Dual Solutions of Boundary Layer Flow and Heat Transfer in Hybrid Nanofluid over a Stretching/Shrinking Cylinder

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ARTICLE INFO	ABSTRACT
	The boundary layer flow over a stretching/shrinking cylinder in hybrid nanofluid with the effects of suction, partial slip and convective boundary condition is studied. Hybrid nanoparticles Al_2O_3 and TiO_2 with water as based fluid are considered in the study. The partial differential equations are transformed to ordinary differential equations by employing the similarity variables. The numerical results are obtained using the bvp4c solver in MATLAB software. The influence of nanoparticles volume fraction (Al_2O_3-TiO_2 in water-based fluid), curvature parameter, suction parameter, partial slip parameter and Biot number on the velocity profile, temperature profile, skin friction coefficient and heat transfer rate are discussed. The numerical results indicate that for shrinking surface case, the dual solutions exist for a certain range of curvature parameter and suction parameter.

1. Introduction

Previously, a large number of studies on the boundary layer flow in nanofluid have been discussed. Choi [1] was the first who introduced the term of nanofluid. After many years, the discovery of hybrid nanofluid for heat transfer enhancement has sparked a huge trend. Hybrid nanofluid is utilized as an alternative to conventional coolant in radiators as an example use in the vehicle sector. Othman *et al.,* [2] stated that the radiators often use traditional fluids like water, oil, and kerosene to transport heat from the engine or cylinder to the surrounding region. The several papers about nanofluids also can be referred to Othman *et al.,* [3], Bakar *et al.,* [4] and Yashkun *et al.,* [5].

Hybrid nanofluids are also frequently used in experimental and numerical fluid dynamics research. Khashi'ie *et al.,* [6] list out of couple of nanoparticles likes metal oxides (i.e., Fe_2O_3/hematite, Fe_3O_4/magnetite, Al_2O_3/alumina, CuO/cupric oxide), carbon materials (i.e., CNT/carbon nanotube, graphite, MWCNT/multi-walled carbon nanotubes), metals (i.e., Ag/silver, Cu/copper), or metal carbide are the most common used to make hybrid nanoparticles.

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Numerous studies have attempted to explain slip problem of boundary layer. Navier [7] proposed a slip boundary condition where the slip depends linearly on the shear stress. Zainal *et al.,* [8] explored unsteady stagnation point flow in hybrid nanofluid (Al₂O₃-Cu/H₂O) across a convectively heated stretching/shrinking sheet and the velocity slip condition is considered. Wahid *et al.,* [9] investigated on the hybrid nanofluid slip flow in the existence of heat generation over an exponentially stretching/shrinking permeable sheet. Hybridization of alumina and copper with water as the base fluid is considered. The mathematical model is simplified through the similarity transformation. Ali *et al.,* [10] explores a numerical analysis to inspect the peristaltically driven MHD flow of hybrid nanofluid (TiO₂–Cu/H₂O) along with slip conditions. They found that the volume friction of nanoparticles enhanced and reduces both temperature and velocity. Wahid *et al.,* [11] modelled and explored the properties of flow and heat transfer of mixed convection stagnation point flow for hybrid nanofluid (alumina-copper/water) on a vertical plate with slips and suction. The aim of this study is to investigate the effects of slip and thermal convective boundary condition on boundary layer flow and heat transfer over a stretching/shrinking cylinder in hybrid nanofluid.

2. Problem Formulation

Consider a hybrid nanofluid boundary layer flow past a stretching/shrinking sheet cylinder with a radius *R*. It is assumed that the stretching/shrinking velocity is $U_w = cx/L$ where *c* is constant with $c > 0$ corresponds to stretching constant, meanwhile $c < 0$ is shrinking constant, *x* is the coordinate measured along the cylinder and *L* is the characteristics length. A flow with convective heat transfer coefficient h_f and temperature T_f is flowing below the stretching/shrinking cylinder. The surface temperature T_w is the result of a convective heating process characterized by hot fluid. Therefore, the equations of hybrid nanofluid flow are given as follows (Waini *et al.,* [12]),

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0 , \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{\mu_{hnf}}{\rho_{hnf}}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{k_{hnf}}{(\rho C_\rho)_{hnf}}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right). \tag{3}$$

The boundary conditions are written as

$$u = \varepsilon U_w(x) + N_{\rho v}\left(\frac{\partial u}{\partial r}\right), \quad V = V_w, \quad -k_{hnf}\left(\frac{\partial T}{\partial r}\right) = h_f\left(T_f - T\right) \quad at \quad r = R,$$
$$u \to 0, \quad T \to T_\infty \quad as \quad r \to \infty. \tag{4}$$

where *u* and *v* represent the velocity components along the *x*- and *r*- axes, *T* is the temperature of the hybrid nanofluid. Since the nanofluid model proposed by Tiwari and Das [13] is considered, we introduced the thermophysical properties of the hybrid nanofluid that defined in Table 1. In addition, the physical properties of nanoparticle copper Cu and alumina Al₂O₃ as well as the water is a based

fluid are provided Table 2. Here, the nanoparticle Cu and Al$_2$O$_3$ volume fraction are given by φ_1 and φ_2 and the subscripts $n1$ and $n2$ correspond to their solid components, respectively. Meanwhile the fluid, nanofluid and hybrid nanofluid are presented by the subscripts *f*, *nf* and *hnf*, respectively.

The partial differential equations (PDEs) are transformed to a set of ordinary differential equations (ODEs) by using similarity transformation. The independent variable η is introduced and written as $\eta = \dfrac{r^2 - R^2}{2R}\left(\dfrac{c}{vL}\right)^{\frac{1}{2}}$ (Bakar *et al.,* [14]). The stream function ψ is given as

$\psi = \left(\dfrac{vc}{L}\right)^{\frac{1}{2}} xRf(\eta)$, where ψ is defined in the classical forms *u* and *v* are as follows:

$u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$.

Next, the dimensionless variable for the temperature $\theta(\eta)$ is given by

$$\theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty} \text{ .} \tag{5}$$

Table 1
Thermophysical Properties of Nanofluid and Hybrid Nanofluid (Oztop and Abu-Nada, [15])

Thermophysical Properties	Hybrid Nanofluid
Density	$\rho_{hnf} = (1-\varphi_2)\left[(1-\varphi_1)\rho_f + \varphi_1\rho_{n1}\right] + \varphi_2\rho_{n2}$
Heat capacity	$(\rho C_p)_{hnf} = (1-\varphi_2)\left[(1-\varphi_1)(\rho C_p)_f + \varphi_1(\rho C_p)_{n1}\right] + \varphi_2(\rho C_p)_{n2}$
Dynamic viscosity	$\mu_{hnf} = \dfrac{\mu_f}{(1-\varphi_1)^{2.5}(1-\varphi_2)^{2.5}}$
Thermal conductivity	$\dfrac{k_{nf}}{k_f} = \dfrac{k_{n2} + 2k_{nf} - 2\varphi_2(k_{nf} - k_{n2})}{k_{n2} + 2k_{nf} + \varphi_2(k_f - k_{n2})}$ where $\dfrac{k_{nf}}{k_f} = \dfrac{k_{n1} + 2k_f - 2\varphi_1(k_f - k_{n1})}{k_{n1} + 2k_f + \varphi_1(k_f - k_{n1})}$

Table 2
Thermophysical Properties of Nanoparticle and Water
(Oztop and Abu-Nada, [15])

Thermophysical Properties	Al$_2$O$_3$	TiO$_2$	Cu	Water
$\rho \left(kg/m^3 \right)$	3970	4250	8933	997.1
$C_p \left(J/kgK \right)$	765	686.2	385	4179
$k \left(W/mK \right)$	40	8.9538	400	0.613
Prandtl number, Pr				6.2

The new momentum and energy equations are as follows, respectively

$$\frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f}\left[\left(1+2\eta\gamma \right) f''' + 2\gamma f'' \right] + ff'' - f'^2 = 0, \tag{6}$$

$$\frac{1}{\text{Pr}}\frac{k_{hnf}/k_f}{\left(\rho C_\rho \right)_{hnf}/\left(\rho C_\rho \right)_f}\left[\left(1+2\eta\gamma \right)\theta'' + 2\gamma\theta' \right] + f\theta' = 0, \tag{7}$$

where $\dfrac{r^2}{R^2} = 1+2\eta\gamma$, $\gamma = \left(\dfrac{vL}{cR^2} \right)^{\frac{1}{2}}$ is the curvature parameter and Prandtl number $\text{Pr} = \dfrac{\left(\mu C_\rho \right)_f}{k_f}$.

The new boundary conditions are considered as

$$f'(0) = \varepsilon + \sigma f''(0),\ f(0) = S,\ \ \theta'(0) = Bi(\theta - 1) \quad at \quad r = R,$$
$$f'(\infty) \to 0, \quad \theta(\infty) \to 0 \quad as \quad r \to \infty. \tag{8}$$

The physical quantities of physical interest are the skin friction coefficient or shear stress C_f and the local Nusselt number Nu_x which are given as follows

$$C_f = \frac{T_w}{\rho_f U_w^2}, \tag{9}$$

$$Nu_x = \frac{xq_w}{k_f \left(T_f - T_\infty \right)}, \tag{10}$$

where τ_w is the surface shear stress and q_w is the heat flux from the stretching or shrinking surface, which are defined as,

$$\tau_w = \mu_{hnf} \left(\frac{\partial u}{\partial r} \right)_{r=R}, \qquad (11)$$

$$q_w = -k_{hnf} \left(\frac{\partial T}{\partial r} \right)_{r=R}. \qquad (12)$$

By replacing Eq. (9) and Eq. (10) into Eq. (11) and Eq. (12), we obtained

$$\mathrm{Re}_x^{\frac{1}{2}} C_f = \frac{\mu_{hnf}}{\mu_f} f''(0), \qquad (13)$$

$$\left(\mathrm{Re}_x \right)^{-\frac{1}{2}} Nu_x = \frac{-k_{hnf}}{k_f} \theta'(0), \qquad (14)$$

where $\mathrm{Re}_x = \dfrac{U_w x}{v_f}$ is the local Reynolds number and $v = \dfrac{\mu}{\rho}$.

3. Results and Discussion

The ordinary differential Eq. (6) and Eq. (7) with the boundary conditions in Eq. (8) are solved numerically by using bvp4c method. The main numerical results consist of the velocity profile $f'(\eta)$, temperature profile $\theta(\eta)$, the skin friction coefficient $f''(0)$ and the heat transfer rate $-\theta'(0)$ for different parameters. The constant values of parameters are considered such as Pr = 6.2 (water-based fluid), $\varphi_1 = 0.1$ (nanoparticle Al$_2$O$_3$), $\varphi_2 = 0.1$ (nanoparticle TiO$_2$), $S = 2$ (suction), $\sigma = 0.2$ (partial slip), Bi = 0.2 (Biot number), $\varepsilon = -1$ (stretching /shrinking) and $\gamma = 0.2$ (cylindrical surface).

Table 3 shows the method validated with the previous work by Bakar *et al.,* [14] when the hybrid nanofluid is absence $(\varphi_1 = \varphi_2 = 0)$. A good agreement is found, and thus it is confidence that the present numerical output is accurate. Table 4 represents the values skin friction coefficient $f''(0)$ and heat transfer rate $-\theta'(0)$ for some values of γ and *S* with Pr = 6.2, $\varepsilon = -1$, $\varphi_1 = 0.1$, $\varphi_2 = 0.1$, and $\sigma = 0.2$.

Table 3
Comparison of $f''(0)$ when hybrid nanoparticle is absence $(\varphi_1 = \varphi_2 = 0)$

S	$f''(0)$			
	(Bakar *et al.,* [14])		Present Results	
	First solution	Second solution	First solution	Second solution
2	1.140551	0.5820935	1.14055	0.58210
2.1	1.2356765	0.5303445	1.23570	0.53030

Table 4
The values of $f''(0)$ and $-\theta'(0)$ for some values of γ and S

γ	S	$f''(0)$		$-\theta'(0)$	
		First solution	Second solution	First solution	Second solution
0.01	2.4	1.37836	0.38662	0.19492	0.19482
	2.2	1.24052	0.46181	0.19442	0.19431
	2	1.06212	0.56675	0.19380	0.19370
0.1	2.4	1.57922	-0.05258	0.19493	0.19476
	2.2	1.47151	0.10652	0.19445	0.19423
	2	1.35017	0.24135	0.19385	0.19359
0.2	2.4	1.75002	-0.59313	0.19495	0.19461
	2.2	1.65744	-0.33707	0.19446	0.19401
	2	1.55638	-0.11819	0.19387	0.19321

Figure 1 and Figure 2 exhibit the effects of nanoparticle volume fraction φ_2 (TiO₂) on the skin friction coefficient $f''(0)$ and the heat transfer rate $-\theta'(0)$ with nanoparticle volume fraction $\varphi_2 = 0.01,\ 0.1,\ 0.2$ (TiO₂) when a shrinking case, $\varepsilon = -1$ is considered. The value of nanoparticle volume fraction $\varphi_1 = 0.1$ (Al₂O₃) is a fixed value. In these figures, the dual solutions are found to exist for a certain value of γ where the solid line corresponds to the first solution (upper) and the dashed line corresponds to the second solution (lower). The dual solutions exist for the range $\gamma > \gamma_c$, where γ_c is critical value (turning point) of γ. By considering $\varphi_2 = 0.01,\ 0.1,\ 0.2$, the values of γ_c are given by $-0.0266, -0.0150$, and 0.0100, respectively. There is no solution exist when $\gamma > \gamma_c$. It is worth mentioning that the small value of φ_2 has tendency to increase the movement of hybrid nanofluids and its temperature. Moreover, the range of solutions widens even though the value of φ_2 is small.
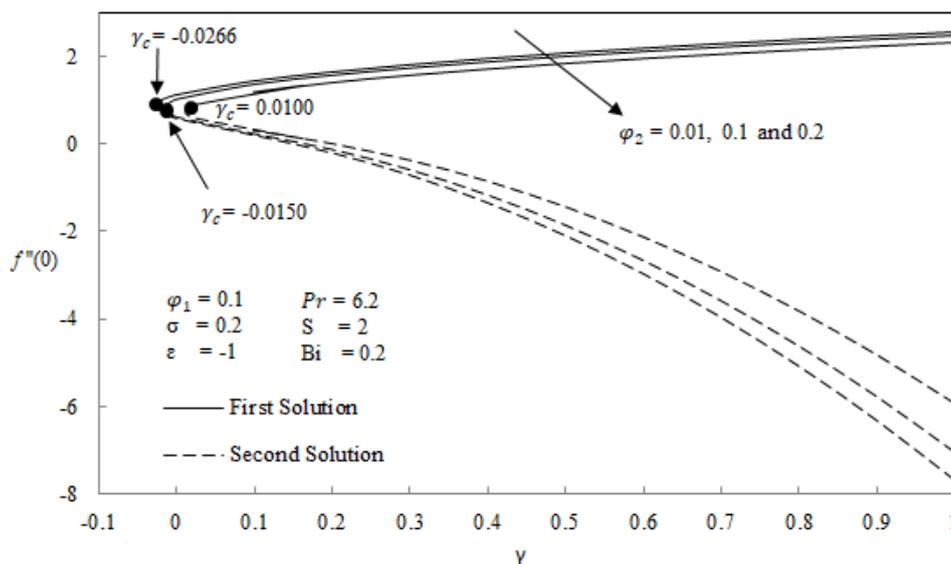
Fig. 1. Variation of $f''(0)$ with γ for different φ_2 when $S = 2$ and $\text{Pr} = 6.2$
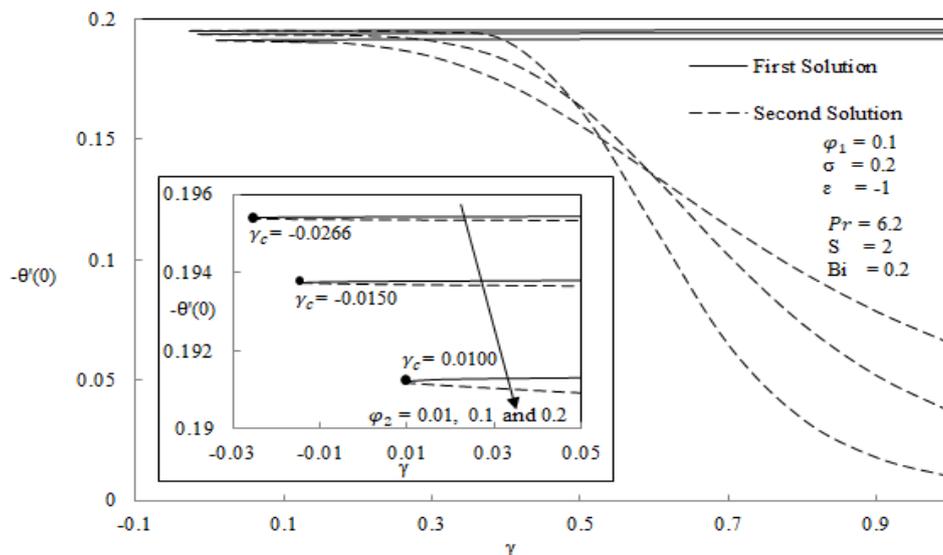
Fig. 2. Variation of $-\theta'(0)$ with γ for different φ_2 when $S = 2$ and $\mathrm{Pr} = 6.2$

The influence of cylindrical parameter γ on the skin friction coefficient $f''(0)$ and the heat transfer rate $-\theta'(0)$ with suction parameter S when a shrinking case, $\varepsilon = -1$ is considered are presented in Figure 3 and Figure 4 respectively. The chosen values of γ are 0.01, 0.1 and 0.2. From these figures, the dual solutions exist within a certain range of S. The dual solutions exist for the range $S > S_c$, where S_c is critical value (turning point) of S. The values of S_c for $\gamma = 0.01$, 0.1 and 0.2 are given by 1.1300, 1.4735, and 1.8614 respectively. There is no solution exist when $S > S_c$. The dual solutions occurs when S is positive value (suction). The decreasing value of γ widens the range of existence of dual solutions. Furthermore, the small value of γ gives the higher values of the skin friction coefficient $f''(0)$ the heat transfer rate $-\theta'(0)$ for TiO$_2$ - Al$_2$O$_3$ water hybrid nanofluid.
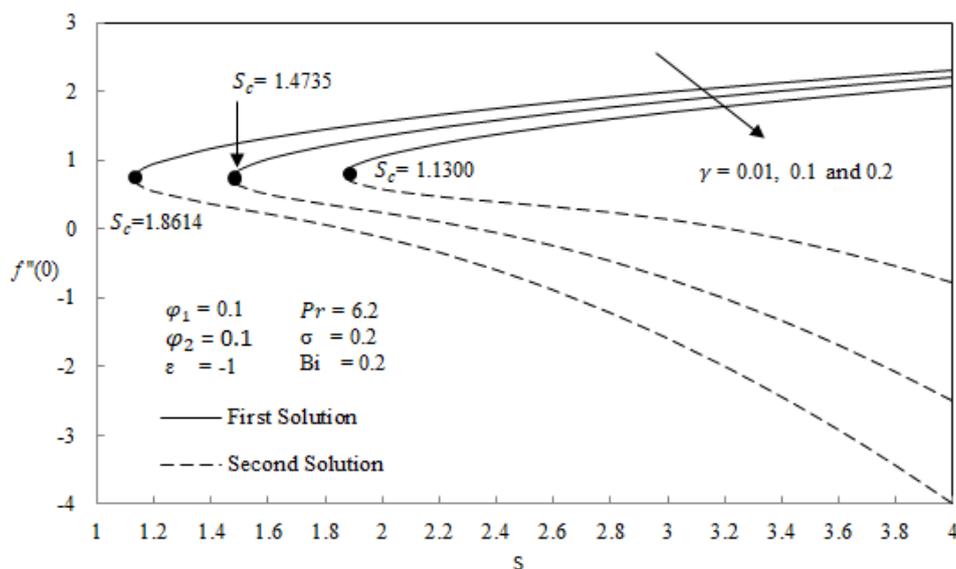
Fig. 3. Variation of $f''(0)$ with S for different value of γ when $\mathrm{Pr} = 6.2$
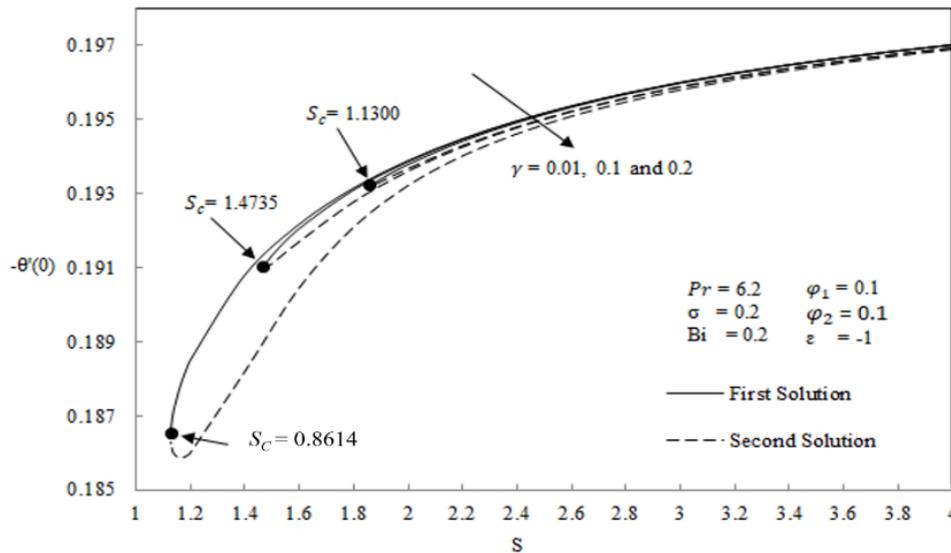
Fig. 4. Variation of $-\theta'(0)$ with *S* for different value of γ when $\mathrm{Pr} = 6.2$

The velocity and temperature profiles for different slip parameter σ are illustrated in Figure 5 and Figure 6, respectively. It is quite different output for temperature profile where the larger value of σ is needed to increase the velocity, meanwhile the increasing value of σ tends to decrease the temperature. In general, the partial slip effect in hybrid nanofluid has tendency to remove the heat away from the surface and gives the lower value of temperature profile. Lastly, the temperature profile for different value of Biot number is depicted in Figure 7. In this figure, it is shows that the increasing value of Bi will decreases the temperature. The boundary layer thickness is thicker and hence, decreases the temperature gradient at the surface. Then, the heat transfer rate decreases. All these profiles in Figure 5 – Figure 7 asymptotically satisfied all the boundary conditions. Physically, for the existence of dual solutions, it means that the separation of the flow from the plate is occurred.
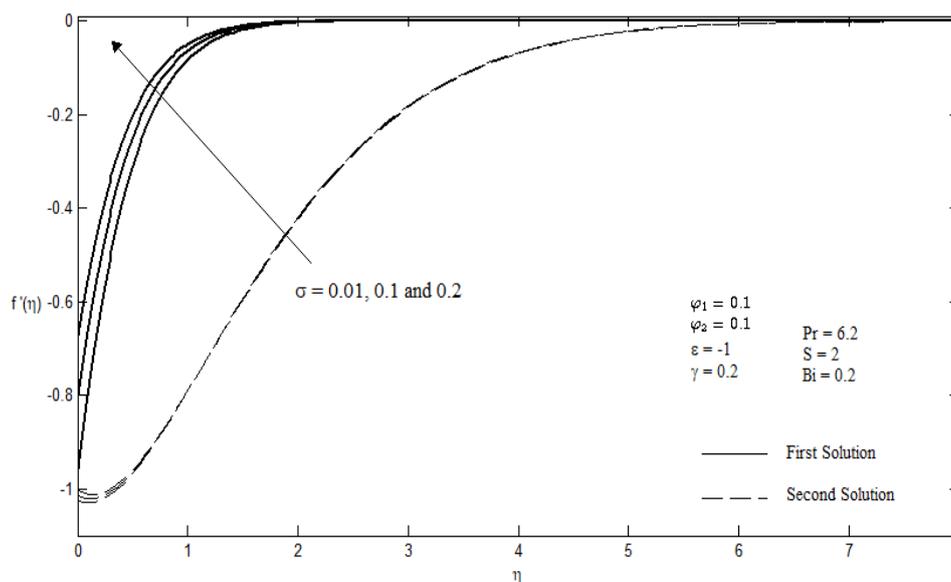
Fig. 5. Velocity profile $f'(\eta)$ for different σ when $\varepsilon = -1$ and $\mathrm{Pr} = 6.2$
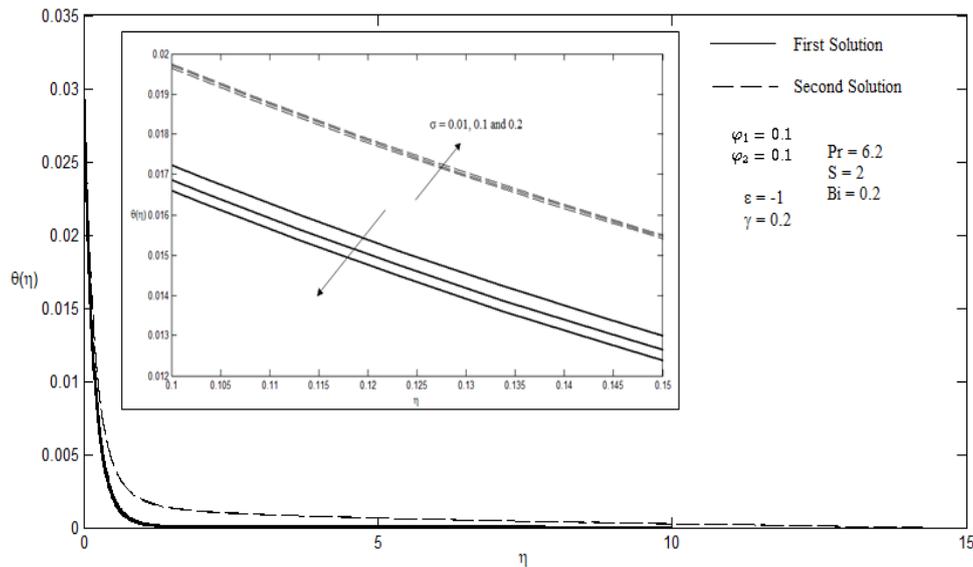
Fig. 6. Temperature profile $\theta(\eta)$ for different σ when $\varepsilon = -1$ and $\text{Pr} = 6.2$
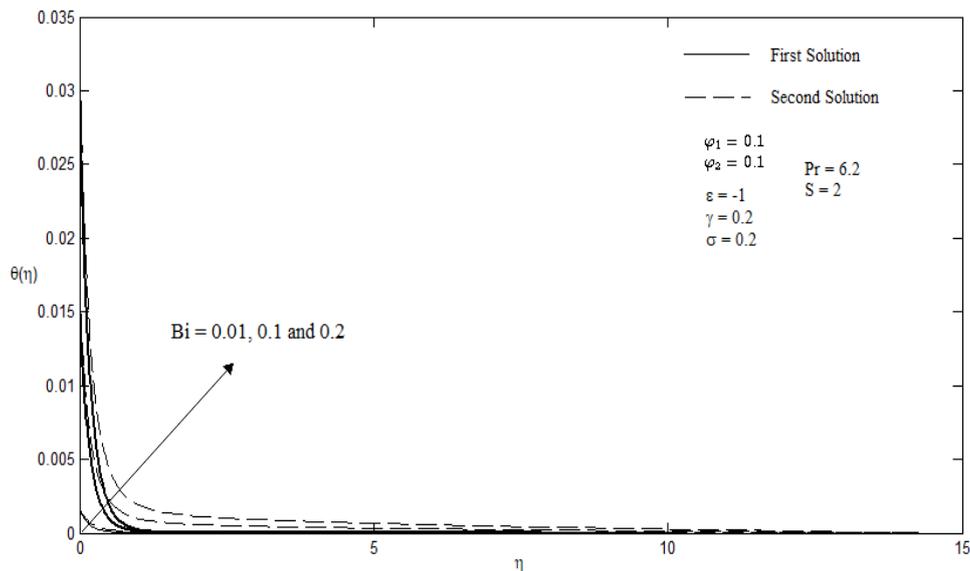
Fig. 7. Temperature profile $\theta(\eta)$ for different Bi when $\varepsilon = -1$ and $\text{Pr} = 6.2$

4. Conclusion

The study on boundary layer flow and heat transfer in hybrid nanofluid over a stretching/shrinking cylinder is investigated. The non-unique solution (dual) is obtained in the study with the presence of shrinking cylinder ($\varepsilon = -1$) and suction parameter ($S > 0$). The dual solution exists in a certain range of critical values, γ_c and S_c. The decreasing values of nanoparticle volume fraction φ_2 (TiO₂) and curvature parameter γ widen the dual solution exist in the study. The decreasing values of nanoparticle volume fraction φ_2 (TiO₂) and curvature parameter γ tend to increase the fluid flow (skin friction coefficient). The increasing value of σ will increase the fluid flow. The decreasing values of nanoparticle volume fraction φ_2 (TiO₂), curvature parameter γ, partial slip parameter σ and Biot number tend to increase the fluid temperature (heat transfer rate).

Acknowledgement

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