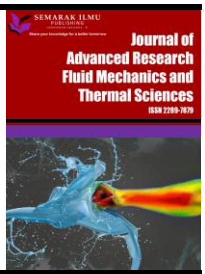
SEMARAK ILMU PUBLISHING

Journal of Advanced Research in Fluid Mechanics and Thermal Sciences

Journal homepage:
https://semarakilmu.com.my/journals/index.php/fluid_mechanics_thermal_sciences/index
ISSN: 2289-7879



On Exact Solutions to Problems of Steady Flow of Incompressible Viscous Fluid in a Channel and in a Circular Pipe

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ARTICLE INFO

Article history:
Received 28 March 2023
Received in revised form 10 June 2023
Accepted 18 June 2023
Available online 5 July 2023

Keywords:
Steady flow; incompressible viscous fluid; Navier-Stokes equations; Couette flow; Hagen-Poiseuille flow; exact solutions

ABSTRACT

Turning to the well-known Couette flow, we again, but now in a new formulation, consider the steady motion of an incompressible viscous fluid in a channel with two parallel flat walls, one of which is stationary while the other moves at a constant speed in its plane. In the classical formulation of this problem on a layered (laminar) fluid flow in a channel, the wall velocity and the constant pressure drop along the channel are independent quantities and are considered predetermined. However, when a solid body moves along the fluid surface, the fluid particles are set in motion. This occurs due to the arising pressure generated in the fluid caused by the movement of the solid body. Therefore, here the pressure or pressure drop is not predetermined but is determined as a result of solving the corresponding two-dimensional boundary value problem of a steady flow of a viscous fluid in a channel. For this, the corresponding boundary value problem of a fluid flow in the channel is formulated based on the linearized Navier-Stokes equations obtained from the general nonlinear Navier-Stokes equations for a steady flow of a viscous fluid neglecting the convective terms, which is true for small Reynolds numbers. Using the Fourier integral transform method, an exact (closed) solution to this boundary value problem is constructed; the velocity and pressure components are determined. Then it is shown that the resulting solution, different from the known Couette flow solution, also satisfies the original nonlinear Navier-Stokes equations, the continuity equation, and boundary conditions. A comparative analysis of the new solution with the Couette flow solution is carried out. The well-known Hagen-Poiseuille problem is also discussed from the point of view of establishing the conditions for the laminar axisymmetric and steady flow of an incompressible viscous fluid in a straight circular pipe According to the study's results, to implement laminar flow under the indicated conditions, the pressure drop along the pipe must be constant. The study demonstrates that this condition is simultaneously sufficient, and therefore, the necessary and sufficient condition is established for a laminar flow in a straight circular pipe.

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https://doi.org/10.37934/arfmts.107.2.6883

1. Introduction

The Navier-Stokes partial differential equations describing the flow of a viscous fluid are generally nonlinear and so complicated that they cannot be solved exactly. If the flow is steady and the viscous fluid is incompressible, these equations are somewhat simplified, but remain nonlinear. They are greatly simplified turning into linear differential equations when the nonlinear convective terms are neglected in them. Such a simplification is justified if the velocities are very small or if the dynamic coefficient of viscosity of the fluid is very large. Then the friction forces significantly exceed the inertia forces, and, consequently, the latter can be neglected in comparison with the former. This linearization approach is characterized by small Reynolds numbers *Re*, when usually $Re < 1$. The solutions to the Navier-Stokes equations obtained in this way are called slow or creeping solutions.

Despite these and other difficulties, exact solutions of a sufficiently large number model problems of viscous fluid mechanics are obtained by a semi-inverse method, when, based on the characteristics of the problem under consideration, the mode of distribution of velocities in a viscous fluid medium is set in advance, and then the governing Navier-Stokes equations, the continuity equation, and the corresponding boundary conditions are satisfied. These solutions are of considerable theoretical and practical interest. They are summarized in monographs and in numerous articles [1-6]. In a study by Fefferman [7], the existence and smoothness of the Navier-Stokes solutions is studied. The flow of nanofluids was studied by modern numerical methods [8,9]. Recent Deep Neural Network solutions to the three-dimensional potential problems in non-homogeneous media are presented by Guo *et al.,* [10]. The fundamentals of fluid dynamics are described in detail by Batchelor [11], where methods for simplifying the basic equations of viscous fluid mechanics as well as a comparative analysis of experimental and theoretical results are given. In the paper by Savenkov [12], the linear stage of three-dimensional disturbance development in the Poiseuille-Couette flow in the case when both walls move in the lateral direction was studied based on the asymptotic triple-deck theory. It is shown that the lateral wall motion has no effect on the velocity of the wave packet in the longitudinal direction, and the packet itself does not bifurcate. A class of viscous parallel flow problems, Couette and Hagen-Poiseuille flows, have been computationally solved and plotted using Python. SymPy and NumPy modules by Pawar *et al.,* [13]. In the study by Lopes and Siqueira [14], an analytical solution for steady Couette, Poiseuille, and Couette–Poiseuille flows of incompressible Newtonian fluids in semi-elliptical channels under the no-slip condition at the boundaries is constructed. A mapping function to rewrite the problem in an elliptical coordinate system coupled with Fourier's method for the solution of a Laplace equation with Dirichlet-type boundary conditions is applied. The closed solutions to the incompressible Navier-Stokes equations are obtained by Babu [15] for fluid parallel flow as well as using the lubrication approximation. In study by Marušić-Paloka [16], an exact solution for a steady fluid flow through a channel with upper wall attached to an elastic spring is derived. The upper wall displacement caused by the fluid-wall interaction is calculated from a quadratic equation. A new exact solution to the Navier-Stokes equations is also given Prosviryakov [17]; it is a superposition of the Couette flows for each component of the velocity vector. In research done by Sarukhanyan *et al.,* [18], the regularities of changes in the hydrodynamic characteristics of a viscous incompressible fluid in flat diffusers were studied depending on the diffuser opening angle and the Reynolds number. For fixed values of the opening angle and the Reynolds number, the conditions for separation of the flow from a stationary wall, where the flow velocity changes sign, are derived. The monograph is devoted to the study of mathematical issues of solving boundary value problems in the mechanics of an incompressible viscous fluid [19].

In the present paper, based on the ideas and approaches outlined, we again, but in a new formulation, consider the classical Couette flow - the problem of a steady flow of an incompressible viscous fluid in a channel with two parallel flat walls, the lower one of which is stationary, while the upper one moves in its plane at a constant speed. In the classical formulation of this problem, within the framework of the semi-inverse method of solving it, the wall velocity and pressure or pressure drop along the length of the channel are independent quantities and are considered prescribed. However, since when a solid body moves along the surface of a fluid, its particles are set in motion due to the pressure that has arisen in the fluid, it becomes necessary to consider such formulation of the problem when the pressure in the channel is generated by the velocity of the moving wall. For this purpose, assuming the flow to be planar (two-dimensional), first, for the linearized Navier-Stokes equations for the channel section, the corresponding two-dimensional boundary value problem is considered. For this purpose, assuming the flow to be planar (two-dimensional), first, we considered the corresponding two-dimensional boundary value problem for the linearized Navier-Stokes equations for the channel section. By the Fourier integral transform method, we obtained an exact solution to this problem. As a result, the velocity and pressure components in the channel are determined. Then we showed that this solution also satisfies the governing nonlinear Navier-Stokes equations, the continuity equation, and boundary conditions. Ultimately, as far as we know, a new solution to the problem under consideration is obtained, different from the Couette flow solution.

The well-known Hagen-Poiseuille solution is also discussed in order to establish the necessary and sufficient condition for a laminar axisymmetric flow in a steady-state regime of an incompressible viscous fluid in a straight circular pipe.

2. Methodology

2.1 Formulation of the Problem and Derivation of the Basic Equations

Let a channel $\Omega = \{-\infty < x, z < \infty; -h \le y \le h\}$ of height $2h$ and infinite length in the direction of the *0z* axis, referred to the right rectangular coordinate system *Oxyz*, be filled with an incompressible viscous fluid with dynamic viscosity coefficient μ and density $\rho = const.$ Further, let the lower wall of channel $y = -h$ be fixed, and the upper wall, parallel to it, move in its plane in the direction of the *0x* axis with a constant speed of U. We will assume that the fluid flow is steady and that there are no body forces. Since the pattern of the distribution of velocities and pressure in all sections of the channel Ω perpendicular to the *0z* axis is the same, we arrive at a two-dimensional flow in a planar channel $\omega = \{-\infty < x < \infty; -h \le y \le h\}$ located in the *0xy* plane with $u = u(x, y),\ v = v(x, y),\ w = w(x, y) = 0,\ p = p(x, y),$ where u, v, w are velocity components along the axes *0x*, *0y*, *0z*, respectively, and p is the pressure. Then the Navier-Stokes equations will take the form [2]

$$
\begin{aligned}
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\\
&u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \left((x, y) \in \omega\right),
\end{aligned} \tag{1}
$$

and the continuity equation will take the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \big((x,y)\in\omega\big). \tag{2}$$

Eq. (1) and Eq. (2) should be considered under the boundary conditions

$$\begin{aligned} &u(x,y)\big|_{y=-h}=0,\ \ v(x,y)\big|_{y=-h}=0,\\ &u(x,y)\big|_{y=h}=U,\quad v(x,y)\big|_{y=h}=0 \qquad (-\infty<x<\infty). \end{aligned} \tag{3}$$

Now, according to the model adopted above, we neglect the nonlinear convective terms on the left-hand sides of Eq. (1). As a result, we get

$$\begin{cases} \dfrac{\partial p}{\partial x}=\mu\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right),\\ \dfrac{\partial p}{\partial y}=\mu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right). \end{cases} \qquad \big((x,y)\in\omega\big) \tag{4}$$

If we differentiate the first of these equations with respect to *x*, the second with respect to *y*, add the results and take into account the continuity of Eq. (2), then we find that the pressure $p(x,y)$ is a harmonic function in ω:

$$\Delta p=\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=0 \qquad \big((x,y)\in\omega\big). \tag{5}$$

Next, we construct a solution to the boundary value problem consisting of differential Eq. (4), Eq. (5), Eq. (2) and boundary conditions (3). For this purpose, we introduce the Fourier transformants in the coordinate *x*:

$$\left\{\overline{u}(\lambda,y);\overline{v}(\lambda,y);\overline{p}(\lambda,y)\right\}=\int_{-\infty}^{\infty}\left\{u(x,y);v(x,y);p(x,y)\right\}e^{i\lambda x}dx,$$

where λ is the spectral Fourier parameter, and the Fourier transform is treated in the sense of the theory of generalized functions [20]. After applying the Fourier transform to both sides of the indicated equations and boundary conditions, we arrive at the following one-dimensional boundary value problem in Fourier transformants:

$$\begin{cases} \mu\left(\dfrac{d^2\overline{u}}{dy^2}-\lambda^2\overline{u}\right)=-i\lambda\overline{p}; \\ \mu\left(\dfrac{d^2\overline{v}}{dy^2}-\lambda^2\overline{v}\right)=\dfrac{d\overline{p}}{dy}; \\ \dfrac{d\overline{v}}{dy}-i\lambda\overline{u}=0; \qquad (-h<y<h), \\ \dfrac{d^2\overline{p}}{\partial y^2}-\lambda^2\overline{p}=0; \\ \overline{u}(\lambda,y)\Big|_{y=-h}=0;\ \overline{v}(\lambda,y)\Big|_{y=\pm h}=0; \\ \overline{u}(\lambda,y)\Big|_{y=h}=2\pi U\delta(\lambda) \qquad (-\infty<\lambda<\infty), \end{cases} \tag{6}$$

where $\delta(\lambda)$ is the known Dirac delta function.

The general solution of the fourth equation from (6) is represented by the formula

$$\overline{p}(\lambda,y)=A\cosh(\lambda y)+B\sinh(\lambda y) \qquad (-h\le y\le h), \tag{7}$$

where A and B are arbitrary constants. We substitute this solution into the right-hand sides of the first two equations of (6). We have

$$\frac{d^2\overline{u}}{dy^2}-\lambda^2\overline{u}=-\frac{i\lambda}{\mu}\left[A\cosh(\lambda y)+B\sinh(\lambda y)\right],$$
$$\frac{d^2\overline{v}}{dy^2}-\lambda^2\overline{v}=\frac{\lambda}{\mu}\left[A\sinh(\lambda y)+B\cosh(\lambda y)\right] \qquad (-h<y<h).$$

The general solutions of these differential equations, consisting of the sums of general solutions of homogeneous equations and particular solutions of inhomogeneous equations, are given by the formulas

$$\overline{u}(\lambda,y)=C\cosh(\lambda y)+D\sinh(\lambda y)-\frac{iy}{2\mu}\left[B\cosh(\lambda y)+A\sinh(\lambda y)\right] \qquad (-h<y<h); \tag{8}$$

$$\overline{v}(\lambda,y)=E\cosh(\lambda y)+F\sinh(\lambda y)+\frac{y}{2\mu}\left[B\sinh(\lambda y)+A\cosh(\lambda y)\right]. \tag{9}$$

Now we substitute these expressions for $\overline{u}(\lambda,y)$ and $\overline{v}(\lambda,y)$ into the third equation of (6) – into the continuity equation in Fourier transformants. Hence, we obtain the following dependencies between the constants included in (8) and (9):

$$E=iD-\frac{B}{2\lambda\mu}, \qquad F=iC-\frac{A}{2\lambda\mu}.$$

Substituting these expressions for the constants E and F into (9), we find

$$\bar{v}(\lambda, y) = \left(iD - \frac{B}{2\lambda\mu} \right)\cosh(\lambda y) + \left(iC - \frac{A}{2\lambda\mu} \right)\sinh(\lambda y) + \frac{y}{2\mu}\left[B\sinh(\lambda y) + A\cosh(\lambda y) \right] \qquad (-h \le y \le h). \tag{10}$$

We require that (8) and (10) satisfy the boundary conditions from (6). Then, to determine the constants *A, B, C*, and *D*, we obtain the following system of linear algebraic equations:

$$\begin{cases} C\cosh(\lambda h) - D\sinh(\lambda h) + \dfrac{ih}{2\mu}\left[B\cosh(\lambda h) - A\sinh(\lambda h) \right] = 0, \\ C\cosh(\lambda h) + D\sinh(\lambda h) - \dfrac{ih}{2\mu}\left[B\cosh(\lambda h) + A\sinh(\lambda h) \right] = 2\pi U\delta(\lambda), \\ \left(iD - \dfrac{B}{2\lambda\mu} \right)\cosh(\lambda h) + \left(iC - \dfrac{A}{2\lambda\mu} \right)\sinh(\lambda h) + \dfrac{h}{2\mu}\left[B\sinh(\lambda h) + A\cosh(\lambda h) \right] = 0, \\ \left(iD - \dfrac{B}{2\lambda\mu} \right)\cosh(\lambda h) - \left(iC - \dfrac{A}{2\lambda\mu} \right)\sinh(\lambda h) - \dfrac{h}{2\mu}\left[A\cosh(\lambda h) - B\sinh(\lambda h) \right] = 0. \end{cases} \tag{11}$$

From the first two equations of system (11), by addition and subtraction, we easily find

$$C = \frac{\pi U}{\cosh(\lambda h)}\delta(\lambda) + \frac{ih}{2\mu}A\tanh(\lambda h), \qquad D = \frac{\pi U}{\sinh(\lambda h)}\delta(\lambda) + \frac{ih}{2\mu}B\coth(\lambda h). \tag{12}$$

Substituting these expressions for the constants *C* and *D* into the third and fourth equations of system (11), after simple transformations, we obtain

$$A = \frac{4\pi i\mu U}{\sinh(2\lambda h) - 2\lambda h}\lambda\sinh(\lambda h)\delta(\lambda), \qquad B = \frac{4\pi i\mu U}{\sinh(2\lambda h) + 2\lambda h}\lambda\cosh(\lambda h)\delta(\lambda). \tag{13}$$

Finally, substituting (13) into (12) and the expressions for the constants *A, B, C*, and *D* into (7), (8), and (10), after simple calculations, we finally obtain for the Fourier transformants of the main characteristics of the problem under discussion:

$$\bar{p}(\lambda y) = 4\pi i\mu U\delta(\lambda)\left[\frac{\lambda\sinh(\lambda h)\cosh(\lambda y)}{\sinh(2\lambda h) - 2\lambda h} + \frac{\lambda\sinh(\lambda y)\cosh(\lambda h)}{\sinh(2\lambda h) + 2\lambda h} \right]; \tag{14}$$

$$\bar{u}(\lambda y) = \left\{ \frac{\sinh(2\lambda h) - \lambda h\cosh(2\lambda h) - \lambda h}{\cosh(\lambda h)\left[\sinh(2\lambda h) - 2\lambda h \right]}\cosh(\lambda y) - \frac{\lambda h\cosh(2\lambda h) - \sinh(2\lambda h) - \lambda h}{\sinh(\lambda h)\left[\sinh(2\lambda h) + 2\lambda h \right]}\sinh(\lambda y) \right\} \times \pi U\delta(\lambda) + 2\lambda y\left[\frac{\sinh(\lambda h)\sinh(\lambda y)}{\sinh(2\lambda h) - 2\lambda h} + \frac{\cosh(\lambda h)\cosh(\lambda y)}{\sinh(2\lambda h) + 2\lambda h} \right]\pi U\delta(\lambda) \quad (-h \le y \le h); \tag{15}$$

$$\bar{v}(\lambda, y) = i\pi U\delta(\lambda)\left\langle y\left[\frac{2\lambda\cosh(\lambda h)\sinh(\lambda y)}{\sinh(2\lambda h)+2\lambda h}+\frac{2\lambda\sinh(\lambda h)\cosh(\lambda y)}{\sinh(2\lambda h)-2\lambda h}\right]\right.$$
$$\left.-\left\{\frac{\lambda h\cosh(2\lambda h)-\sinh(2\lambda h)-\lambda h}{\sinh(\lambda h)\left[\sinh(2\lambda h)+2\lambda h\right]}+\frac{2\cosh(\lambda h)\cosh(\lambda y)}{\sinh(2\lambda h)+2\lambda h}\right\}\cosh(\lambda y)\right\rangle \tag{16}$$
$$+i\pi U\delta(\lambda)\left\{\frac{\sinh(2\lambda h)-\lambda h\cosh(2\lambda h)-\lambda h}{\cosh(\lambda h)\left[\sinh(2\lambda h)-2\lambda h\right]}-\frac{2\sinh(\lambda h)}{\sinh(2\lambda h)-2\lambda h}\right\}\sinh(\lambda y).$$

Now, by the formula for the inverse Fourier transform, according to formulas (14), (15), and (16), we have

$$\left\{p(x,y);u(x,y);v(x,y)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\bar{p}(\lambda,y);\bar{u}(\lambda,y);\bar{v}(\lambda,y)\right\}e^{-i\lambda x}d\lambda.$$

To calculate these integrals, let us find out the behaviour of functions (14), (15), and (16) as $\lambda \to 0$. Taking into account (14), we can write:

$$\frac{\lambda\, sinh(\lambda h)\, cosh(\lambda y)}{sinh(2\lambda h)-2\lambda h} = \frac{\lambda\left(\lambda h+\frac{\lambda^3h^3}{3!}+\dots\right)\left(1+\frac{\lambda^2h^2}{2!}+\dots\right)}{2\lambda h+\frac{4}{3}\lambda^3h^3-2\lambda h} \sim \frac{3}{4\lambda h^2}\,(\lambda \to 0); \tag{17}$$

$$\frac{\lambda\, cosh(\lambda h)\, sinh(\lambda y)}{sinh(2\lambda h)-2\lambda h} = \frac{\lambda\left(1+\frac{\lambda^2h^2}{2!}+\dots\right)\left(\lambda h+\frac{4}{3}\lambda^3h^3+\dots\right)}{\frac{4}{3}\lambda^3h^3} \sim \frac{3}{4\lambda h^2}\,(\lambda \to 0). \tag{18}$$

Then,

$$p(x,y) = 2i\mu U\left[\int_{-\infty}^{\infty}\frac{\lambda\sinh(\lambda h)\cosh(\lambda y)}{\sinh(2\lambda h)-2\lambda h}\delta(\lambda)e^{-i\lambda x}d\lambda+\int_{-\infty}^{\infty}\frac{\lambda\cosh(\lambda h)\sinh(\lambda y)}{\sinh(2\lambda h)+2\lambda h}\delta(\lambda)e^{-i\lambda x}d\lambda\right]$$
$$= 2i\mu U\left[\int_{-\infty}^{\infty}(-i)\frac{\lambda^2\sinh(\lambda h)\cosh(\lambda y)}{\sinh(2\lambda h)-2\lambda h}\delta(\lambda)\frac{e^{-i\lambda x}-1}{-i\lambda}d\lambda+\int_{-\infty}^{\infty}\frac{\lambda\sinh(\lambda h)\cosh(\lambda y)}{\sinh(2\lambda h)-2\lambda h}\delta(\lambda)d\lambda\right.$$
$$\left.+\int_{-\infty}^{\infty}(-i)\frac{\lambda^2\cosh(\lambda h)\sinh(\lambda y)}{\sinh(2\lambda h)+2\lambda h}\delta(\lambda)\frac{e^{-i\lambda x}-1}{-i\lambda}d\lambda\right].$$

Since Gelfand and Shilov [20], in terms of weak convergence

$$\lim_{N\to\infty}\frac{1}{\pi}\frac{\sin(N\lambda)}{\lambda} = \delta(\lambda),$$

then we have for the second integral

$$\int_{-\infty}^{\infty}\frac{\lambda\sinh(\lambda h)\cosh(\lambda y)}{\sinh(2\lambda h)-2\lambda h}\delta(\lambda)d\lambda = \lim_{N\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda\sinh(\lambda h)\cosh(\lambda y)}{\sinh(2\lambda h)-2\lambda h}\frac{\sin(N\lambda)}{\lambda}d\lambda = 0$$

in the sense of the principal value of the integral due to the fact that the integrand is odd. Now, taking into account (17), (18), and the known limit

$$\lim_{\lambda\to 0}\frac{e^{-i\lambda x}-1}{-i\lambda} = \mathrm{x},$$

we have

$$p(x,y)=\frac{3\mu U}{2h^2}x \qquad (-\infty<x<\infty;\ -h\le y\le h). \tag{19}$$

Let us turn to (15). Let us find out the behaviour of the terms in (15) as $\lambda\to 0$. We can write

$$\frac{sinh(2\lambda h)-\lambda h\,cosh(2\lambda h)-\lambda h}{cosh(\lambda h)[sinh(2\lambda h)-2\lambda h]}cosh(\lambda y)$$
$$=\frac{(2\lambda h+4\lambda^3h^3/3+\dots)-\lambda h(1+2\lambda^2h^2)-\lambda h}{4\lambda^3h^3/3}(1+\lambda^2y^2/2+\dots)$$
$$=\frac{4\lambda^3h^3/3-2\lambda^3h^3+\dots}{4\lambda^3h^3/3}\sim-\frac{1}{2}(\lambda\to 0);$$

$$\frac{\lambda h\,cosh(2\lambda h)-sinh(2\lambda h)-\lambda h}{sinh(2\lambda h)+2\lambda h}=\frac{\lambda h(1+2\lambda^2h^2+\dots)-2\lambda h-\lambda h}{4\lambda h}$$
$$=\frac{-2\lambda h+\dots}{4\lambda h}\sim-\frac{1}{2}(\lambda\to 0);$$

$$2\lambda y\frac{sinh(\lambda h)\,sinh(\lambda y)}{sinh(2\lambda h)-2\lambda h}=\frac{2\lambda y(\lambda h+\dots)(\lambda y+\dots)}{4\lambda^3h^3/3}=\frac{2\lambda^3hy^2}{4\lambda^3h^3/3}\sim\frac{3}{2}\frac{y^2}{h^2}(\lambda\to 0);$$

$$2\lambda y\frac{cosh(\lambda h)\,cosh(\lambda y)}{sinh(2\lambda h)+2\lambda h}\sim\frac{y}{2h}(\lambda\to 0).$$

Then, using these relations, we have from (15)

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{u}(\lambda,y)e^{-\lambda x}d\lambda=\left(\frac{3y^2}{4h^2}+\frac{y}{2h}-\frac{1}{4}\right)U \quad (-\infty<x<\infty;\ -h\le y\le h). \tag{20}$$

Passing to the determination of $v(x,y)$ and proceeding in exactly the same way as above, from (16) we obtain:

$$\frac{iUy}{2}\int_{-\infty}^{\infty}\frac{2\lambda\,cosh(\lambda h)\,sinh(\lambda y)}{sinh(2\lambda h)+2\lambda h}\delta(\lambda)e^{-i\lambda x}d\lambda=0;$$

$$\frac{iUy}{2}\int_{-\infty}^{\infty}\frac{2\lambda\,sinh(\lambda h)\,cosh(\lambda y)}{sinh(2\lambda h)-2\lambda h}\delta(\lambda)e^{-i\lambda x}d\lambda=\frac{3}{4}\frac{xy}{h^2};$$

$$\frac{iU}{2}\int_{-\infty}^{\infty}\frac{\lambda h\,cosh(2\lambda h)-sinh(2\lambda h)-\lambda h}{sinh(\lambda h)[sinh(2\lambda h)+2\lambda h]}cosh(\lambda y)\delta(\lambda)e^{-i\lambda x}d\lambda=-\frac{U}{4}\frac{x}{h};$$

$$iU\int_{-\infty}^{\infty}\frac{cosh(\lambda h)\,cosh(\lambda y)}{sinh(2\lambda h)+2\lambda h}\delta(\lambda)e^{-i\lambda x}d\lambda=\frac{U}{4h}x;$$

$$\frac{iU}{2}\int_{-\infty}^{\infty}\frac{sinh(2\lambda h)-\lambda h\,cosh(2\lambda h)-\lambda h}{cosh(\lambda h)[sinh(2\lambda h)-2\lambda h]}sinh(\lambda y)\delta(\lambda)e^{-i\lambda x}d\lambda=0;$$

$$iU\int_{-\infty}^{\infty}\frac{sinh(\lambda h)\,sinh(\lambda y)}{sinh(2\lambda h)-2\lambda h}\delta(\lambda)e^{-i\lambda x}d\lambda=\frac{3U}{4h^2}xy.$$

Eventually,

$$v(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{v}(\lambda,y)e^{-\lambda x}d\lambda=\frac{3U}{4h^2}xy+\frac{U}{4}\frac{x}{h}-\frac{U}{4}\frac{x}{h}\ -\frac{3U}{4h^2}xy=0.$$

So

$$v(x,y)=0 \qquad (-\infty<x<\infty;\ -h\le y\le h). \tag{21}$$

Now it is easy to see that the functions (19), (20), and (21) satisfy not only the boundary value problem (4)-(5)-(3), but also the nonlinear Navier-Stokes equations (1), the continuity equation (2), and the boundary conditions (3). Consequently, these functions represent the new elementary solution to the problem of the steady flow of an incompressible viscous fluid in a channel with flat parallel walls. This solution, like other exact elementary solutions, is theoretically valid for any Reynolds numbers. However, in reality, these solutions are valid up to a certain value of the Reynolds number, called the critical value.

2.2 Comparison of the Solution (19)-(21) with the Couette Solution

The solution, or Couette flow in a channel $\Sigma=\{-\infty<x,z<\infty;\ 0\le y\le h\}$ with parallel walls, depending on the constant velocity U of the upper wall, is given by the formula [2]

$$u=\frac{y}{h}U-\frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}\left(1-\frac{y}{h}\right) \qquad (0\le y\le h), \tag{22}$$

where

$$\frac{dp}{dx}=-\frac{p_1-p_2}{l}=const$$

is the constant pressure drop on the horizontal part of the channel Σ of length l. Here U and dp/dx are prescribed quantities. The same formula (22) in channel $\Omega=\{-\infty<x,z<\infty;\ -h\le y\le h\}$ is represented as

$$u = u(y) = \frac{U}{2}\left(1+\frac{y}{h}\right) - \frac{h^2}{2\mu}\left(1-\frac{y^2}{h^2}\right)\frac{dp}{dx} \qquad (-h \le y \le h). \tag{23}$$

We also transform formula (20), for which we determine U from (19),

$$U = \frac{2h^2}{3\mu}\frac{dp}{dx},$$

and substitute this expression into (20). We get

$$u = u(x,y) = \frac{h^2}{6\mu}\frac{dp}{dx}\left(\frac{3y^2}{h^2}+\frac{2y}{h}-1\right),$$

that is,

$$u = u(x,y) = \frac{h^2}{2\mu}\frac{dp}{dx}\left(\frac{y}{h}+1\right)\left(\frac{y}{h}-\frac{1}{3}\right) \qquad (-h \le y \le h). \tag{24}$$

Evidently, (24) does not coincide with Couette's solution (23), although both solutions give a parabolic distribution of horizontal velocities in the channel sections in the direction of movement of the upper wall. In addition, in (22) and (23) U and dp/dx are predetermined, while in the solution (19) to (21) only U is predetermined.

Solution (20) does not depend on the dynamic coefficient μ, while (22) or (23) depend on it. However, here the pressure (19) depends on μ; it is generated by the velocity U and is not given in advance.

Next, in (19) and (20), we introduce dimensionless quantities, setting

$$\xi = x/h,\ \eta = y/h,\ p_0(\xi) = p_0(\xi,\eta) = 2p(h\xi, h\eta)/\rho U^2,$$
$$u_0(\eta) = u_0(\xi,\eta) = u(\xi h, \xi\eta)/U,$$

as well as the Reynolds number

$$\mathrm{Re} = hU/\nu = \rho hU/\mu; \qquad \mu = \rho\nu.$$

Here ν is the kinematic coefficient of the viscous fluid and $\rho U^2/2$ is the dynamic pressure. As a result, (19) is transformed into the form

$$p_0(\xi) = \frac{3}{\mathrm{Re}}\xi \qquad (-\infty < \xi < \infty), \tag{25}$$

and in the coordinate plane (ξ,η) $(\eta = p_0(\xi))$, it is represented by a straight line passing through the origin, with slope $k = 3/\mathrm{Re}$. Formula (20) can be written as

$$u_0(\eta) = \frac{1}{4}\left(3\eta^2 + 2\eta - 1\right) \qquad (-1 \le \eta \le 1), \tag{26}$$

and in the coordinate plane (η, u_0), it is depicted as a parabola with an axis of symmetry $\eta = -1/3$ and with a vertex at the point $A(-1/3, -1/3)$. Thus, according to (26), it follows that in any vertical section of the channel, horizontal dimensionless velocities are distributed by the law of parabola with branches directed toward the movement of the channel upper wall. In this case, according to (19), the dimensionless pressure gradient

$$P = \frac{h^2}{2\mu U}\left(-\frac{d\,p}{dx}\right) = -\frac{3}{4} < 0$$

and for $-1 < \eta < 1/3$ or in the strip $-h < y < h \quad u_0(\eta) < 0$, and, consequently, there is a reverse movement of a viscous fluid, that is, movement in the opposite direction of the channel upper wall movement. This is explained by the fact that, by (9), the pressure gradient in the direction of the $0x$ axis is positive, i.e., the pressure in the channel increases as the x-coordinate increases. Since the fluid flows in the channel from places with high pressure to places with low pressure, particles near the stationary wall of the channel cannot overcome the pressure resistance, and, therefore, a backflow of the fluid occurs. The latter is compensated by the movement of the upper wall along the abscissa axis, and, moreover, near the upper wall, the fluid flows in the direction of the velocity U. However, this compensation occurs partially, namely for $1/3 < \eta < 1$ or in the strip $h/3 < y < h.$. For $-1 < \eta < 1/3$ or in the strip $-h < y < h/3$, fluid particles are not able to overcome the resistance of the pressure that arises there, as a result of which a reverse movement of the fluid occurs.

Thus, the main characteristics of the analysed new solution to the problem of the steady flow of an incompressible viscous fluid in a channel with parallel walls in a dimensionless form are represented by formulas (25), (26), as well as by the formula $v_0(\xi, \eta) = v(h\xi, h\eta)/U = 0.$

It is obvious that for $U = 0$ the velocity and pressure components are identically equal to zero. This is due to the fact that the system of Eq. (11) has only a trivial zero solution and, consequently, the original two-dimensional boundary value problem (2) to (5) has a unique solution.

3. On the Hagen-Poiseuille Flow

Let us also briefly consider the Hagen-Poiseuille problem of an axisymmetric steady flow of an incompressible viscous fluid in a straight circular pipe of infinite length with radius a. In the case of laminar fluid flow and axial symmetry, the solution to this problem in a cylindrical coordinate system (r, φ, z) is given by formulas [2]

$$\begin{gathered} v_r(r,z) = v_\varphi(r,z) = 0; \;\; v_z(r,z) = -\frac{1}{4\mu}\frac{d\,p}{d\,z}\left(a^2 - r^2\right); \\ \frac{d\,p}{d\,z} = const \quad \left(0 \le r \le a; \;\; -\infty < z < \infty; \;\; 0 \le \varphi \le 2\pi\right), \end{gathered} \tag{27}$$

according to which there is a parabolic velocity distribution along the pipe. Here v_r, v_φ, and v_z are the velocity components in the radial, circumferential, and axial directions, respectively, i.e., along the coordinate axes r,φ,z. Solution (27) satisfies Schlichting [2] the nonlinear stationary Navier-Stokes equations for an incompressible viscous fluid in a cylindrical coordinate system, the continuity equation, and the boundary condition of fluid particles sticking to the cylindrical pipe surface. In this case, the pressure in each cross section is constant, and the pressure drop along the length of a straight pipe is also a constant quantity, that is, $dp/dz = const \quad (0 \le r \le a;\ -\infty < z < \infty)$. It follows that for the implementation of a laminar (layered) flow of a viscous fluid in a circular pipe, the condition $dp/dz = const$ is a necessary condition. Let us show that this is also a sufficient condition for laminar flow.

To this end, in a cylindrical coordinate system r,φ,z, we consider the following boundary value problem for the reduced Navier-Stokes equations (without convective nonlinear terms) in the case of an axisymmetric steady flow of an incompressible viscous fluid in a straight circular pipe of radius a in the absence of body forces:

$$\begin{cases} \dfrac{dp}{dr} = \mu\left(\dfrac{\partial^2 v_r}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v_r}{\partial r} - \dfrac{v_r}{r^2} + \dfrac{\partial^2 v_r}{\partial z^2}\right); \\ \dfrac{\partial p}{\partial z} = \mu\left(\dfrac{\partial^2 v_z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v_z}{\partial r} + \dfrac{\partial^2 v_z}{\partial z^2}\right) \qquad (0 < r < a; -\infty < z < \infty); \\ \dfrac{\partial v_r}{\partial r} + \dfrac{v_r}{r} + \dfrac{\partial v_r}{\partial z} = 0; \\ \Delta p = \dfrac{\partial^2 p}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial p}{\partial r} + \dfrac{\partial^2 p}{\partial z^2} = 0; \\ v_r(r,z)\big|_{r=a} = v_z(r,z)\big|_{r=a} = 0 \qquad (-\infty < z < \infty). \end{cases} \tag{28}$$

Using the method of the integral Fourier transform with respect to the variable z, as above, we show that in this case the solution of the boundary value problem (28) under the condition $\partial p/\partial z = const$ is indeed expressed by formulas (27). However, to simplify the calculations, we will prove this assertion using the following considerations. Since by the condition

$$\frac{\partial p}{\partial z} = p_0 = const \qquad (0 \le r \le a),$$

from here, by integration, we get

$$p(r,z) = p_0 z + h(r) \qquad (0 \le r \le a),$$

where $h(r)$ is a yet arbitrary function. But $p(r,z)$ is a harmonic function in $D = \{0 < r < a; -\infty < z < \infty\}$. Therefore, it must satisfy the fourth equation of (28), the Laplace equation of the boundary value problem. Then the function $h(r)$ is determined from the equation

$$h''(r) + \frac{1}{r}h'(r) = 0 \qquad (0 < r < a),$$

having a general solution

$$h(r) = C\ln r + C_1 \qquad (0 \le r \le a).$$

From the condition of boundedness of this function for $r = 0$, it follows that $C = 0$. We can also put $C_1 = 0$. As a result, we will have

$$\frac{\partial p}{\partial z} = p_0 = const; \quad p = p(r,z) = p_0 z \qquad (-\infty < z < \infty).$$

Hence the pressure does not depend on r. Therefore, from (28) it follows that the radial velocities v_r are determined by the boundary value problem

$$\begin{cases} \dfrac{\partial^2 v_r}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v_r}{\partial r} - \dfrac{v_r}{r^2} + \dfrac{\partial^2 v_r}{\partial z^2} = 0 \qquad (0 < r < a; \ -\infty < z < \infty), \\ v_r\big|_{r=a} = 0. \end{cases} \tag{29}$$

Hence, based on the maximum principle for partial differential equations of elliptic type $v_r(r,z) \equiv 0 \ (r,z) \in D.$ The same result can be easily obtained directly by solving the boundary value problem (29). Indeed, we introduce the Fourier transformant with respect to the z coordinate

$$\overline{v}_r(r,\lambda) = \int_{-\infty}^{\infty} v_r(r,z) e^{i\lambda z} dz .$$

Then the two-dimensional boundary value problem (29) is transformed into the following one-dimensional boundary value problem:

$$\begin{cases} \dfrac{d^2\overline{v}_r}{dr^2} + \dfrac{1}{r}\dfrac{d\overline{v}_r}{dr} - \dfrac{\overline{v}_r}{r^2} - \lambda^2 \overline{v}_r = 0 \qquad (0 < r < a), \\ \overline{v}_r(r,\lambda)\big|_{r=a} = 0. \end{cases} \tag{30}$$

Next in (30) setting $r = x/|\lambda|$, we arrive at the boundary value problem:

$$\begin{cases} \dfrac{d^2\overline{V}_r}{dx^2} + \dfrac{1}{x}\dfrac{d\overline{V}_r}{dx} - \left(1 + \dfrac{1}{x^2}\right)\overline{V}_r = 0, \\ \overline{V}_r\big|_{x=a} = 0, \qquad \overline{V}_r = \overline{v}_r(x/|\lambda|). \end{cases} \tag{31}$$

The differential equation in (31) has two linearly independent solutions $I_1(x),\ K_1(x)$ − modified Bessel functions of the first and second kinds, respectively. But for $x = 0 \ (r = 0)$, only the function $I_1(r)$ is bounded. Therefore, the general solution of this equation has the form

$$\overline{V}_r = CI_1(x) \quad \Rightarrow \quad \overline{v}_r(r,\lambda) = CI_1(|\lambda| r) \qquad (0 \le r \le a).$$

Subjecting this solution to the boundary condition from (30) or (31), we find $C=0,$ since the equation $I_1(|\lambda| r)=0$ has no real roots. Hence

$$\bar{v}_r(r,\lambda)=0 \;\;\Rightarrow\;\; v_r(r,z)=0 \qquad \big((r,z)\in D\big).$$

Based on this, from the third equation in (28), the continuity equation, it follows that v_z does not depend on the z coordinate. Then, according to (28), to determine the axial velocity component v_z , we obtain the following one-dimensional boundary value problem:

$$\begin{cases} \dfrac{d^2\bar{v}_z}{dr^2}+\dfrac{1}{r}\dfrac{d\bar{v}_z}{dr}=p_0 \qquad (0<r<a), \\ \bar{v}_z\big|_{r=a}=0. \end{cases}$$

The solution to this problem is represented by the second formula in (27). The assertion has been proven.

As a result, the following theorem is proved:

Theorem 1. In order for the axisymmetric steady flow of an incompressible viscous fluid in an infinite straight circular pipe to be laminar, it is necessary and sufficient for the pressure drop along the pipe to be constant,

$$\frac{dp}{dz}=p_0=const.$$

Hence, under the same conditions, it follows:

Theorem 2. In order for the flow of a viscous fluid in an infinite straight circular pipe to be turbulent, it is necessary and sufficient for the pressure drop along the pipe to be nonconstant,

$$\frac{dp}{dz}\neq const\,.$$

4. Conclusion

For the problem of a steady flow of an incompressible viscous fluid in a channel with parallel walls, one of which is stationary and the other one moves at a constant speed in its plane, a new elementary solution is obtained, which differs from the well-known Couette flow solution. Unlike the Couette flow problem, where the velocity and the pressure drop along the length of the channel are predetermined, here, the pressure is generated by the velocity of the moving wall, and the pressure distribution law is determined from the linearized Navier-Stokes equations by Fourier analysis. A comparative analysis of these two solutions is carried out. Thus, it is shown that the reserves of mathematical analysis to obtain new simple solutions to problems in the mechanics of a viscous fluid have not yet been exhausted.

The well-known problem of Hagen-Poiseuille is again considered, and the necessary and sufficient condition for the laminar flow of a viscous incompressible fluid in an infinite straight circular pipe is established. The new solution to the Couette flow problem that we have obtained, as well as the Couette solution itself and the Hagen-Poiseuille solution does not depend on the Reynolds number, which is in conflict with the experimental data. It should also be noted that for the two problems considered here, creeping solutions, that is, the solutions of the linearized Navier-Stokes equations, also simultaneously satisfy the nonlinear Navier-Stokes equations for a steady flow of an incompressible viscous fluid.

As a result, the known solutions to two problems of viscous fluid mechanics are supplemented with new elements.

Acknowledgement
This research was not funded by any grant.

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