Unsteady Magnetohydrodynamic Flow of Second Grade Nanofluid (Ag-Cu) With CPC Fractional Derivative

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ABSTRACT

Unsteady magnetohydrodynamics (MHD) flow of fractionalized second grade nanofluid (Ag-Cu) over a vertical plate is obtained. The model is solved by using CPC fractional derivative. The novelty of present study is to generalized the model by using Fourier’s and Fick’s laws. The nanofluids are formed by dispersing two different nanoparticles, silver (Ag) and copper (Cu), into a based fluid. The governing dimensionless equations for velocity, concentration, and temperature profiles are solved using Laplace transform method and compared graphically. The effects of different parameters like fractional parameter, second grade parameter and magnetic parameter M are discussed through numerous graphs. From figures, it is observed that second grade and magnetic field have decreasing effect on velocity profile, whereas mass Grashof number have increasing effect on velocity of fluid.

Keywords:
Free convection; second grade fluid; magnetic field; CPC fractional parameter

1. Introduction

Convection flow in the presence of porosity has numerous important applications such as flows in soils, solar power collectors, heat transfer correlated with geothermal systems, heat source in the field of agricultural storage system, heat transfer in nuclear reactors, heat transfer in aerobic and anaerobic reactions, heat evacuation from nuclear fuel detritus, and heat exchangers for porous material. Heat and mass transfer occurs mostly in nature due to temperature and concentration differences respectively. Magnetohydrodynamics (MHD) is branch of Mechanics which examines the movement of electro-conductive fluid in the existence of magnetic field. Today research work in Magnetohydrodynamics (MHD) has substantial significance as these flows are absolutely prevailing in nature. Hsiao [1] worked on MHD heat transfer thermal extrusion system using non-Newtonian Maxwell fluid with radiative and viscous dissipation effects. Chamkha et al., [2] discussed the hall and ion slip effects on MHD rotating boundary layer flow of nanofluid. Chamkha et al., [3] analyzed the hall and ion slip effects on MHD free convective rotating flow through a porous medium. Ramzan et
al., [4] analyzed the MHD flow of fractionalized Jeffrey fluid with non-Newtonian heating and thermal radiation over a vertical plate. Ahmad et al., [5] compared the generalized form of Jeffrey fluid flow acquired by contemplating fractional derivative of singular kernel (Caputo) and non-singular kernel (Caputo-Fabrizio). During the last decade, different generalized fractional derivatives have appeared in the literature that are derivatives of Caputo, Caputo-Fabrizio, constant proportional Caputo [6-7]. Some studies of free convection on an inclined plane in various thermal and mechanical situations have recently been presented by mathematicians [8-14]. Some mathematical models of second grade fluids are industrial oils, slurry streams, and dilute polymer solutions with different geometry and boundary conditions. The solution of unsteady second grade fluid at plate with the assistance of the Fourier sine transformation was described by Fetecau et al., [15]. Ahmed et al., [16] has analyzed MHD heat transfer into convective boundary layer with a minimal pressure gradient. Convective mixed MHD flow studied by Narayana [17], while Chamkha et al., [18] worked Hall and ion slip impacts on unsteady MHD convective rotating flow of heat generation/absorption second grade fluid. Because of its rising significance, engineering needs to incorporate non-Newtonian fluid. Khan et al., [19] presented a fractional flow of the second grade fluid on a vertical surface driven by temperature as well as concentration gradients. Chamkha et al., [20] discussed heat transfer analysis of unsteady hybrid nanofluid flow with thermal radiation. Seth et al., [21] discussed the MHD convection flow over a vertical plate with ramped temperature. Tran et al., [22] studied on mandatory stability of fractional derivatives for fractional calculus equations, and the mathematical model used for transference of COVID-19 with Caputo fractional derivatives also discussed by Tuan et al., [23].


In this problem, the model of MHD flow of second grade nanofluid (Ag-Cu) over a plate is considered. Firstly, the governing equations have been made non-dimensional and then solved semi-analytically. The results for velocity profile, temperature profile, and concentration profile are obtained and then analyzed graphically. Various graphs are plotted and discussed for different parameters, which are used in the flow model.

2. Mathematical Description of The Model

The magnetohydrodynamic flow of second grade nanofluid over a plate is considered. The fluid is flowing along $y_1^*$-axis. The motion of fluid depends on $x_1^*$-axis and time $t_1^*$. The plate and fluid have concentration $C_\infty^*$ and temperature $T_\infty^*$ at constant $t_1^* = 0$ with zero velocity. But for $t_1^* > 0$, the plate starts to move in the plane with uniform velocity $U_1 e^{at_1^*}$. The level of concentration is lowered or raised to $t_1^*(C_\infty^* + C_W^*)/t_0^* + C_W^*$ for $t_0^* \leq t_1^*$ and $C_W^*$ for $t_0^* \leq t_1^*$. The temperature of the plate is lowered or raised to $t_1^*(T_\infty^* + T_W^*)/t_0^* + T_W^*$ for $t_0^* \leq t_1^*$ and $T_W^*$ for $t_0^* \leq t_1^*$. A uniform magnetic field $\beta_0$ is acting in the transverse direction to the flow. The transversely applied magnetic field and magnetic Reynolds number are assumed to be very small so that the induced magnetic field the Hall effect are negligible. Normally. In view of above assumption and using Boussinesq’s approximation, the convection flow of second grade fluid with chemical reaction, and magnetic field through a plate, linear momentum equation by using [28,32] is
\[
\rho_{nf} \frac{\partial u(x^*_1,t^*_1)}{\partial t^*_1} = \mu_{nf} (1 + \frac{\alpha_1}{\rho} \frac{\partial}{\partial t^*_1}) \frac{\partial^2 u(x^*_1,t^*_1)}{\partial x_1^*} + (\rho \beta \eta^{-1})_{nf} g(T^* - T_{\infty}^*) - \frac{\sigma_{nf} \beta \delta u(x^*_1,t^*_1)}{\rho} + (\rho \beta C^{-1})_{nf} g(C^* - C_{\infty}^*)
\]

(1)

Thermal equation is
\[
(\rho c_p)_{nf} \frac{\partial T^*(x^*_1,t^*_1)}{\partial t^*_1} = -\frac{\partial q(x^*_1,t^*_1)}{\partial x_1^*}
\]

(2)

According to Fourier’s Law, \(q_1(x^*_1,t^*_1)\) is given by
\[
q_1(x^*_1,t^*_1) = -K_{nf} \frac{\partial T^*(x^*_1,t^*_1)}{\partial x_1^*}
\]

(3)

Diffusion Eq. is
\[
\frac{\partial C^*(x^*_1,t^*_1)}{\partial t^*_1} = -\frac{\partial J(x^*_1,t^*_1)}{\partial x_1^*}
\]

(4)

According to Fick’s Law, \(J_1(x^*_1,t^*_1)\) is given by
\[
J_1(x^*_1,t^*_1) = -D_{nf} \frac{\partial C^*(x^*_1,t^*_1)}{\partial x_1^*}
\]

(5)

The conditions for the model [27] are
\[
\begin{align*}
&u_1(x^*_1,t^*_1) = 0, \quad T^*(x^*_1,t^*_1) = T_{\infty}^*, \quad C^*(x^*_1,t^*_1) = C_{\infty}^*, \quad x^*_1 > 0, \quad t^*_1 = 0, \\
u_1(0,t^*) = U_1 f(t^*_1), \quad T(0,t^*_1) = \begin{cases} T_{\infty}^* & t^*_1 > 0; \\
T_w^* + \frac{t^*_1(T_w^* - T_{\infty}^*)}{t_0} & 0 < t^*_1 \leq t_0,
\end{cases} \\
C(0,t^*_1) = \begin{cases} C_w^* & t^*_1 > 0; \\
C_{\infty}^* + \frac{t^*_1(C_w^* - C_{\infty}^*)}{t_0} & 0 < t^*_1 \leq t_0,
\end{cases}
\end{align*}
\]

(6) \hspace{1cm} (7) \hspace{1cm} (8)

\[
\begin{align*}
u_1(x^*_1,t^*_1) & \to 0, \quad T^*(x^*_1,t^*_1) \to 0, \quad C^*(x^*_1,t^*_1) \to 0, \quad x^*_1 \to \infty, \quad t^*_1 > 0
\end{align*}
\]

(9)

Dimensionless form of the variables are
\[
\begin{align*}
x^* &= \frac{x^*_1}{v}, & t^* &= \frac{t^*_1}{v}, & T^* &= \frac{T^* - T_{\infty}^*}{T_w^* - T_{\infty}^*}, & \eta &= \frac{u_1}{U^*}, \\
Gr^* &= \frac{\nu \beta (T_w^* - T_{\infty}^*)}{U^3}, & C^* &= \frac{C^* - C_{\infty}^*}{C_w^* - C_{\infty}^*}, & Grm^* &= \frac{\nu \beta (C_w^* - C_{\infty}^*)}{U^3}
\end{align*}
\]

(10)

Using parameter of Eq. (10) into Eq. (1)-(9), we have
\[
\frac{\partial u(x,t)}{\partial t} = a_0 (1 + S \frac{\partial}{\partial t}) \frac{\partial^2 u(x,t)}{\partial x^2} - M_0 u(x,t) + Gr_0 T(x,t) + Grm_0 C(x,t),
\]

(11)
where $a_0 = ((1 - \phi + \frac{\phi(p_s)}{\rho_f})(1 - \phi)^{2.5})^{-1}$, $Gr_0 = \frac{Gr}{\phi_0}$, $Gm_0 = \frac{Gm}{\phi_0}$, $M_0 = \frac{M}{\phi_0}$.

\[
\frac{\partial T(x,t)}{\partial t} = \frac{1}{b_0} \frac{\partial A}{\partial x},
\]

with

\[
A(x,t) = -\frac{\partial T}{\partial x},
\]

where $b_0 = \frac{\phi_1 Pr}{\phi_2}$, $Pr = \frac{\nu \rho c_p}{K_f}$, $A = \frac{q_v}{U K_f(T_w - T_\infty)}$.

\[
\frac{\partial C(x,t)}{\partial t} = \frac{1}{c_0} \frac{\partial B}{\partial x},
\]

with

\[
B(x,t) = -\frac{\partial C}{\partial x},
\]

where $c_0 = \frac{Sc}{1 - \phi}$, $Sc = \frac{\nu}{D_n f}$, $B = \frac{q_v}{U D_n f(C_w - C_{\infty})}$, $\phi_0 = 1 - \phi + \frac{\phi(p_s)}{\rho_f}$, $\phi_1 = 1 - \phi + \frac{\phi(p_c p_s)}{(\rho c p)_f}$, $\phi_2 = \frac{2k_f + k_s - 2(k_f - k_s)\phi}{2k_f + k_s + (k_f - k_s)\phi}$.

3. Generalized Model

Eq. (11) is

\[
\frac{\partial u(x,t)}{\partial t} = a_0 (1 + S \frac{\partial}{\partial t} \frac{\partial^2 u(x,t)}{\partial x^2} - M_0 u(x,t) + Gr_0 T(x,t) + Gm_0 C(x,t),
\]

Eq. (13) and Eq. (15) is generalized fractionally by [33,34]

\[
A(x,t) = -D_t^\gamma \frac{\partial T(x,t)}{\partial x}, \quad 1 \geq \gamma > 0,
\]

\[
B = -D_t^\gamma \frac{\partial C(x,t)}{\partial x}, \quad 1 \geq \gamma > 0
\]

Put Eq. (17) into Eq. (12) and Eq. (18) into Eq. (14) and making non-dimensional results, we have

\[
\frac{\partial T(x,t)}{\partial t} = \frac{1}{b_0} D_t^\alpha \frac{\partial^2 T(x,t)}{\partial x^2},
\]

\[
\frac{\partial C(x,t)}{\partial t} = \frac{1}{c_0} D_t^\alpha \frac{\partial^2 C(x,t)}{\partial x^2},
\]

Initial and boundary conditions are

\[
u(x,t) = T(x,t) = C(x,t) = 0, \quad t = 0,
\]
\( u(0, t) = f(t), \quad T(0, t) = C(0, t) = \begin{cases} 1, & t > 1; \\ t, & 0 < t \leq 1, \end{cases} \) (22)

\( u(x, t) \to 0, \quad T(x, t) \to 0, \quad C(x, t) \to 0, \quad x \to \infty, \quad t > 0, \) (23)

where \( Gm, S, M, Gr, \) and \( u \) represents the mass Grashof number, second grade parameter, magnetic field, mass Grashof number, and motion of fluid respectively and \( D_\\tau^\beta u(x, t) \) is the CPC derivative of \( u(x, t) \) given by

\[
D_\\tau^\beta u(x, t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \left[ K_1(\beta)u(x, \tau) + K_0(\beta)u'(x, \tau) \right] (t-\tau)^{-\beta} d\tau
\] (24)

4. Solution of Problem

Eq. (16), (19), (20) with conditions have been solved semi-analytically via technique of Laplace transform can be solved numerically by using Stehfest’s and Tzou’s algorithms [35,36] in case of complex expression.

4.1 Calculation of Temperature

By applying Laplace transform on Eq. (19), we have

\[
q \tilde{T}(x, q) = \frac{1}{b_0} \left[ K_1(\alpha) \frac{q}{q} + K_0(\alpha) \right] q^\alpha \frac{\partial^2 \tilde{T}(x, q)}{\partial x^2}
\] (25)

By applying Laplace transform on Eq. (19), we have with boundary conditions

\[
\tilde{T}'(0, q) = \frac{1-e^{-q}}{q^2}, \quad \tilde{T}(x, q) \to 0, \quad x \to \infty \] (26)

Put Eq. (26) in Eq. (25)

\[
\tilde{T}(x, q) = \left( \frac{1-e^{-q}}{q^2} \right) e^{-x} \left( \frac{K_1(\alpha) q b_0}{q + K_0(\alpha) q^\alpha} \right), \] (27)

4.2 Calculation of Concentration

By applying Laplace transform on Eq. (20), we have

\[
q \tilde{C}(x, q) = \frac{1}{c_0} \left[ K_1(\alpha) \frac{q}{q} + K_0(\alpha) \right] q^\alpha \frac{\partial^2 \tilde{C}(x, q)}{\partial x^2}
\] (28)

with boundary conditions

\[
\tilde{C}(0, q) = \frac{-e^{-q} + 1}{q^2}, \quad \tilde{C}(x, q) \to 0, \quad x \to \infty \] (29)

Put Eq. (29) in Eq. (28)
\[
\bar{\mathbf{C}}(x, q) = \left(\frac{-e^{-q+1}}{q^2}\right)e^{-x\sqrt{\frac{(qc_0)}{q}K_1(\alpha q+K_0(\alpha))q^\alpha}},
\]

(30)

4.3 Calculation of Velocity

By applying Laplace transform on Eq. (16), we have

\[
q\bar{u}(x, q) = a_0(1 + Sq)\frac{\partial^2\bar{u}(x, q)}{\partial x^2} - M_0\bar{u}(x, q) + \text{Gr}_0\bar{T}(x, q) + \text{Gm}_0\bar{C}(x, q),
\]

(31)

with boundary conditions

\[
\bar{u}(0, q) = f(q), \quad \bar{u}(x, q) \to 0, \quad x \to \infty
\]

(32)

Putting Eq. (32) in Eq. (31) we have

\[
\bar{u}(x, q) = \frac{1}{q-a}e^{-x\sqrt{\frac{a_0[1+Sq]}{(K_1(\alpha q+K_0(\alpha))q^\alpha)}}}[e^{-x\sqrt{\frac{q+M_0}{a_0[1+Sq]}} - e^{-x\sqrt{\frac{(qc_0)}{K_1(\alpha q+K_0(\alpha))q^\alpha}}}] +
\]

\[
\frac{\text{Gm}_0}{a_0[1+Sq]}[e^{-x\sqrt{\frac{q+M_0}{a_0[1+Sq]}} - e^{-x\sqrt{\frac{(qc_0)}{K_1(\alpha q+K_0(\alpha))q^\alpha}}}]}
\]

(33)

5. Results and Discussion

The solution for the impact of second grade nanofluid past over a vertical plate are developed by using Laplace transform technique. The effect of numerous parameters used in the governing equations of velocity fields have been analyzed in figures.

The impact of Gm and Gr on fluid velocity \(u(x, t)\) is illustrate in Figure 1. It is highlighted that fluid motion raises as values of Gm increasing. Physically higher the values of Gm increase the concentration gradients which make the buoyancy force significant and hence it is examined that velocity field is raising. And fluid motion rises up with maximizing the values of Gr, and it represents the impact of thermal buoyancy force to viscous force. Therefore maximizing the values of Gr exceed the temperature gradient due to which velocity field rises.

The impact of \(M\) and Pr on \(u(x, t)\) is reported in Figure 2. Graph shows that fluid velocity \(u(x, t)\) is reduced with accelerating values of parameter \(M\). Resistivity becomes dominant with raising \(M\) which reduced the speed of fluid. The \(u(x, t)\) increases with decreasing the values of Pr. As Pr is the ratio of momentum diffusivity to the thermal diffusivity. Physics behind this effect is that larger the momentum diffusion, slows down the fluid motion. From the graph, Figure 3 represents the effect of \(S\) and \(t\) on \(u(x, t)\). The \(u(x, t)\) increases with decreasing the values of \(S\) and increases with increasing values of time \(t\). Figure 4 indicates the effect of Sc and \(\phi\) on \(u(x, t)\). The fluid velocity rises by falling the magnitude of Sc and \(\phi\). Physics behind this fact is that increase in Schmidt number, Kinematic viscosity rises which declines the molecular diffusion, therefore, velocity of the fluid falls down and it is found that enhancing volume fraction of nanoparticles boosts the viscous effect for the nanofluid.
consequently it decreases the fluid velocity. Figure 5 indicates the effect of $\alpha$ on $u(x, t)$ and $T(x, t)$. It is clear that as $\alpha$ increases, the fluid velocity and temperature are increase. Figure 6 indicates the impact of $Pr$ and $\phi$ on $T(x, t)$. As we increased the value of $Pr$, heat diffusion is reduced which slow down the fluid motion and temperature increases with increasing values of $\phi$ as reported in figure. Because thermal conductivity of nanofluid increases with the increasing volume fraction $\phi$ of nanoparticles. The behavior of $\alpha$ and $Sc$ on $C(x, t)$ are shown in Figure 7. The $C(x, t)$ is increases with increasing value of $\alpha$ and decreases with increasing values of $Sc$ as depicted in graph. Figure 8 and 9 show the comparison of present work with Khalid et al., [26]. If we put $\beta = \gamma = \alpha \rightarrow 1, S = Gm = 0$, and in the absence of Casson parameter of Khalid et al., [26] work, the both fluid are identical. The velocity distributions overlap which shows the authenticity of inversion algorithms as presented in Figure 10. The authenticity of inverse algorithms for temperature and concentration distributions as presented in Figure 11.

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**Fig. 1.** Velocity diagram $u(x, t)$ for various values of parameters $Gm$ and $Gr$ at $S = 2, M = 0.3, Sc = 1.2, Pr = 2.5, t = 1.5$

**Fig. 2.** Velocity distribution $u(x, t)$ for various values of parameter $M$ and $Pr$ at $S = 2, Gm = 15, Sc = 1.2, Gr = 15, t = 1.5
Fig. 3. Velocity distribution $u(x, t)$ for various values of parameter $S$ and $t$ at $M = 0.3, \ Gm = 15, \ Sc = 1.2, \ Gr = 15, \ Pr = 2.5$

Fig. 4. Velocity diagram $u(x, t)$ for various values of parameters $Sc$ and $\phi$ at $S = 2, \ M = 0.3, \ Gr = 15, \ Pr = 2.5, \ t = 1.5, \ Gm = 15$

Fig. 5. Graph of fractional parameter $\alpha$ for Velocity and temperature distribution at $S = 2, \ Pr = 2.5, \ Sc = 1.2, \ Gr = 15, \ t = 1.5$
Fig. 6. Temperature distribution $T(x, t)$ for various values of parameter $Pr$ and $\phi$

Fig. 7. Concentration profile $C(x, t)$ for various values of parameter $\alpha$ and $Sc$

Fig. 8. Comparison of velocity profile for different values of $\alpha$
6. Conclusion

The flow of fractional second grade nanofluid (Ag-Cu) has been taken and solved using Laplace transform with solution. The conditions of flow problem are satisfied by the results. Different graphs have been plotted for flow parameters and then discussed. The key points of this flow model are

i. With higher Magnetic values, the velocity distribution slows down.
ii. With decreasing values of $\alpha$, the velocity distribution slows down.
iii. Thermal buoyancy forces accelerate the fluid velocity.
iv. The fluid velocity decreased for increasing values of $S$.
v. The fluid velocity decreased for increasing values of $Pr$.
vi. The fluid velocity decreased for increasing values of $\phi$.
vii. The Temperature of fluid decays down for larger values of $Sc$.
viii. The Temperature of fluid rises up for larger values of $\phi$.
ix. The concentration level is an increasing function of fractional parameter.
x. The concentration level is a decreasing function of $Sc$. 
References


