [15] studied the effect of magnetic parameter, micro rotation velocity for suction and injection. In which it is observed that the magnetic parameter raises the velocity profiles and for the case of micro rotation, velocity initially has the highest value for higher values of magnetic parameter considered in the study, further results to a lowest value as a dimensionless parameter η exceeds the value 1. Hsu *et al.,* [16] studied the flow of a magnetohydrodynamic fluid by varying magnetic parameter, power - law index, suction parameter and noted that for the increasing values of magnetic parameter with increased values of power - law index and decreased value of suction parameter leads the velocity profile to converge faster to a steady case.

Li *et al.,* [17] analysed that the magnetic number boosts up the temperature gradient for the convective non-Newtonian fluid flow through Darcy-Forchheimer's porous space. Iqbal *et al.,* [18] analysed the effect of magnetic parameter on base fluid (water) and ferrofluid flow on non - isothermal vertical surface. The results reveals that the velocity profile increases and decreases for the case of assisting and opposing flow respectively for enhanced values of magnetic parameter and it is noted that the thickness of boundary layer increases for base fluid in comparison to ferrofluid. Animasaun *et al.,* [19] in the study of stagnation nanofluid flow with dust particles in the presence of magnetic field asserts the effect of magnetic parameter on velocity profiles of fluid and dust phase for a copper and copper oxide water based nanofluid, which specifies that velocity profiles for both the phases are amplified for increasing values of magnetic parameter. Maranna *et al.,* [20] studied the silver and copper nano materials based blood flow in the presence of various parameters such as magnetic parameter, permeability and so on over shrinking/stretching sheet. Ullah *et al.,* [21] studied the effects of various parameters on magnetized-nanofluid flow. In particular, they specified that the magnetic parameter shows dual nature on radial velocity. The profile of velocity initially decreases and further increases.

Mohamed *et al.,* [22] analysed the effects of various parameters on Nusslet number and skin friction in the study of magnetohydrodynamic ferrofluid flow on a flat plate with permeability. Kudenatti *et al.,* [23] analysed the effects of various parameters on a velocity profiles for a flow through constant wedge in a porous medium. In particular, it is observed that due to adverse pressure gradient the far field solutions leads to oscillatory velocity profiles. Hussain and Ahmed [24] discussed magnetohydrodynamic convection ferrofluid flow in a backward facing step around a rotating cylinder. Here the fluid fills the geometry with strong flow near the top wall and the bottom of rotating cylinder in the absence of Hartmann number. Further the increase of Hartmann number, the forced flow under the rotating cylinder disappears gradually indicating the Lorentz force retards the forced convection/flow. Nargund and Asha [25] analysed the load carrying capacity of ferrofluid bearings through Shliomis model. Asha and Achala [26] studied the effect of magnetic field on viscosity and rheology for the ferrofluid flow between two inclined planes.

In the study of third grade fluid flow subjected to thermal radiation and Lorentz force by Hamad *et al.,* [27] emphasis that the magnetic parameter creates a resistive force which opposes the fluid velocity. Astanina *et al.,* [28] studies that the increased values of Hartmann number provides space for heat conduction whereas results in reduction of ferrofluid flow rate in a partial porous medium. Mehrez *et al.,* [29] asserts that the entropy generation and Nusselt number pivots on Hartmann values and angle of inclination of magnetic field in the study of flow of a nanofluid in an cavity. Jakati *et al.,* [30] investigated the effect of magnetic field applied at an inclined angle on velocity, concentration of nano particles and temperature for a fluid flow over a stretching sheet. Animasaun *et al.,* [31] in the study of flow of visco elastic fluid analysed that increasing values of magnetic parameter generated through Lorentz force reduces both the longitudinal and transverse velocity profiles.

Hosseinzadeh *et al.,* [32] analysed the effect of various parameters such as Brownian motion, thermophoresis phenomenon, magnetic parameter and so on with suction and injection for the flow past the moving plates. In which they reveals that the increasing magnetic parameter enhances the velocity profiles whereas reduces the temperature. The velocity profiles and temperature was studied for various parameters by Ramzan *et al.,* [33] in the study of Hall current effect on the nano ferrofluid flow due to low oscillating magnetic in which they assert that the increased value of Hall parameter decreases magnetic damping leading to enhancement of radial velocity and temperature. Sheikholeslami [34] discussed the influence of velocity ratio parameter, radiation parameter, magnetic parameter and temperature index parameter and specified that magnetic parameter induced by Lorentz force retards the flow velocity with the increasing values of the magnetic parameter. The coefficient of skin friction and Nusselt number increases and decreases with the increase of magnetic parameter respectively. Khan *et al.,* [35] analysed the hybrid nano flow over a stretching disk in view of varying parameters like Hall current, magnetic parameter, volume fraction etc.

Shoaib *et al.,* [36] explains that the parameters involved in the work of magnetohydrodynamic flow of hybrid nanofluid over a stretching sheet with thermal radiation as either an increasing or decreasing effect on velocity profiles and temperature. Nandeppnavar *et al.,* [37] in their study showed that a separation exists in laminar boundary layer flow for the case of injection in the presence of magnetic field. Hassan *et al.,* [38] exhibits through graphs that the structure of nano particles impacts the profiles of velocity and temperature for a ferrofluid flow in the presence of oscillating magnetic field. Ellahi *et al.,* [39] interpreted the velocity and temperature profiles variation in the view of concentration of particles and magnetization parameter. They specify that axial velocity has more impact of parameters in comparison to radial and tangential velocity. Bhandari [40] analysed the impact of rotational viscosity, in consideration of constant values of other parameters for a ferrofluid flow based on intensity and frequency fluctuating magnetic field.

Turkyimazoglu [41] imposed that the increasing magnetic field generates a drag force which tends the flow to slow down around the disk in the study of magnetohydrodynamic effect on flow of fluid due to rotating disk with stretching. Bhandari [42] ensures the presence of the parameter like magnetic field dependent viscosity reduces the radial velocity due to the increased drag of the flow with increasing parameter values in their study of ferrofluid flow with magnetoviscous effect. The aforementioned literature survey provides us with the concept to analyse the present work for a ferrofluid flow over a rotating disk through Shliomis model in the presence of alternating magnetic field and Lorentz force as it is new to best of our knowledge.

2. Mathematical Formulation

The flow of axi-symmetric, non-conducting, incompressible ferrofluid flow past a rotating disk with an angular velocity $\vec{\Omega}$ in the z direction is analysed on Shliomis model in consideration with the Lorentz force effect due to the magnetic field applied in the radial direction, as shown in Figure 1. The geometry, equations of motion, momentum of the system in reference to Bhandari [42] are as follows

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{q}) = 0 \tag{1}$$

$$\rho \frac{d\vec{q}}{dt} = -\nabla p + \mu_0\left(\vec{M} \cdot \nabla\right)\vec{H} + \mu\nabla^2\vec{q} + \frac{I}{2\tau_s}\nabla \times \left(\vec{\theta} - \vec{\Omega}\right) + \sigma_m(\vec{J} \times \vec{B}) \tag{2}$$

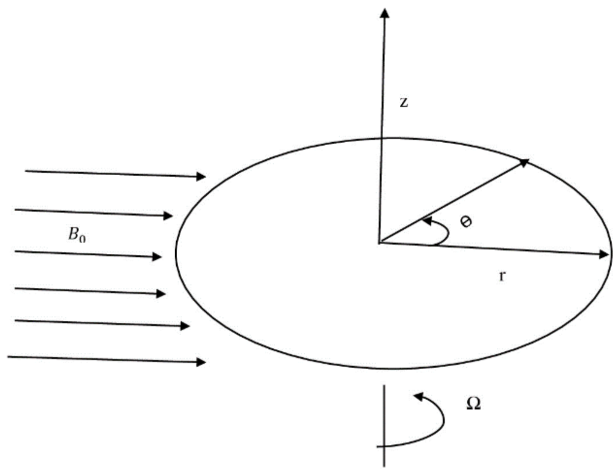

Fig. 1. Geometrical representation of ferrofluid flow

$$I\frac{d\vec{\theta}}{dt} = \mu_0(\vec{M} \times \vec{H}) - \frac{I}{\tau_s}(\vec{\theta} - \vec{\Omega}) \tag{3}$$

$$\frac{d\vec{M}}{dt} = \vec{\theta} \times \vec{M} - \frac{1}{\tau_B}(\vec{M} - \vec{M}_0) \tag{4}$$

$$\nabla \times \vec{H} = 0 \tag{5}$$

$$\vec{J} = \vec{E} + \vec{q} \times \vec{B} \tag{6}$$

$$\vec{B} = \mu_B \vec{H} \tag{7}$$

The Langevin function subjected to instantaneous equilibrium magnetization $\vec{M}_0$ at $\tau_B = 0$ is defined as

$$\vec{M}_0 = nmL(\xi)\frac{\vec{H}}{H}, L(\zeta) = coth\xi - \xi^{-1}, \xi = \frac{mH}{kT} \tag{8}$$

In comparison to relaxation term, inertial term in Eq. (3) can be neglected $\left(I\frac{d\vec{\theta}}{dt} \ll I\frac{\vec{\theta}}{\tau_s}\right)$, which can be written as

$$\vec{\theta} = \vec{\Omega} + \mu_0\frac{\tau_s}{I}(\vec{M} \times \vec{H}) \tag{9}$$

In view of Eq. (9), The altered form of Eq. (2) and Eq. (4) are written as

$$\rho\left[\frac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right] = -\nabla p + \mu\nabla^2\vec{q} + \mu_0(\vec{M}\cdot\nabla)\vec{H} + \frac{\mu_0}{2}\nabla \times (\vec{M} \times \vec{H}) \tag{10}$$

$$\frac{d\vec{M}}{dt} = \vec{\Omega} \times \vec{M} - \frac{1}{\tau_B}(\vec{M} - \vec{M}_0) - \mu_0\frac{\tau_s}{I}\vec{M} \times (\vec{M} \times \vec{H}) \tag{11}$$

The system is subjected to the magnetic field in radial direction, which is the form as

$$H_r = H_0 cos\omega_0 t, H_\theta = 0 = H_z \tag{12}$$

where ω_0 is the angular frequency of applied magnetic field. We assume the amplitude of magnetic field is small, that is $mH_0 << kT$. Considering series in $coth$ and neglecting terms of order greater than one, the Eq. (8) can be written as

$$\vec{M}_0 = \chi'\vec{H}, \chi' = \frac{\mu_0 n m^2}{3kT} \tag{13}$$

Here χ' is the magnetic susceptibility.

Due to superposition of two rotating fields, Eq. (12) can be written as

$$\vec{H} = \frac{1}{2}\left(\vec{H}_+ + \vec{H}_-\right) \tag{14}$$

Where

$$\vec{H}_+ = H_0 cos\omega_0 t\hat{\imath} + H_0 sin\omega_0 t\hat{\jmath} + 0\hat{k} \tag{15}$$

$$\vec{H}_- = H_0 cos\omega_0 t\hat{\imath} - H_0 sin\omega_0 t\hat{\jmath} + 0\hat{k} \tag{16}$$

For a quiescent state of a fluid, there exists phase angle α_0 due to the lag in the magnetization and the magnetic field. Then

$$\vec{M}_+ = Mcos(\omega_0 t - \alpha_0)\hat{\imath} + Msin(\omega_0 t - \alpha_0)\hat{\jmath}, \quad \vec{\theta}_+ = +\theta\hat{k} \tag{17}$$

$$\vec{M}_- = Mcos(\omega_0 t - \alpha_0)\hat{\imath} - Msin(\omega_0 t - \alpha_0)\hat{\jmath}, \quad \vec{\theta} = -\theta\hat{k} \tag{18}$$

Eq. (9) and Eq. (4) indeed of Eq. (14) to Eq. (18) are obtained as

$$M = \chi' H_0 cos\alpha_0, \theta = \left(\mu_0 \frac{\tau_s}{I}\right) M H_0 sin\alpha_0, tan\alpha_0 = (\omega_0 - \theta)\tau_B \tag{19}$$

Here $\xi = \frac{mH_0}{kT\sqrt{2}}$, Langevin parameter acting over the magnetic field $\frac{H_0}{\sqrt{2}}$, $\frac{I}{\tau_s} = 6\mu\phi$, where $\phi = nV$ is the volume fraction of the particles, V is the volume of the particles and $\tau_B = \frac{3\mu V}{kT}$. The angle α_0 is eliminated from Eq. (19) and including terms upto ξ^2 results in

$$M = \frac{\chi'\xi H_0}{\sqrt{1+\omega_0^2\tau_B^2}}, \theta = \frac{\omega_0\xi^2/3}{1+\omega_0^2\tau_B^2}, tan\alpha_0 = \omega_0\tau_B\left(1 - \frac{\xi^2/3}{1+\omega_0^2\tau_B^2}\right) \tag{20}$$

In view of hydrodynamical vortex $\Omega = (0,0,\Omega)$ and in difference of rotation rates of left and right polarized fields the Eq. (8) and Eq. (19) can be written as

$$M_+ = \chi' H_0 cos\alpha_{0_+}, \theta_+ = \left(\frac{\tau_s}{I}\right) M_+ H_0 sin\alpha_{0_+} + \Omega, tan\alpha_{0_{\mp}} = (\omega_0 - \theta_+)\tau_B \tag{21}$$

$$M_- = \chi' H_0 cos\alpha_{0_-}, \theta_- = \left(\frac{\tau_s}{I}\right) M_- H_0 sin\alpha_{0_-} - \Omega, tan\alpha_{0_-} = (\omega_0 - \theta_-)\tau_B \tag{22}$$

The same sign of ω_0 and Ω ensures the rotation of magnetic particles are faster. Hence the last terms of Eq. (21) and Eq. (22) are written as

$$tan\alpha_{0_+} = (\omega_0 - \Omega)\tau_B, tan\alpha_{0_-} = (\omega_0 + \Omega)\tau_B \tag{23}$$

The magnetization components of θ due to alternating magnetic field is written as

$$M_\theta^+ = \chi' H_0 cos\alpha_{0_+} sin(\omega_0 t - \alpha_{0_+}) = \frac{\chi' H_0}{1+(\omega_0-\Omega)^2\tau_B^2}[sin\omega_0 t - (\omega_0 - \Omega)\tau_B cos\omega_0 t] \tag{24}$$

$$M_\theta^- = -\chi' H_0 cos\alpha_{0\_} sin(\omega_0 t - \alpha_{0\_}) = -\frac{\chi' H_0}{1+(\omega_0-\Omega)^2\tau_B^2}[sin\omega_0 t - (\omega_0 + \Omega)\tau_B cos\omega_0 t] \tag{25}$$

The diffusion time for a Brownian rotational motion of the particles with a diameter 10^{-6} cm is usually to not exceed $10^{-5}s$, hence $\omega\tau_B \ll 1$ valid apart from high viscous fluid.

Thus for the linearly polarized field along the radial direction the magnetization in tangential component is written as

$$\vec{M}_\theta = \frac{1}{2}\left(\vec{M}_\theta^+ + \vec{M}_\theta^-\right) = \chi'\Omega\tau_B H_0 cos^2\alpha_0 cos(\omega_0 t - 2\alpha_0) \tag{26}$$

The magnetization perpendicular (along z -axis) to the fields impedes the magnetic torque upon the particle for a magnetic field polarized in θ direction, leads Eq. (26) to following form

$$\begin{aligned}\mu_0\left(\vec{M} \times \vec{H}\right) &= -\mu_0\vec{\Omega}\tau_B \frac{\chi}{\mu_0} H_0^2 cos^2\alpha_0 cos\omega_0 t cos(\omega_0 t - 2\alpha_0) \\ &= -2\Omega\mu\phi\xi^2 cos^2\alpha_0(cos^2\omega_0 t cos2\alpha_0 + sin\omega_0 t cos\omega_0 t sin2\alpha_0)\end{aligned} \tag{27}$$

On averaging the terms in Eq. (27), for the field variation $\frac{2\pi}{\omega_0}$, we get

$$\frac{1}{\left(\frac{2\pi}{\omega_0}-0\right)}\int_0^{\frac{2\pi}{\omega_0}} cos^2\omega_0 t dt = \frac{1}{2}, \qquad \frac{1}{\left(\frac{2\pi}{\omega_0}-0\right)}\int_0^{\frac{2\pi}{\omega_0}} sin\omega_0 t cos\omega_0 t dt = 0 \tag{28}$$

Thus, Eq. (27) becomes

$$\mu_0\overline{\left(\vec{M} \times \vec{H}\right)} = -\Omega\mu\phi\xi^2 cos^2\alpha cos2\alpha_0 \tag{29}$$

Let us consider

$$\nabla p + \mu_0\left(\vec{M}\cdot\nabla\right)\vec{H} = -\nabla\tilde{p} \tag{30}$$

$$\vec{B} = (\mu_B H_0 cos\omega_0 t, 0, 0) \tag{31}$$

Then $\vec{J} \times \vec{B}$ with a negligible electric field takes the form as

$$\vec{J} \times \vec{B} = (0, -\mu_B^2 H_0^2 cos^2\omega_0 t v_\theta, -\mu_B^2 H_0^2 cos^2\omega_0 t v_z) \tag{32}$$

Subjected to the field variation of period $\frac{2\pi}{\omega_0}$, we get components of $\vec{J} \times \vec{B}$

$$\vec{J} \times \vec{B} = (0, -\mu_B^2 H_0^2 v_\theta, -\mu_B^2 H_0^2 v_z) \tag{33}$$

Eq. (10) in view of Eq. (29) and Eq. (30) can be written as

$$\rho \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q}\right] = -\nabla p + \mu \left(1 + \frac{1}{4}\phi\xi^2 cos^2\alpha cos2\alpha\right)\nabla^2\vec{q} + \sigma_m(\vec{J} \times \vec{B}) \tag{34}$$

Now in the cylindrical form, for a steady, axially symmetric flow of an incompressible ferrofluid the Eq. (34) with Eq. (33) is written as

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 \tag{35}$$

$$-\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial r} + \nu\left(1 + \frac{1}{4}\phi\xi^2 cos^2\alpha_0 cos2\alpha_0\right)\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_r}{r}\right) + \frac{\partial^2 v_r}{\partial z^2}\right] = v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \tag{36}$$

$$\nu\left(1 + \frac{1}{4}\phi\xi^2 cos^2\alpha_0 cos2\alpha_0\right)\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right] - \frac{\sigma_m}{\rho}\mu_B^2 H_0^2 v_\theta = v_r\frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + v_r\frac{\partial v_\theta}{\partial z} \tag{37}$$

$$-\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z} + \nu\left(1 + \frac{1}{4}\phi\xi^2 cos^2\alpha_0 cos2\alpha_0\right)\left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right] - \frac{\sigma_m}{\rho}\mu_B^2 H_0^2 v_z = v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} \tag{38}$$

where v_r, v_θ, v_z are the velocities in radial, tangential and axial direction and ν is the kinematic viscosity. With the boundary conditions in view of Von Karman similarity transformation given by Kármán [43]

$$v_r = 0, v_\theta = r\Omega, v_z = 0 \quad \text{for} \quad z = 0; v_r = 0, v_\theta = 0 \quad \text{for} \quad z \to \infty \tag{39}$$

To non-dimensionalize the above set of equations, we consider the following

$$\begin{cases} v_r = r\Omega E(\alpha), v_\theta = r\Omega F(\alpha), v_z = \sqrt{\frac{\mu}{\rho}\Omega}G(\alpha) \\ \tilde{p} = \rho\Omega\frac{\mu}{\rho}P(\alpha), \alpha = z\sqrt{\frac{\rho}{\mu}\Omega} \end{cases} \tag{40}$$

Eq. (35) to Eq. (38), with similarity transformation as in Eq. (4) reduces to non-dimensional ordinary differential equations with the boundary conditions as follows

$$G' + 2E = 0 \tag{41}$$

$$kE'' - E'G - E^2 + F^2 = 0 \tag{42}$$

$$kF'' - GF' - 2EF - MnF = 0 \quad (43)$$

$$P' - kG' + GG' + MnG = 0 \quad (44)$$

$$E(0) = 0, F(0) = 1, G(0) = 0, P(0) = 0; E(\infty) = 0, F(\infty) = 0. \quad (45)$$

where $k = 1 + \frac{1}{4}\phi\xi^2 cos^2\alpha_0 cos2\alpha_0$ is the dimensionless viscosity parameter dependent on magnetic field and $Mn = \frac{\sigma_m \mu_B^2 H_0^2}{\rho\Omega}$ is the magnetic parameter. The solution are evaluated for varying values of the parameters k and Mn through shooting method.

2.1 Skin Friction Coefficients

The radial and tangential shear stress are as follows from Sparrow and Cess [1]

$$\tau_r = \left[\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial v_r}\right)\right]_{z=0} \qquad \tau_t = \left[\mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial v_\theta}\right)\right]_{z=0} \quad (46)$$

with the Eq. (39) and Eq. (40), skin friction coefficients in radial and tangential direction are

Radial direction

$$C_{f_r} = \frac{\tau_r}{\rho r\sqrt{(\nu\Omega^3)}} = E'(0) \quad (47)$$

Tangential direction

$$C_{f_t} = \frac{\tau_t}{\rho r\sqrt{(\nu\Omega^3)}} = F'(0) \quad (48)$$

The displacement thickness of the boundary layer for a rotating disk is calculated as

$$d = \frac{1}{r\Omega}\int_0^\infty v_\theta dz = \int_0^\infty F(\alpha)d\alpha \quad (49)$$

3. Method of Solution

To solve the set of reduced nonlinear ordinary differential equations we adapt the shooting method which has restrictions on requirement of an initial conditions. That is the number of initial conditions must be same as the order of equations. Firstly, in the shooting technique equations of higher order Eq. (41) to Eq. (44) are reduced to the system of first order ordinary differential equations using following assumptions.

$$f_1 = E, \quad f_3 = F, \quad f_5 = G, \quad f_7 = P \quad (50)$$

$$f_1' = f_2 \quad (51)$$

$$f_2' = \frac{1}{k}(f_2 f_5 + f_1^2 - f_3^2) \quad (52)$$

$$f_3' = f_4 \tag{53}$$

$$f_4' = \frac{1}{k}(f_4 f_5 + 2f_1 f_3 + Mn f_3) \tag{54}$$

$$f_6 = -2f_1 \tag{55}$$

$$f_7' = kf_6 - f_5 f_6 - Mn f_5 \tag{56}$$

with the boundary conditions

$$f_1(0) = 0, f_3(0) = 1, f_5(0) = 0, f_7(0) = 0, f_1(\infty) = 0, f_3(\infty) = 0. \tag{57}$$

Further in our system we are short of two initial conditions which are obtained initially by guess. That is, the method requires the appropriate initial guess for $f_2(0)$ and $f_4(0)$. These initial guess should be guessed such that it satisfies the boundary conditions for a considered α_∞. The good convergences is obtained by improvising the initial guess by Secant method. When the difference between consecutive values of solutions are upto a tolerance of 10^{-6}, the iterative process terminates after the resultant ordinary differential equations which are integrated using Runge - Kutta method of fourth order.

4. Results and Discussion

In the present study of steady, incompressible axi-symmetric ferrofluid flow over a infinite rotating disk in the presence of external magnetic field. We analyse the velocity profiles and pressure in view of magnetic field dependent viscosity parameter k dependent on magnetic field and magnetic parameter Mn generated by Lorentz force.

Figure 2 to Figure 5 represents the variation in velocity profiles and pressure for different values of magnetic field dependent viscosity with constant value of the magnetic parameter. In Figure 2, the radial velocity attains the same maximum value and position of this maximum value shifts away from the disk for the increasing value of the magnetic field dependent viscosity parameter. From Figure 3 it is notified that the tangential velocity increases with increasing values of k. Physically the magnetic field viscosity dominates the inertial forces for increasing values of α, as a result velocity profile increases with the increasing k. Figure 4 represents the increased negative values of axial velocity for increasing values of the magnetic field dependent viscosity. Practically the negative value indicates the inward flow in the axial direction. Physically, near the disk, in the presence of magnetic field dependent vicosity, the fluid particles and fluid exerts difference in its velocity which results in adding up more resistance to the flow. This additional resistance initially reduces the number of molecules colliding the surface as a result the pressure decreases which is evident through Figure 5. we notice that the pressure decreases for increasing values of k in comparison to $k = 1$, which represents the case of no magnetic field. Further we notice that the pressure initially decrease near to the disk and for increasing values of non - dimensional parameter α the pressure increases.

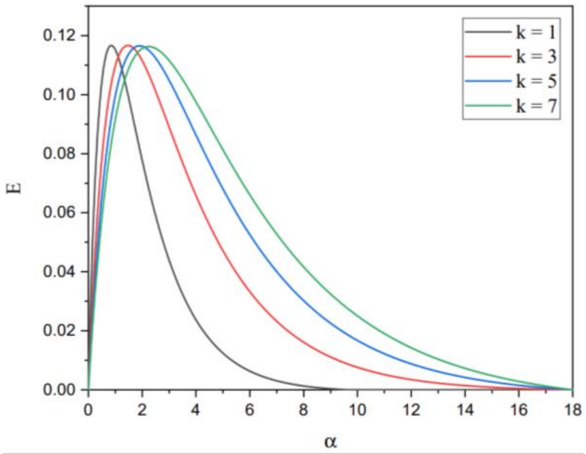

Fig. 2. Radial velocity for varying k and $Mn = 0.8$

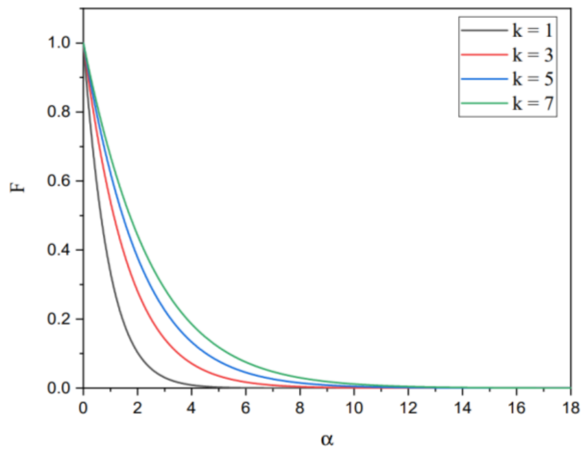

Fig. 3. Tangential velocity for varying k and $Mn = 0.8$

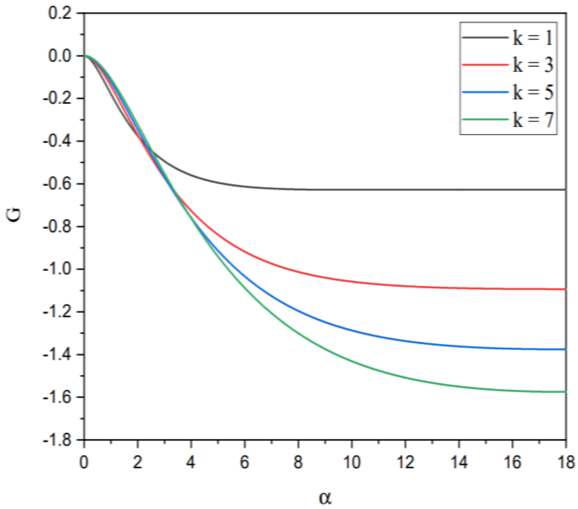

Fig. 4. Axial velocity for varying k and $Mn = 0.8$

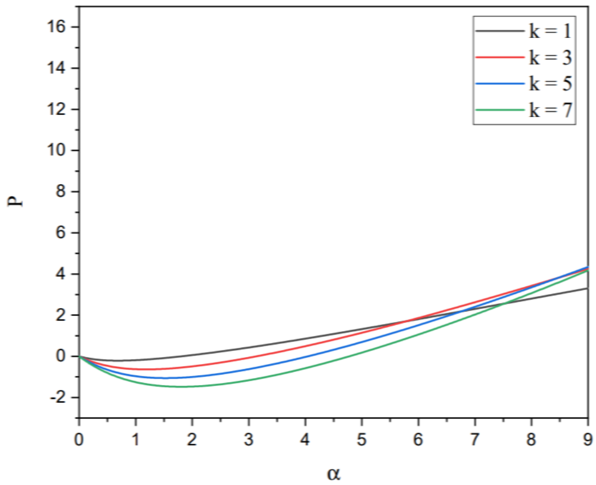

Fig. 5. Pressure for varying k and $Mn = 0.8$

Figure 6 and Figure 7 represent the derivatives of radial and tangential velocities. In particular radial skin friction coefficient $E'(0)$ decreases, tangential skin friction coefficient $F'(0)$ increases which are evident from the Table 1 at $z = 0$ for increasing values of magnetic field dependent viscosity parameter. Table 2 represents the boundary layer thickness which increases with rising values of k.

Table 1
Radial and tangential skin friction coefficients for $Mn = 0.8$

	$k = 1$	$k = 3$	$k = 5$	$k = 7$
C_{f_r}	0.3907	0.2256	0.1747	0.1475
C_{f_t}	-1.0134	-0.5851	-0.4532	-0.3829

Table 2
Boundary layer displacement thickness $Mn = 0.8$

	$k = 1$	$k = 3$	$k = 5$	$k = 7$
d	-1.7750	1.5550	2.0082	2.4357

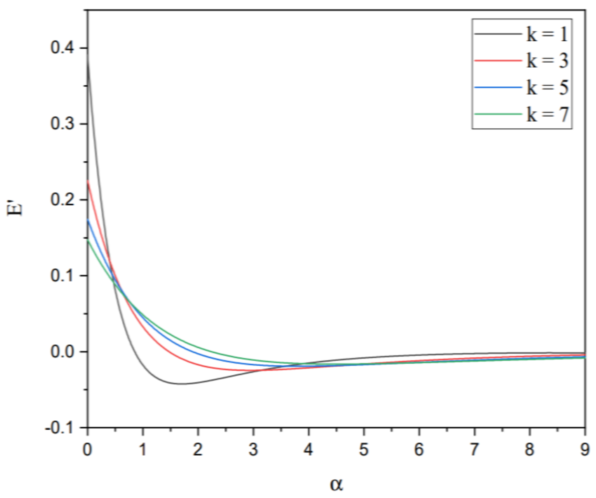

Fig. 6. Derivative of radial velocity for varying k and $Mn = 0.8$

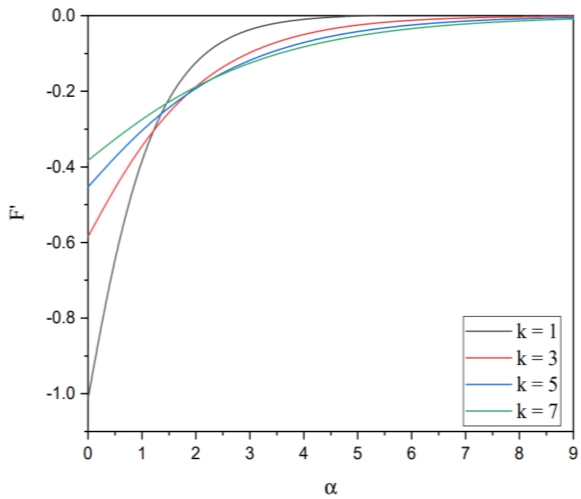

Fig. 7. Derivative of tangential velocity for varying k and $Mn = 0.8$

Figure 8 to Figure 11 represents the graphs plotted for varying magnetic parameter for a fixed value of k. Figure 8 represents the variation in the radial velocity for different values of magnetic parameter. The radial velocity decreases for increasing values of magnetic parameter. Figure 9 indicates the decrease in the tangential velocity for increasing values of magnetic parameter. From the graph of axial velocity as in Figure 10 the negative values of the velocity decreases with increasing magnetic parameter. The negative value asserts the fluid is moved towards disk, which are flown outward in the radial direction. The physical reason behind the decrease of velocity profiles is the magnetic parameter obtained through Lorentz force. This Lorentz forces creates a drag force which has the tendency to reduce the fluid flow. Further pressure in Figure 11 represents the increase in its profile with increase of magnetic parameter. This is physically evident through graphs that, as velocity decreases the pressure increases.

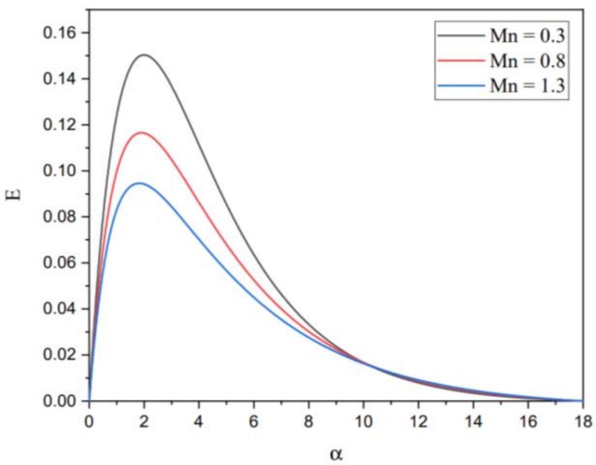

Fig. 8. Radial velocity for varying Mn and $k = 5$

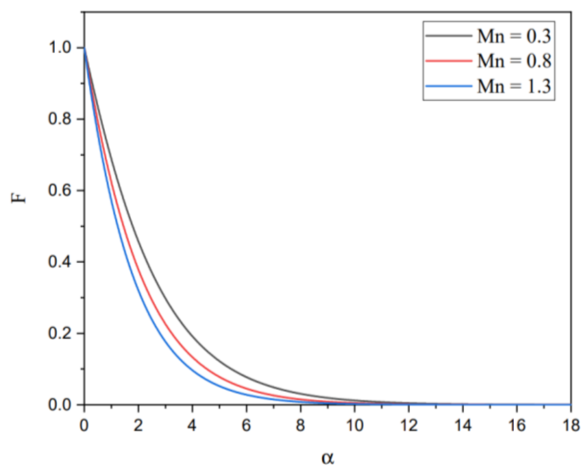

Fig. 9. Tangential velocity for varying Mn and $k = 5$

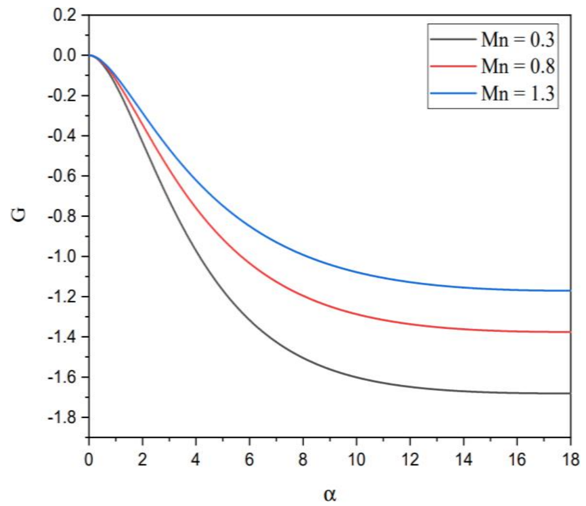

Fig. 10. Axial velocity for varying Mn and $k = 5$

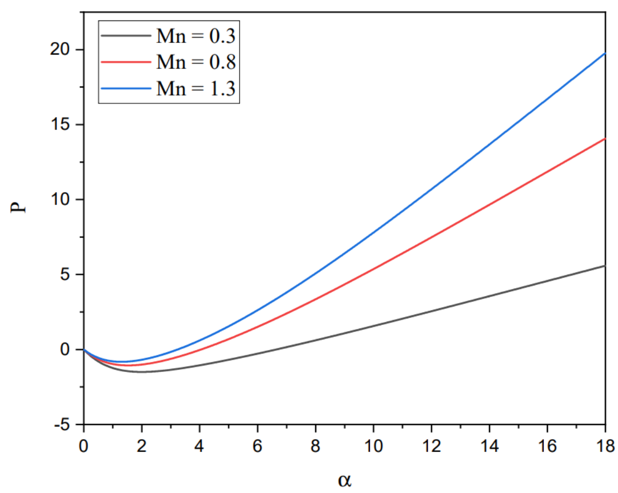

Fig. 11. Pressure for varying Mn and $k = 5$

Figure 12 and Figure 13 represent derivatives of radial and tangential velocities for varying magnetic parameter. Table 3 represents skin friction coefficients in radial and tangential skin friction which decreases with increasing magnetic parameter. The increasing values of Mn decreases the boundary layer thickness, which is evident from Table 4. Table 5 provides the limiting case of the present work to that of the Bhandari [42] for Mn=0, which is in quantative argeement with the available result.

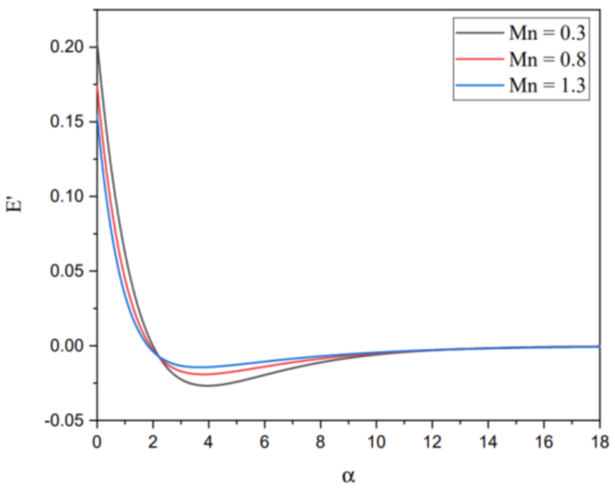

Fig. 12. Derivative of radial velocity for varying Mn and $k = 5$

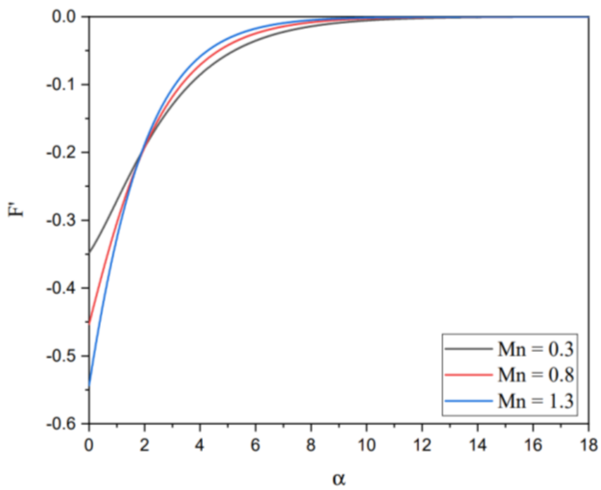

Fig. 13. Derivative of tangential velocity for varying Mn and $k = 5$

Table 3
Radial and tangential skin friction coefficients for $k = 5$

	$\boldsymbol{Mn = 0.3}$	$\boldsymbol{Mn = 0.8}$	$\boldsymbol{Mn = 1.3}$
C_{f_r}	0.2037	0.1747	0.1545
C_{f_t}	-0.3481	-0.4532	-0.5437

Table 4
Boundary layer displacement thickness $k = 5$

	$\boldsymbol{Mn = 0.3}$	$\boldsymbol{Mn = 0.8}$	$\boldsymbol{Mn = 1.3}$
$\boldsymbol{d}$	2.4357	2.0082	1.7326

Table 5
Comparison of the present result with Bhandari [42] for Mn = 0

	$k = 5$	
	Bhandari	Present
C_{f_r}	0.17513	0.18800
C_{f_t}	-0.23764	-0.29281

5. Conclusions

Shliomis model is considered to analyse the ferrofluid flow on a rotating disk with the Lorentz effect. This model is considered to be an appropriate model to describe flow of a ferrofluid in the system. In the present paper, analysis of parameters leads to the following observations.

i. There exists slow convergence due to increase in the radial and tangential velocity profile with the increase of magnetic field dependent viscosity parameter.
ii. The axial velocity converges to a constant negative value at the far distance from the disk. The negative value of axial velocity, positive value of pressure increases with the increasing values of magnetic field dependent viscosity.
iii. Profiles of radial, tangential and axial velocities decreases whereas pressure increases with increase of magnetic parameter.
iv. The presence of Lorentz force reduces the flow due to increased drag in the fluid.
v. The skin friction coefficient decreases in radial direction and increases in tangential direction for varying magnetic field dependent viscosity parameter.
vi. We observe decrease of skin friction coefficient for both in radial and tangential direction for increasing values of the magnetic parameter.
vii. The boundary layer thickness increases and decreases for magnetic field dependent viscosity and magnetic parameter respectively.

The present study will be applicable to enhance the life of hard disk drive, to improve the capturing capacity of wavelength in optics and to analyse minute contrast in MRI scans by maintaining a suitable level of viscosity and inclination of magnetic field in ferrofluid.

Acknowledgement
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