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Effect of Graded Mesh Number on the Solution of Convection-Diffusion Flow Problem with Quadratic Source

Aslam Abdullah[1,*]

[1] Department of Aeronautical Engineering, Faculty of Mechanical and Manufacturing Engineering, Universiti Tun Hussein Onn Malaysia, 86400 Parit Raja, Batu Pahat, Johor, Malaysia

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ABSTRACT

Convection-diffusion phenomena are fundamentally modelled in a variety of engineering fields and physical sciences. A suitable meshing strategy is needed in computational fluid dynamics to solve model problems numerically. Unintentional implementation of the approach could lead to subpar solutions such as erroneous oscillations, over- or under-predictions, and excessive computing time. This paper highlights the significance of the influence of mesh structure on the solution of convection-diffusion flow problem with quadratic source for flow parameters of interest. Particularly, it presents the accuracy of the solution of the flow problem at low Peclet number with respect to graded mesh number, where the expansion factor used is based on an established linear logarithmic model involving Peclet number derived in previous works. The problem is solved by assigning several mesh intervals to graded mesh against each Peclet number of interest. Based on the values of the interval and Peclet number, 16 test cases are considered. Quantitative results lead to orders of accuracy of the solution of the flow problem. The effect of graded mesh number on the accuracy thus serves as a reference for a more structured decision-making and improves the heuristic process in choosing the computational domain mesh with expected order of accuracy for the numerical solution of the problem particularly in the calculation of scalar concentration. The orders of accuracy confirm the profiles of concentration.

1. Introduction

1.1 Engineering Applications

In fluid mechanics, the most fundamental phenomena include the transport of heat, mass and momentum since it is a universally and fundamentally natural problem [1,2]. It is extremely important to model and describe the phenomena in various engineering disciplines [3,4], aviation [5-10], meteorology, and physical sciences [1,2,11,12]. The mathematical framework for heat and mass transfer are of same kind, and basically encompassed by advection and diffusion effects. An initially discontinuous profile is propagated by diffusion and convection (or advection), the latter with a

* *Corresponding author.*
E-mail address: aslam@uthm.edu.my

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speed [1,11,12], in such general scalar transport equations. These equations are frequently used in computational simulations, such as wake vortex simulation in aviation [7,8], petroleum reservoir simulation, and global weather prediction.

It is possible to utilise the convection-diffusion equation to describe the dynamics of aircraft wake vortices, which are a highly major aviation hazard [7]. These vortices are created as an aircraft takes off and flies through the air. More advanced study includes the wake vortex interaction with turbulent jet which shows that, by solving the convection–diffusion equation, the passive scalar of interest's ability to penetrate inside the vortex core is recognized depending on the distance between the jet and the vortex axis [8].

Invasive sampling of jet engine exhaust gases is another use in aviation where particle losses due to convection, diffusion, and thermophoresis inside a particulate probe are studied and measured [9]. Additionally, the equation may be used to describe how well hot fluid injection works to melt ice off aircraft wings when combined with a nonlinear singular integro-differential equation and Stefan condition [10].

The equation regulates, for example, tracer transport as one of the most typical issues, and is particularly helpful in many aspects of petroleum reservoir engineering. By injecting the tracers into the subsurface porous medium, qualitative data on flow barriers, directional flow trends, and reservoir communication are gathered [13].

Given a velocity field which is known a priori, the method for solving convection-diffusion problem of the second order which initially appeared in 1990 in the environmental science and engineering literature is generic, linear, and of steady-state. It was then independently developed and published in engineering literature in 1993 [2].

If the expression of derivatives depends only on the local characteristic of the function, then the governing differential convection-diffusion equations are of integer order. In a more advanced method, nonlinear fractional differential equations have a so-called memory effect, where the whole information of the function is accumulated in a weighted form [12]. The equations' efficient numerical solutions are a popular research topic due to its widespread usage [5].

1.2 Mesh Challenges

The usage of graded mesh is mainly important in finite element (FEM) [14-18], finite-difference (FDM) [19], exponential B-spline [14], and Newton methods [18]. In particular, the mesh is highly useful in the numerical experiment of reaction-diffusion problems [14,16,17], singularly perturbed problems with two parameters [15], sub-diffusion problems with nonlocal diffusion term [18], and evolution problems with a weakly singular kernel [19]. Non-graded mesh on which these problems are solved might include, for instance, a *local* algorithm which has been proposed [20] to obtain an optimal shape parameter for the infinitely smooth Radial Basis Functions (RBF) under grid-free environment.

In fact, the non-uniform mesh has long been taken into consideration for solving integro-differential equations. For instance, research has been done on the numerical collocation on graded mesh solutions of weakly singular Volterra integral equations. The implicit finite difference with non-uniform time steps in the time fractional diffusion equations has also been a subject of study up until recently [19].

The non-uniform, layer-adapted, exponentially graded mesh was created by optimising certain upper bounds on the error, among the many others meshes that have been presented [16]. These produced meshes [14-17] are graded rather than essentially equidistant.

The mesh is either exponentially graded and constructed optimally for the approximation of exponential layers in the numerical solution of the reaction-diffusion equation [16], with a stronger limitation on the graduation parameter [17], or it is produced iteratively using Newton's algorithm and some implicitly defined function [14]. A thorough comparison with different adaptive meshes was offered after several test samples were collected. Additionally, comparisons between different numerical approaches may be found in the literature. It has been demonstrated that parameter uniform convergence of the optimum order is possible. Researchers [14,16,17] established resilient, optimum convergence rates in a range of norms and used numerical calculations to demonstrate their theoretical conclusions.

On a Shishkin mesh, the findings for one- and two-dimensional reaction-diffusion problems that were presented in smooth domains [16] were compared to FEM. In one-dimensional reaction-diffusion and convection-diffusion problems given in the unit square, it was an extension of the development of resilient and optimum rates of convergence where the numerical testing revealed the mesh outperformed the similarly optimum Bakhvalov-Shishkin mesh.

Using higher-order finite element methods, graded meshes are preferable than equidistant meshes for singularly perturbed reaction–diffusion problems [14]. In order to construct the mesh iteratively, some upper limits on the error were manipulated which leads to the optimum order parameter uniform convergence. Comparison of results was made with those attained over Shishkin mesh, Vulanovi´c mesh, Bakhvalov–Shishkin mesh, and modified Bakhvalov mesh. Further, the comparison of the results obtained with various finite difference techniques and exponential B-spline techniques were made. The benefit of a graded mesh is that a transition point is not necessary in order for it to function, contrary to, for instance, Shishkin mesh [21]. It should be noted, however, that one has to predefine mesh expansion factor r_e in using graded mesh. The improved graded mesh appeared as the best in numerical trials and is therefore a viable option.

Supercloseness results were demonstrated in the numerical solution of a model reaction-diffusion problem using the conventional Q1 finite element approximation [17]. An essentially optimum order error estimate with a constant independent of the singularly perturbed parameter ϵ was produced in a more comprehensive study on the usage of graded meshes for singularly perturbed model problems. The outcomes were evaluated against those found on Shishkin mesh. In numerical studies, it was shown that Shishkin meshes created for specific values of ϵ do not provide reasonable approximations for larger values of ϵ, but graded meshes are independent of ϵ and may thus be utilised for a range of ϵ. This can be useful, for instance, in the numerical approximation of equation systems with various order diffusion parameters. It should be noted, however, that one has to be very careful in selecting mesh expansion factor r_e for different ϵ. Some works even proposed the logarithmic relationship between both parameters to prevent spurious oscillation in the solution [22,23]. Furthermore, boundary conditions also play important roles on the accuracy of the relationship. It was found that it is more stringent for boundary conditions of $\xi(0) = 0, \xi(1) = 1$, in comparison to $\xi(0) = \xi(1) = 0$, where the mesh number has to be greater in the former case, for the same value of r_e. Note that ξ is the concentration scalar.

It was established that when the Duran-Lombardi and Duran-Shishkin type meshes used to solve the reaction-diffusion problem numerically, the solution is of first order error estimates and uniform convergence in an energy norm [15]. It was found that the numerical errors were small than those corresponding to the well-known Shishkin mesh. Therefore, the layer-adapted meshes offer intriguing options. It is reasonable to anticipate that a mesh created for a certain value of a small parameter would perform well for larger values of the parameter when a singularly perturbed problem is approximated using an a priori adapted mesh. The Duran-Lombardi and Duran-Shishkin type meshes exhibit even better performance in this regard.

In the second-order accurate numerical solution of partial differential evolution diffusion equations with nonlocal diffusion term, which generally exhibit a weakly singular kernel near the initial time, the use of L1 scheme on graded mesh was shown to be promising compared to uniform mesh [18,19]. A fully discrete difference scheme was constructed with space discretization by fourth-order accuracy compact difference method, while the Riemann-Liouville fractional integral was approximately calculated using the product integration approach for the time discretization, and a generalised second order accuracy Crank-Nicolson compact difference scheme for time-stepping was taken into consideration for smoothness of the solution at $t = 0$. Even second order accuracy, though, was not possible with uniform steps. With the right choice of r_e for smaller time steps on the targeted section of the graded mesh, one may account for the solution's singular behaviour and recover the optimum convergence in the maximum norm with respect to time while analysing a problem that has an initial weak singularity at $t = 0$. The efficacy of the approach is particularly demonstrated by the fact that the resultant difference scheme is stable and convergent, with convergence orders of 2 and 4 for time and space, respectively. The solution on uniform mesh, in contrary, gives lower order of convergence in maximum norm in time [18].

In general, the advantages of having graded mesh for the numerical calculations are as follows. First of all, it is regarded as a development of classic meshes like those in the Bakhvalov and Shishkin classes [14,16]. It has some desirable properties that the latter do not have [15], one of which is the achievement of optimum rate of convergence [18].

Secondly, it is simple to see that the logarithmic factor's impact on linear components is negligible. For instance, it was shown that for coefficient of diffusivity $\varepsilon \geq 0.32$, $r_e = 1$ is sufficient [14,22,23]. Thirdly, graded mesh is more resilient since the neighbouring intervals have comparable widths [14]. It is worth noting that in this case, r_e must fulfil correct relationship with ε. This ensures that numerical results are not strongly affected by variations of flow parameters. In other words, r_e compensates for especially singular behaviour of the solutions.

Fourthly, the effectiveness of graded mesh is demonstrated by a comparison between the method on it and that on uniform mesh [19].

1.3 Model of Interest

In differential form, we define general problem model of interest as

$$Lu: = -\epsilon\xi'' + c(x)\xi' + d(x)\xi = e(y), \text{ for } x \in (0,1), y \in (0,1) \tag{1}$$

where $-\epsilon\xi''$, $c(x)\xi'$, $d(x)\xi$, and $e(y)$ are diffusive, convective, reactive, and sink/source terms, respectively, $c(x)$, $d(x)$, and $e(y)$ are sufficiently smooth functions, and parameter ξ is unknown. It is assumed that

$\epsilon > 0,$

$d(x) \geq 0$ in $[0,1]$

$c(x) > 0$ for all $x \in [0,1]$,

$$e(y) = -4y^2 + 4y \tag{2}$$

The solution of Eq. (1) remains the same even if the variable transformation is applied to x such that it becomes $1-x$, if $c(x)<0$ in $[0,1]$. It is obvious that the solution of Eq. (1) is spatially linear if there are no convection, reaction, and source. We are not interested in such pure diffusion process. The equation is important with regard to convection only if $c(x)$ is nonzero for all $x\in[0,1]$. Singularly perturbed problem arises when the parameter of singular perturbation $\epsilon \ll \|c\|_{L^{\infty}(\Omega)}$.

A boundary layer can manifest in the solution of Eq. (1) and Eq. (2) at $x=1$ when ϵ is small. This has a huge impact on the stability and consistency of these equations, which show the error in the finite difference method. Enhancement of the consistency of the technique is possible if boundary values are given such that the boundary layer vanishes, but its stability is not guaranteed [24,25].

The reactive and source/sink terms being zero and quadratic, respectively, in this paper. Thus for $x\in(0,1)$, Eq. (1) is simplified into

$$Lu:=-\epsilon\xi''+c(x)\xi'=e(y) \tag{3}$$

According to Eq. (3), the scalar concentration ξ is dispersed in the fluids and transported away by the motion of the fluid element in diffusion and convection, respectively. It initially grows slowly in space when ϵ is small and with proper border conditions, and abruptly changes beyond a certain distance. The significant increase of ξ provides challenges for computational fluid dynamics in the aspects of discretization method and computational domain mesh structure.

In this work, we explore a model of a convection-diffusion problem with quadratic source that is discretized using finite difference method and solved on graded mesh with various mesh numbers N and expansion factors r_e. The outcomes of past numerical analysis justify the adoption of the mesh. We observe average error with respect to mesh number N and Peclet number Pe to determine the rate of convergence. Variation of N determines average mesh width. The works on relationship between mesh structure and Peclet number as well as robustness of graded mesh that were debated in several other papers [22,23] are extended in the paper. The choice of r_e for each Pe of interests has to be made with care in order to prevent nonphysical solutions.

Generally, the need to solve the system of equations have certainly sparked broad study on various mesh schemes and structures. The effect of mesh width in graded mesh with mesh expansion factor r_e on the solution of 2-dimensional convection-diffusion flow problem with quadratic source at various Peclet numbers Pe, however, is an open question. Examining such effect is essential to challenge the claimed robustness of graded mesh in solving the governing equation of interest. Quantifying the rate of convergence of the solution is the aim of this research.

2. Methodology

The following are the boundary conditions for the model problem's formulation in Eq. (3)

$$\begin{aligned}\xi(0)&=0\\ \xi(1)&=0\end{aligned} \tag{4}$$

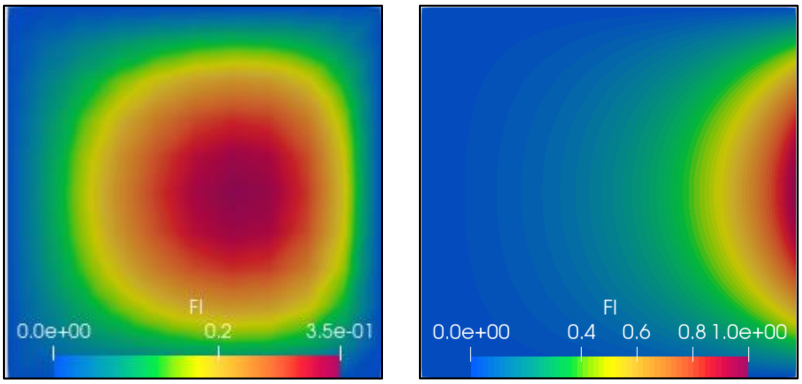
In the relevant domain of solution, graded mesh is employed. The interval number is given by $(N-1)$, where an odd integer N is the mesh number. In order to define the atoms for the mesh, let first discretize a defined independent variable x domain in such a way that $x=[0,1]$.

The atoms $x_0,\ldots,x_{N-1}$ for the mesh is defined as

$$x_{i+1}=x_i+r_e\Delta x_i, \tag{5}$$

where $0 \leq i \leq (N-1), i \in \mathbb{Z}$, and mesh expansion factor $r_e > 0$.

Clearly $\sum \Delta x_i = 1$. Illustration of the mesh is presented in Figure 1.



(a) (b)

Fig. 1. Plots of ξ for $Pe = 3.125$ with different boundary conditions (a) $\xi(0) = \xi(1) = 0$ (b) $\xi(0) = 0, \xi(1) = 1$

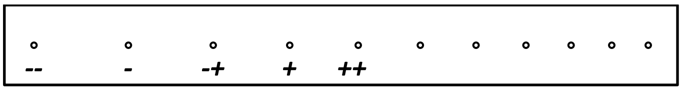


Fig. 2. Graded mesh with computational atoms

The algebraic equations can be used to approximate the solution of Eq. (3) by assigning a particular atom and its neighbors to the variables of interest. This is done by approximating partial derivatives at every single atom by nodal algebraic expression resulting from discretizing Eq. (3) as

$$C_{-+}\xi_{-+} + \sum_m C_m \xi_m = Q_{-+} \tag{6}$$

Assigned to Eq. (6) are the atoms indicated by $'-+'$. The instant left and right atoms are denoted by m. Three $n \times n$ array allocates the matrix C elements, where C is a bidiagonal matrix (the nonzero elements are represented by C_{ii}, C_-, and C_+). Thus, Eq. (6) becomes

$$C_{-+}\xi_{-+} + C_+\xi_{i+1} + C_-\xi_{i-1} = Q_{-+} \tag{7}$$

after using three-point computational atoms.

The outer and inner derivatives of the diffusive term, as well as the derivative of the convective term in Eq. (3), may all be discretized using central difference approach [24,25].

By applying block elimination approach to solve the approximate algebraic Eq. (7), the scalar concentration ξ in Eq. (3) is numerically determined. Note that Eq. (7) represents a linear system of differential equation, where it contains only linear terms. Thus, there is no requirement for linearization of Eq. (3) solution. We choose that

$$\begin{aligned} &c = 1.0, \\ &N = 11{,}21{,}41{,}81 \\ &Pe = 3.125, 6.25, 12.5, 25 \end{aligned} \tag{8}$$

It was found that the expansion factor r_e is inversely proportional to the logarithm of the Peclet number Pe [23] as shown in Figure 3, for a low Peclet number convection-diffusion flow with quadratic source. The relationship is expressed as [23]

$$r_e = m \lg Pe + b, \tag{9}$$

where

$$m = \frac{.5}{(\lg .03125)}, \tag{10}$$

and

$$b = 1. -(m \lg 3.125), \tag{11}$$

are curve slope and a constant, respectively, in order to systematically set the values of r_e. The relationship in Eq. (9) was initially used for the solution of convection-diffusion equation with zero source, and boundary conditions of $\xi(0) = 0, \xi(1) = 1$. Generalization was successfully made [23] to extent the equation when quadratic source term and boundary conditions of $\xi(0) = \xi(1) = 0$ are considered. In this work, we test its validity against a wide range of N given in Eq. (8). Using the ordered pairs

$$\{(Pe_1, re_1), (Pe_2, re_2), \ldots\ldots, (Pe_4, re_4)\} = \{(3.125, 1.), (6.25, .9), (12.5, .8), (25, .7)\},$$

there is no spurious oscillation in the solution [23]. The solution should neither have more than a turning point nor be negative when Pe of interest is greater than one as considered in this paper. This is confirmed by Figure 4 showing the theory-based ξ^* profiles for a few ranges of Pe, where $\xi^* = \xi/\xi_{max}$.

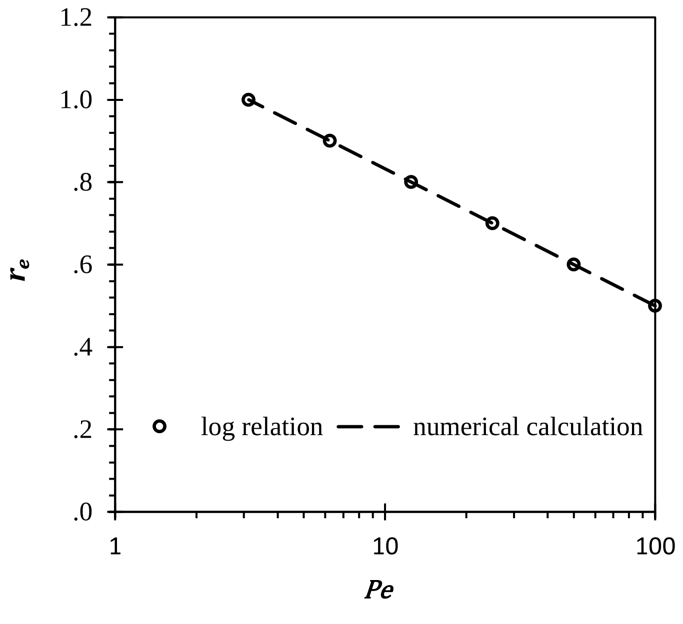


Fig. 3. Logarithmic relation between r_e and Pe

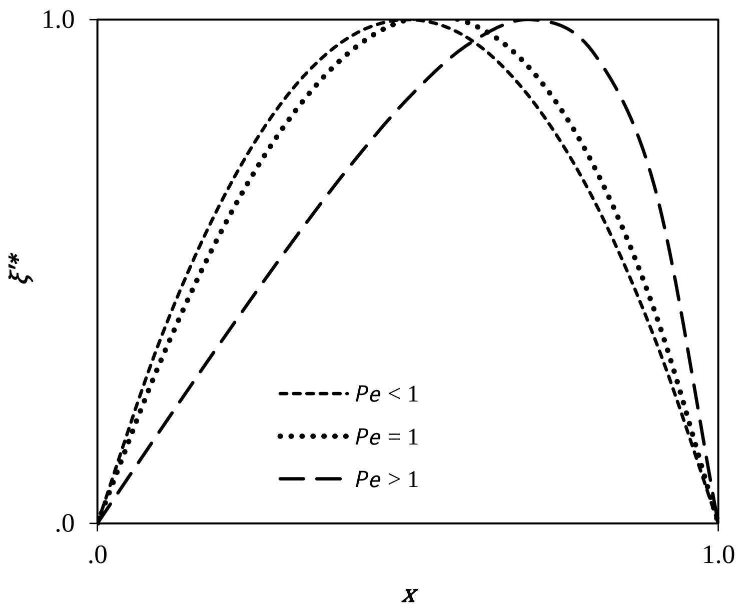


Fig. 4. Theory-based profiles where boundary conditions are fixed for a few ranges of Pe

An instance of spurious oscillation in the solution of Eq. (3) is shown in Figure 5, caused by inappropriate choice of r_e and/or N.

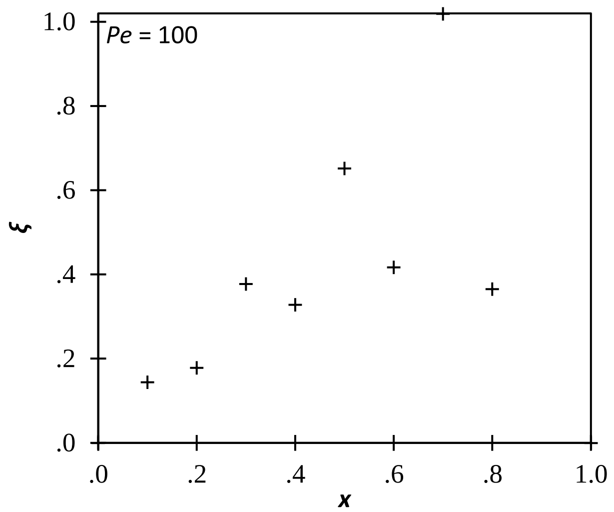


Fig. 5. A physically accurate profile does not fluctuate, but an improper mesh expansion factor and/or mesh number results in an erroneous ξ profile over the computational domains

3. Results

Graded mesh is only applied in *x*-coordinates, while uniform mesh in both *x*- and *y*-coordinates. This is due to the derivatives in the Eq. (3) are those with respect to *x* only, thus non-uniform mesh in *y*-coordinates is unnecessary. Both mesh number N and mesh expansion factor r_e affect the distance between two neighboring computational atoms Δx as illustrated in Figure 6 and Figure 7, respectively, of graded meshes. While Eq. (9) works well for a small N, there has been no attempt to generalize it for larger values of N. The solutions on computational atoms in Figure 6 and Figure 7 need to be analyzed to prove whether such generalization is valid and accurate.

Meshes in Figure 6 show that the mesh width on both coarse and fine part of mesh increases when N decreases and r_e is fixed. On the other hand, the mesh width on coarse part of mesh increases, while that on fine part decreases when r_e decreases and N is fixed as shown in Figure 7. Generally, the change in the mesh width in each individual mesh is exponential. For $r_e = 1$, graded mesh is identical to uniform mesh, where all neighboring computational atoms are equally spaced from one another for all N.

Over-reduction of r_e would result in significant lost of information on the coarse part. This occurs in two ways; firstly, the line curvature over the part is insufficient, and secondly, the overall profile of ξ is under-predicted. Furthermore, under-reduction of r_e would cause a more serious accuracy issue involving spurious fluctuation/s in the solution. Note that even for the smallest N of interest (i.e. $N = 11$), computational atoms on the fine part of mesh are extremely densed for $r_e = .5$ such that they are not easily visually distinguishable (see Figure 7). For $r_e \to 0$ and $r_e \to 1$, $(\Delta x)_{coarse} \to 1$ and $x \to 1/(N-1)$, respectively, where $(\Delta x)_{coarse}$ is the mesh width on the coarse part. For $N \to 2$ and $N \to \infty$, $\Delta x \to 1$ and $\Delta x \to 0$, respectively.

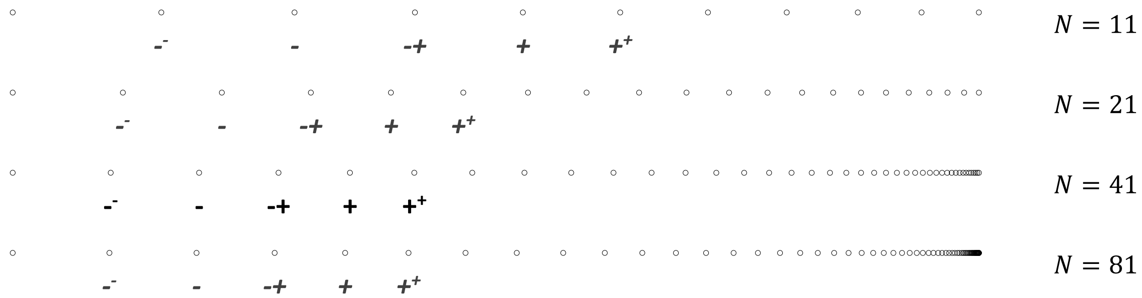


Fig. 6. Computational atoms for various mesh numbers when $r_e = .9$

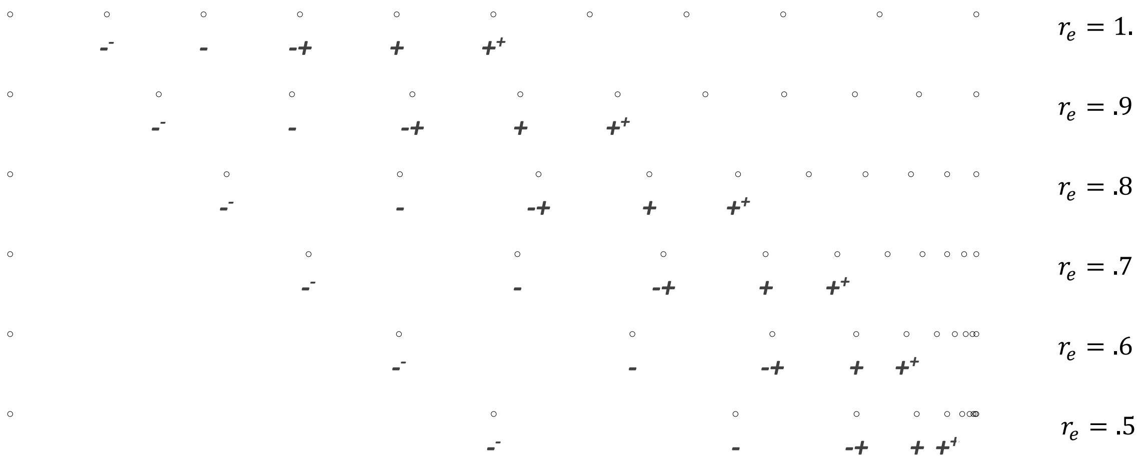


Fig. 7. Computational atoms for various mesh expansion factors when $N = 11$

Two-dimensional scalar concentration ξ was numerically and analytically calculated over graded meshes for $N = 11,21,41,81$. The sample plots at $\left(x_i, y|_{e(y)=0.64}\right)$ and $\left(x_i, y|_{e(y)=1}\right)$ against x presented in Figure 8 correspond to the smallest and largest N of interests (i.e. $N = 11$ and $N = 81$). Each plot generally involves six curves corresponding to; numerical solution at $\left(x_i, y|_{e(y)=0.64}\right)$ when $N = 11$; numerical solution at $\left(x_i, y|_{e(y)=1}\right)$ when $N = 11$; exact solution at $\left(x_i, y|_{e(y)=0.64}\right)$; exact solution at $(x_i, y|_{e(y)=1})$; numerical solution at $(x_i, y|_{e(y)=0.64})$ when $N = 81$; numerical solution at $(x_i, y|_{e(y)=1})$ when $N = 81$. Note that the exact solutions serve as benchmarks for validation of the numerical calculations. The concentration at $y|_{e(y)=0.64}$ and $y|_{e(y)=1}$ represents that when the source is relatively small and maximum, respectively. The plot in the figure when $Pe = 3.125$ represents solutions when $r_e = 1$ where graded mesh is trivial (i.e. it is identical to uniform mesh).

During the gradual growth of ξ, the correct curvature of profiles decreases with the increment of Pe until the curves are close to linear. On the other hand, the second part of the curves whose beginnings are marked by maximum ξ experience sharp drops with regard to Pe.

Applying Eq. (9) for the determination of r_e with regard to the Peclet number Pe, the resulting solutions of Eq. (3) on graded mesh corresponding to all N of interests (i.e. $N = 11,21,41,81$) are in very good agreement with the exact solutions. Closer probe confirmed that these numerical solutions are even free from spurious oscillations.

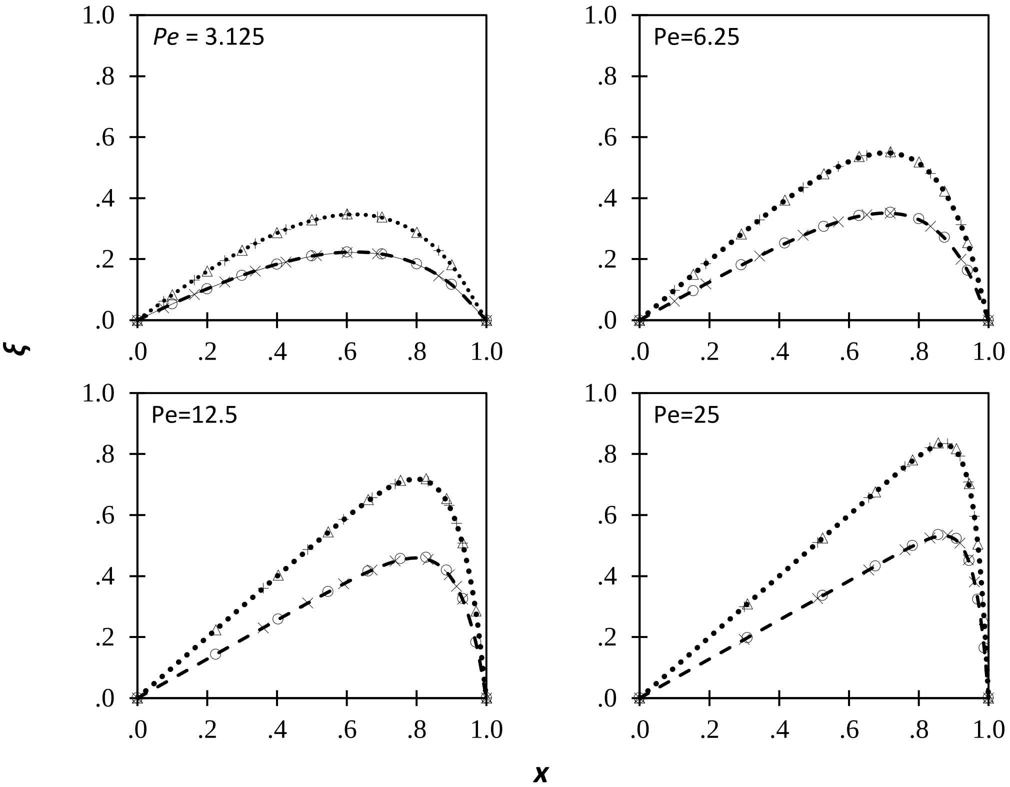


Fig. 8. Profiles of ξ; o numerical solution on graded mesh at $(x_i, y|_{e(y)=0.64})$ when $N = 11$; Δ Numerical solution on graded mesh at $(x_i, y|_{e(y)=1})$ when $N = 11$; exact solution at $(x_i, y|_{e(y)=0.64})$; exact solution at $(x_i, y|_{e(y)=1})$; x numerical solution on graded mesh at $(x_i, y|_{e(y)=0.64})$ when $N = 81$; + Numerical solution on graded mesh at $(x_i, y|_{e(y)=1})$ when $N = 81$

Further investigation on accuracy due to the decrement of average graded mesh width is detailed in Table 1 and Table 2. The average errors are generally related to Pe and mesh interval number $(N-1)$. The following can be observed if average mesh width is reduced by the factor two corresponding to $y|_{e(y)=0.64}$ and $y|_{e(y)=1}$. When $Pe = 3.125, 6.25$, the average error is reduced by the factor four. This behaviour is second order convergence. However, first and a half order convergence behaviour is observed when $Pe = 12.5$. Finally, when $Pe = 25$, the rate of convergence is of first order. It is generalized that the rate of convergence corresponding to all y decreases with Peclet number.

Table 1
Numerical errors at $y|_{e(y)=0.64}$

Pe	$N-1=10$ Error	$N-1=20$ Error	$N-1=40$ Error	$N-1=80$ Error
3.125	8.0×10^{-4}	2.1×10^{-4}	5.4×10^{-5}	1.4×10^{-5}
6.25	1.6×10^{-3}	2.6×10^{-4}	5.7×10^{-5}	2.8×10^{-5}
12.5	2.1×10^{-3}	4.3×10^{-4}	1.9×10^{-4}	9.9×10^{-5}
25	2.1×10^{-3}	6.2×10^{-4}	3.1×10^{-4}	1.6×10^{-4}

Table 2
Numerical errors at $y|_{e(y)=1}$

Pe	$N-1=10$ Error	$N-1=20$ Error	$N-1=40$ Error	$N-1=80$ Error
3.125	1.3×10^{-3}	3.3×10^{-4}	8.5×10^{-5}	2.2×10^{-5}
6.25	2.5×10^{-3}	4.1×10^{-4}	8.9×10^{-5}	4.3×10^{-5}
12.5	3.2×10^{-3}	6.8×10^{-4}	3.0×10^{-4}	1.6×10^{-4}
25	3.3×10^{-3}	9.7×10^{-4}	4.9×10^{-4}	2.5×10^{-4}

2-d plots of concentration ξ are given in Figure 9. The scalar quantity is initially concentrated about the centre of the computation domain especially when $Pe = 3.125$, and moves in the flow direction with respect to Pe. This is due to relatively low diffusivity at higher Pe such that convection becomes more dominant.

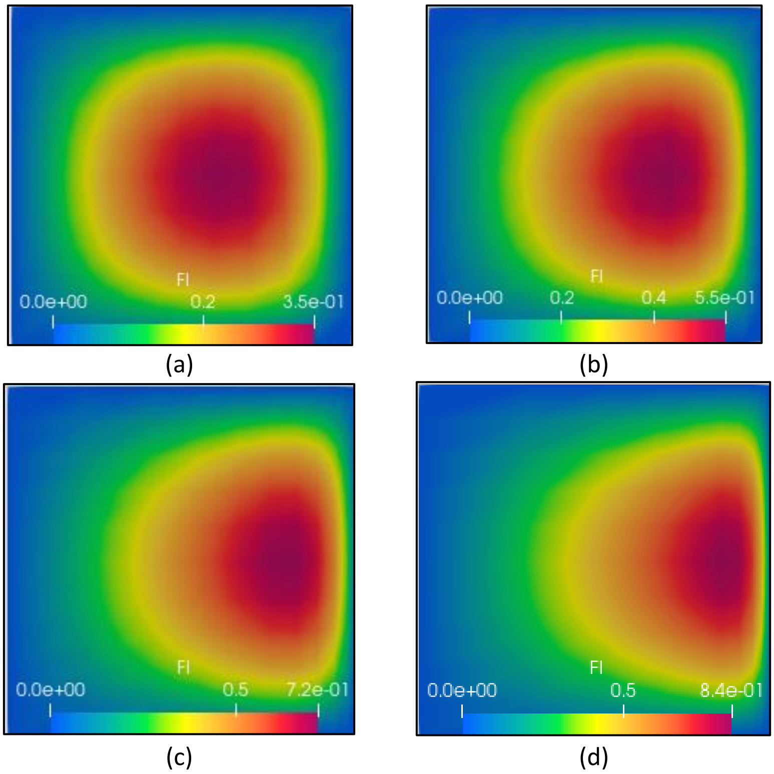


Fig. 9. Plot of ξ on graded mesh for $N = 11$ (a) $Pe = 3.125$ (b) $Pe = 6.25$ (c) $Pe = 12.5$ (d) $Pe = 25$

Note that in the case of extremely high Pe, the scalar would form a boundary layer. It is also interesting to note that the maximum value of ξ (i.e. ξ_{max}) increases with Pe. For instance, ξ_{max} is maximum at $Pe = 25$.

4. Conclusions

Graded mesh numbers N for solving convection-diffusion flow problem in Eq. (1) with quadratic source in Eq. (2) and boundary conditions in Eq. (3) for small Peclet number Pe have been comparatively studied. The findings help us better understand how graded mesh number N affects the rate of convergence as Pe rises, when Eq. (9) is used to determine mesh expansion factor r_e.

In the case of lower Pe, the rate of convergence is of second order, and reduced to first order at higher Pe. Further study might include high range of Peclet number (i.e. $Pe > 25$) to more fully profile the rate of convergence. Whether or not zero order rate of convergence is achievable is still an open question.

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References
[1] Aswin, V. S., Ashish Awasthi, and C. Anu. "A comparative study of numerical schemes for convection-diffusion equation." *Procedia Engineering* 127 (2015): 621-627. https://doi.org/10.1016/j.proeng.2015.11.353

[2] Michael, E. P. E., J. Dorning, and Rizwan-Uddin. "Studies on nodal integral methods for the convection-diffusion equation." *Nuclear science and engineering* 137, no. 3 (2001): 380-399. https://doi.org/10.13182/NSE137-380

[3] Rebhi, Redha, Younes Menni, Giulio Lorenzini, and Hijaz Ahmad. "Forced-Convection Heat Transfer in Solar Collectors and Heat Exchangers: A Review." *Journal of Advanced Research in Applied Sciences and Engineering Technology* 26, no. 3 (2022): 1-15. https://doi.org/10.37934/araset.26.3.115

[4] Al Doori, Wadhah Hussein Abdulrazzaq. "Experiments and Numerical Investigations for Heat Transfer from a Horizontal Plate via Forced Convection Using Pin Fins with Different Hole Numbers." *CFD Letters* 14, no. 9 (2022): 1-14. https://doi.org/10.37934/cfdl.14.9.114

[5] Yüzbaşı, Şuayip, and Murat Karaçayır. "An approximation technique for solutions of singularly perturbed one-dimensional convection-diffusion problem." *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields* 33, no. 1 (2020): e2686. https://doi.org/10.1002/jnm.2686

[6] Bonilla, Jesús, and Santiago Badia. "Maximum-principle preserving space–time isogeometric analysis." *Computer Methods in Applied Mechanics and Engineering* 354 (2019): 422-440. https://doi.org/10.1016/j.cma.2019.05.042

[7] Shen, Chun, Jianbing Li, Yongzhen Li, and Xuesong Wang. "Scattering Characteristics of Aircraft Wake Vortex Based on the FDTD Solution of Convection-Diffusion Equation." In *2019 IEEE International Conference on Computational Electromagnetics (ICCEM)*, pp. 1-3. IEEE, 2019. https://doi.org/10.1109/COMPEM.2019.8779128

[8] Labbe, O., E. Maglaras, and F. Garnier. "Large-eddy simulation of a turbulent jet and wake vortex interaction." *Computers & fluids* 36, no. 4 (2007): 772-785. https://doi.org/10.1016/j.compfluid.2006.06.001

[9] Janakiraman, Arun, K. Isaac, Philip Whitefield, and Donald Hagen. "A numerical thermophoretic model for nanoparticle deposition in an aerosol sampling probe." In *17th AIAA Computational Fluid Dynamics Conference*, p. 5351. 2005. https://doi.org/10.2514/6.2005-5351

[10] Fitt, A. D., and M. P. Pope. "De-icing by slot injection." *Acta mechanica* 147, no. 1 (2001): 73-86. https://doi.org/10.1007/BF01182353

[11] Abbasbandy, Saeid, Saeed Kazem, Mohammed S. Alhuthali, and Hamed H. Alsulami. "Application of the operational matrix of fractional-order Legendre functions for solving the time-fractional convection–diffusion equation." *Applied Mathematics and Computation* 266 (2015): 31-40. https://doi.org/10.1016/j.amc.2015.05.003

[12] Momani, Shaher. "An algorithm for solving the fractional convection–diffusion equation with nonlinear source term." *Communications in Nonlinear Science and Numerical Simulation* 12, no. 7 (2007): 1283-1290. https://doi.org/10.1016/j.cnsns.2005.12.007

[13] Almeida, Alcino Resende, and Renato MacHado Cotta. "Integral transform methodology for convection-diffusion problems in petroleum reservoir engineering." *International journal of heat and mass transfer* 38, no. 18 (1995): 3359-3367. https://doi.org/10.1016/0017-9310(95)00101-E

[14] Kaushik, Aditya, Vijayant Kumar, Manju Sharma, and Nitika Sharma. "A modified graded mesh and higher order finite element method for singularly perturbed reaction–diffusion problems." *Mathematics and Computers in Simulation* 185 (2021): 486-496. https://doi.org/10.1016/j.matcom.2021.01.006

[15] Brdar, Mirjana, Helena Zarin, and Ljiljana Teofanov. "A singularly perturbed problem with two parameters in two dimensions on graded meshes." *Computers & Mathematics with Applications* 72, no. 10 (2016): 2582-2603. https://doi.org/10.1016/j.camwa.2016.09.021

[16] Constantinou, Philippos, Sebastian Franz, Lars Ludwig, and Christos Xenophontos. "Finite element approximation of reaction–diffusion problems using an exponentially graded mesh." *Computers & Mathematics with Applications* 76, no. 10 (2018): 2523-2534. https://doi.org/10.1016/j.camwa.2018.08.051

[17] Durán, Ricardo G., Ariel L. Lombardi, and Mariana Ines Prieto. "Supercloseness on graded meshes for Q1 finite element approximation of a reaction–diffusion equation." *Journal of computational and applied mathematics* 242 (2013): 232-247. https://doi.org/10.1016/j.cam.2012.10.004

[18] Chaudhary, Sudhakar, and Pari J. Kundaliya. "L1 scheme on graded mesh for subdiffusion equation with nonlocal diffusion term." *Mathematics and Computers in Simulation* 195 (2022): 119-137. https://doi.org/10.1016/j.matcom.2022.01.006

[19] Chen, Hongbin, Da Xu, and Jun Zhou. "A second-order accurate numerical method with graded meshes for an evolution equation with a weakly singular kernel." *Journal of Computational and Applied Mathematics* 356 (2019): 152-163. https://doi.org/10.1016/j.cam.2019.01.031

[20] Yedida, Sanyasiraju VSS, and Chirala Satyanarayana. "RBF Based Grid-Free Local Scheme with Spatially Variable Optimal Shape Parameter for Steady Convection-Diffusion Equations." *CFD Letters* 4, no. 4 (2012): 152-172.

[21] Abdullah, Aslam. "Comparative Study of the Condition for Non-Oscillatory Solution of a Singularly Perturbed Problem on Uniform and Piecewise-Uniform Meshes." *CFD Letters* 12, no. 8 (2020): 108-120. https://doi.org/10.37934/cfdl.12.8.108120

[22] Abdullah, Aslam. "Formulation of low Peclet number based grid expansion factor for the solution of convection-diffusion equation." *Eng. Technol. Appl. Sci. Res* 8 (2018): 2680-2684. https://doi.org/10.48084/etasr.1858

[23] Abdullah, Aslam. "Comparative Study of Uniform and Graded Meshes for Solving Convection-Diffusion Equation with Quadratic Source." *Progress in Aerospace and Aviation Technology* 2, no. 1 (2022): 1-9.

[24] Ferziger, Joel H., and M. Perić. "Properties of numerical solution methods." *Computational methods for fluid dynamics* 3, no. 2 (2002): 31-35. https://doi.org/10.1007/978-3-642-56026-2

[25] John, Volker. "Numerical Methods for Scalar Convection-Dominated Problems." (2013).