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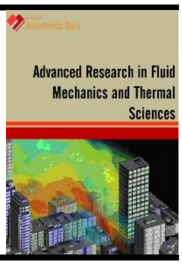

Stable Configuration of Double Horizontal Interfaces via the He-Multiple Scales Method

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ABSTRACT

The present work deals with the temporal instability of three horizontal superposed conducting incompressible fluids. The system is stressed upon by uniform tangential magnetic fields. These fields admit a presence of free-surface currents. In accordance with the importance of the porous media in many applications, the study is carried throughout porous media. To avoid the mathematical manipulation, the viscous potential theory is utilized. Therefore, the viscosity contributions could be demonstrated only on the boundary conditions. The linear stability approach together with the normal modes analysis reveal two coupled differential equations, with complex coefficients, of the Ince's type. Away from the symmetric and anti-symmetric modes of perturbations, the present study presents a general case of the amplitudes of the interface surface waves. To relax the calculations, the matrix approach is used. The stability criteria of the resonance as well as the non-resonance modes are, theoretically, discussed. The analytical perturbed solutions of the interfaces are derived. A set of graphs is depicted to identify the influences of the various parameters on the stability picture. A non-dimension analysis is adopted before the numerical calculations. It observed that the tangential magnetic fields and the porosity have stabilizing effect. In contrast, the streaming has a destabilizing influence.

Keywords:
Multiple-scales homotopy technique; porous media; simultaneous ince's equations; surface currents; viscous potential theory

1. Introduction

A magnetic fluid or simply a ferrofluid is a stable colloidal system formed of very small solid surfactant-coated ferromagnetic particle in a liquid. The fluid displays a considerable magnetic response. This area has a wide attention in many fields, for instance, see Rosensweig [1]. A very important area is the interface stability of ferrofluids. Zelazo and Melcher [2] studied the linear stability of a ferrofluid on a rigid horizontal plane under a tangential magnetic field, theoretically, as well as experimentally. Three experiments are reported which support the theoretical models and emphasize the interracial dynamics as well as the stabilizing effects of a tangential magnetic field.

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They found that the magnetic field has a stabilizing effect on the fluid interface. In addition, they gave a detailed attention to waves and instabilities of a flat interface between two ferrofluids that are acted upon by an arbitrary directed magnetic field. On the other hand, Cowley and Rosensweig [3] found that the normal electric field has a destabilizing influence. They made experiments using a magnetizable fluid at the interfaces with air and water and covered a wide domain of density differences. They found that the interface took a new format which the elevation had a normal hexagonal pattern. Elhefnawy *et al.,* [4] studied the nonlinear stability of two-superposed magnetic fluids through porous media under the action of a uniform normal magnetic field. They employed the multiple-time scale analysis expansion to achieve the non-linear stability analysis. Their analysis resulted in a Ginzburg-Landau equation. Zakaria *et al.,* [5] investigated the stability properties of non-Newtonian fluid layers. Their system is influenced by an oblique uniform magnetic field. They found that the non-Newtonian fluids have a lighter growth rate than the Newtonian ones. Furthermore, the phenomenon of the dual role is found of the magnetic permeability. The stability analysis of a plane interface that is stressed by a periodic electric field is studied by El-Dib [6]. In the presence or absence of the surface charges, El-Dib [6] obtained a Mathieu equation with complex coefficients. He found that the surface currents disappear on the interface when the magnetic field becomes unity.

Inviscid fluids are liquids with zero viscosity. The viscous effects on the motion of fluids were not understood before the definition of viscosity that was introduced earlier by Navier in 1822. The potential flow helps us to simplify the mathematical manipulation and get a good approximation solution. In addition, it represents an idealized flow solution that does not exist in real flows. The theory of viscous potential theory was first introduced by Stokes [7]. All his attenuation is paid of small amplitude waves on a liquid-gas interface. The problem that was introduced by Stokes was solved exactly, later, by using the linearized Navier-Stokes equation, assuming the potential theory by Lamb [8]. Later, Funada and Joseph [9] introduced the viscous potential theory to discuss a stability problem of stratified gas-liquid flow in a horizontal rectangular channel. Their analysis led to an explicit dispersion relation. Awasthi *et al.,* [10] discussed the viscous potential theory on the problem of stability of thin sheets of dielectric and viscous liquid. They found that the liquid viscosity has gained a stabilizing influence in the stability analysis. On the other hand, air viscosity was found to have a destabilizing effect. Moatimid and Hassan [11] investigated the linear electrohydrodynamic instability of an interface between two viscous layers. They considered the viscous potential theory through their analysis. The effects of the different parameters on the stability picture are depicted through a set of figures. They found that the Darcy’s coefficient of the porous layers has a stabilizing influence on the stability configuration. Moatimid *et al.,* [12] have studied the viscous theory to investigate a nonlinear electrohydrodynamic stability of an interface between two porous layers. They obtained a Ginzburg-Landau equation that governs the nonlinear stability analysis. The current paper deals with the stability analysis of two-horizontal interfaces imbedded between three viscous layers. Therefore, the viscous potential theory will be adopted to overcome the mathematical manipulations.

The porous media of fluids has gained a considerable interest in many engineering applications and petroleum production. Bau [13] presented a linear Kelvin-Helmholtz instability of in porous media for Darcian and non-Darcian flows. He showed that when the fluids are streaming parallel to each other, the interface becomes unstable. Furthermore, he discovered the corresponding conditions for marginal stability of the Darcian as well as non-Darcian flows. In both cases, the velocities should exceed some critical values for the stability to manifest itself. For excellent reviews around porous media, see Nield and Bejan [14]. El-Sayed *et al.,* [15] investigated the linear stability of a non-Newtonian liquid jet in a streaming inviscid gas. Their analysis resulted in a transcendental dispersion relation. This equation is numerically solved via the Mathematica software. They found that the system is more unstable in the presence of a porous medium. Moatimid and Hassan [16] presented the Marangoni convection of viscous liquid. In accordance with the complexity of their problem, they obtained a transcendental dispersion relation. By making use of the Mathematica software, they gave numerical estimations for the roots of their transcendental relation. Therefore, they illustrate the relation between the disturbance growth rate and the variation of the wave number. In addition, they found that the existence of the porous structure restricts the flow and, hence, has a stabilizing effect. Gamiel *et al.,* [17] presented the temporal stability of streaming conducting fluids through porous media under the influence of a uniform normal magnetic field. Their linear stability criteria are discussed analytically and numerically by a set of stability pictures. Recently, Bhat and Katagi [18] investigated the incompressible viscous fluid between two permeable plates. Their analysis is based on the homotopy analysis method and computer extended series method. Their examination affirms that the used methods converges the solution for very large values of the Reynolds number.

Many phenomena, in a wide range of sciences and engineering technology, are subjected to periodic forces through linear as well as nonlinear differential equations. Generally, these equations do not involve a small parameter. Therefore, to obtain a uniformly valid-expansion, the multiple-time scales, technique, as presented by Nayfeh [18], fails to obtain an analytical perturbed solution. To overcome this difficulty, the Homotopy perturbation approach does not need to the presence of this small parameter. Therefore, the Homotopy perturbation method provides a universal technique to introduce a perturbative parameter. To the best of our knowledge, the method is first, clearly, illustrated by He in 1999 [19]. It has been successfully applied to a wide range of linear as well as nonlinear differential equations. Furthermore, it is considered as a combination of the Homotopy in topology and classical perturbation techniques. The method provides us, in a convenient way, with an analytic or approximate solution in a wide variety of many problems arising in different fields. Away from the traditional perturbation methods, the Homotopy perturbation method does not need a linearization of the zero-order equation. Therefore, throughout this method, one can put a small parameter$\delta\epsilon[0,1]$, where δ is termed as the embedded Homotopy parameter. It is putted as a coefficient of any term of the problem. When $\delta = 0$, the differential equation takes a simplified form at which it may have an analytical exact solution. As δ increases and eventually take the unity, the equation evolves into the required form. At this step, the perturbed solution will approach the desired value. El-Dib [20] presented a combination between the Homotopy perturbation and the multiple-scale techniques. His modification is very well adapted for nonlinear oscillator problems. In addition, he constructed an approximate solution of the wave amplitude equation. Recently, El-Dib [21] and Moatimid *et al.,* [22] presented a combination between the homotopy perturbation and the multiple time scale techniques to obtain approximate solutions for the delayed Mathieu equation and the coupled Mathieu equations.

The first appearance of the following Ince's equation

$$(1+\Gamma\cos 2t)\,y''(t)+(\Lambda\sin 2t)y'(t)+(\Delta+\Sigma\ \cos 2t)\,y(t)=0 \tag{1}$$

is due to Whittaker's paper [23]. Whittaker has presented the special case in which Γ=0. Later, more details of this special case are reported by Ince [24,25]. The generalized Ince's equation is presented by Moussa [26]. These linear second-order differential equations describe a wide range of many physical and engineering phenomena; for instance, pendulum-like systems, vibrations, wave propagation, and many other topics in the classical mechanics (see for example Recktenwald, and Rand [27]). The current problem may be classified as a branch of the Ince's equation.

Al-Hamdan and Alkarashi [28] presented the stability problem of three horizontal porous fluid layers under the action of a uniform tangential magnetic field. They discussed the basic periodic streaming between the three fluid layers. Therefore, they obtained two simultaneous Ince's equations with complex coefficients. Unfortunately, their stability analysis must be adjusted. For instance, their dispersion relationship represents a quartic equation in the frequency of the surface waves with complex coefficients. Therefore, the four complex roots of this equation may not be having complex conjugates behavior. It follows that the zero-order solutions are, exactly, unstable. So, there is a doubt in their stability analysis. Therefore, the main purpose of the present work is to make a correction on the stability analysis of the previous work [28]. For this purpose, the basic periodic tangential streaming through three liquids, in the presence of tangential uniform magnetic fields, is presented. Because of the instability in the saturated porous media, especially in case of plane geometry, may be of special interest in both geophysics and bio-mechanics, the present study is performed throughout porous media. The aim is to study the dynamical stabilization of the unstable modes. Furthermore, the nature of instabilities that are arising, in accordance with the parametric resonance, will be achieved. The stability analysis depends mainly on the multiple- scales Homotopy perturbation technique [20]. To the best of our knowledge, this is the first attempt to utilize this method to treat the stability analysis of coupled Ince's equations with complex coefficients. To clarify the problem, the plan of this paper is organized as follows: in Section 2, the formulation of the problem is presented. The main objective of this work is to investigate the stability analysis of the resulted simultaneous Ince's equations with complex coefficients. Therefore, to avoid redundancy of the paper, the details of the solution of the boundary-value problem are excluded. To simplify the mathematical calculations, the resulted Ince's equations are put in a matrix approach. The Homotopy perturbation technique of the undisturbed state and the modulated multiple-scales Homotopy technique are presented in their subsections. Section 3 is devoted to depicting the investigation of the stability of the non-resonance cases. Several harmonic resonances are presented in Section 4 and their subsections. The results and discussion are introduced in Section 5. Finally, the concluding remarks are presented in Section 6.

2. Methodology

An irrotational motion of a magnetic horizontal fluid sheet of a finite thickness of $2a$ and embedded between two bounded layers of thicknesses $|h-a|$ for each of them is considered. Both of the three flows are assumed to be conducting and incompressible fluids. The system is influenced by uniform tangential magnetic fields. These fields admit a presence of free-surface currents. In addition, no volume charges are assumed to be present in the bulk of the fluid layers. Without any loss of generality, the motion is considered in two-dimensions only. It is convenient to work with the Cartesian coordinates (x, y). Figure 1 is a sketch of the system under consideration, where the y-axis is taken to be vertically upward, and the x-axis are taken horizontally to be at the center of the middle sheet. The upper fluid occupies the region $a<y<h$ and the middle fluid is contained in the region typified by $-a<y<a$, while the range $-h<y<-a$ is occupied by the lower layer. The three fluids are basically streaming with periodic velocities as $V_j cos\Omega t$, where $j=1,2,3$. Generally, the subscripts, $j=1,2,3$ refers to quantities in the upper fluid, plane sheet and lower fluid, respectively. The system is influenced by a gravitational force that is acting in the negative y-direction. There are two interfaces between the fluids and they are assumed to be well defined and initially flat to form the planes $y=-a$ and $y=a$. In fact, sharp interfaces between the three fluids may not exist. In addition, there is an ill-defined transition region at which the two fluids intermix. The width of this transition zone is usually small compared to the other characteristic length of the motion; hence, for the mathematical analysis, we shall assume that the fluids are separated by sharp interfaces. The two interfaces are assumed to be parallel and the flow in each phase is everywhere parallel to each other. All fluids are assumed to be of a uniform nature, homogeneous and are all saturated in porous media. The structure of the liquids is defined from the following characteristics; density ρ, dynamic viscosity η, Darcy's coefficient V, and porosity ζ.

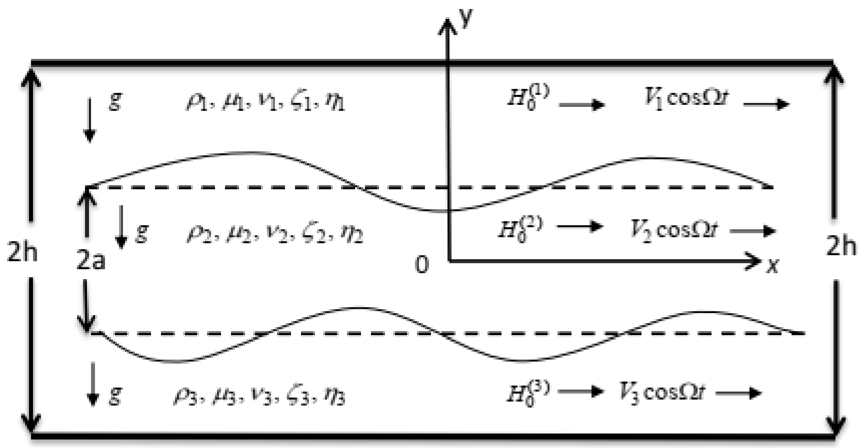


Fig. 1. The physical model

2.1 Derivations of the Ince's Equations

After a small departure from the stationary state, the surface deflections may be expressed as follows

$y(x;t) = \xi_1(x;t)$ at $y = a$ and $y(x;t) = \xi_2(x;t)$ at $y = -a$

Recently, Rusdi *et al.,* [29] utilized a linear stability analysis is to analyze an eigenvalue problem of the perturbed state. Their approach depends mainly on the normal modes analysis together with a numerical technique. Therefore, in the light of the normal modes analysis, the surface deflections $\zeta_j(x,t)$may be given by a sinusoidal wave of finite amplitude where, after disturbance, the interface is represented by

$$y(x;t) = (-)^{j+1} a + \xi_j(x;t) \qquad (j = 1,2) \tag{2}$$

and

$$\xi_j(x;t) = \gamma_j(t)e^{ikx} + c.c. \qquad (j = 1,2) \tag{3}$$

where $\gamma_j(t)$ is an arbitrary time-dependent function. It determines the behavior of the amplitude of the disturbance on the interface. k is the wave number which is assumed to be real and positive and $c.c.$ represents the complex conjugate of the preceding term.

As usual in the problems of linear magnetic stability, e. g. Melcher [30], Chandrasekhar [31] and El-Dib [6], after lengthy, but straightforward calculation, one finds the following characteristic equations

$$\frac{d^2}{dt^2}\gamma_j(t)+\left(r_{2j-1}+2i\,p_{2j-1}\cos\Omega\, t\right)\frac{d}{dt}\gamma_j(t)+$$
$$\left(a_{2j-1}+2i\,(b_{2j-1}\cos\Omega\, t+c_{2j-1}\sin\Omega\, t)+4q_{2j-1}\cos^2\Omega\, t\right)\gamma_j(t)$$
$$+\left(r_{2j}+2i\,p_{2j}\cos\Omega\, t\right)\frac{d}{dt}\gamma_{3-j}(t)$$
$$+\left(a_{2j}+2i\,(b_{2j}\cos\Omega t+c_{2j}\sin\Omega\, t)+4q_{2j}\cos^2\Omega\, t\right)\gamma_{3-j}(t)=0,\quad (j=1,2) \tag{4}$$

where the coefficients r_j, p_j, a_j, b_j, c_j and q_j involve all parameters of the system at hand. These coefficients are well-known from the context. Because they are so long, they are not included in the paper.

Now, we proceed to investigate the stability analysis of the Ince's Eq. (4). For simplicity, these equations may be rewritten in a matrix form. Therefore, they may be written in the following matrix form

$$\ddot{\underline{Y}}(t)+(\underline{R}+2i\underline{P}\cos\Omega t)\dot{\underline{Y}}(t)+(\underline{A}+2i\underline{B}\cos\Omega\, t+2i\underline{C}\sin\Omega\, t+4\underline{Q}\cos^2\Omega\, t)\underline{Y}(t)=0 \tag{5}$$

where Ω is the frequency of the external magnetic field excitation, t is the time, $\underline{Y}$ is a real vector variable having two dependent variables on t which are given by $\gamma_1(t)$ and $\gamma_2(t)$, as in the form

$\underline{Y}(t)=(\gamma_1(t),\gamma_2(t))^T$, $\underline{R}$, $\underline{P}$, $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{Q}$ are constant 2×2 matrices

For simplicity, these matrices may be expressed as follows

$$\underline{R}=\begin{pmatrix} r_1 & r_2 \\ r_4 & r_3 \end{pmatrix},\quad \underline{P}=\begin{pmatrix} p_1 & p_2 \\ p_4 & p_3 \end{pmatrix},\quad \underline{A}=\begin{pmatrix} a_1 & a_2 \\ a_4 & a_3 \end{pmatrix},$$
$$\underline{B}=\begin{pmatrix} b_1 & b_2 \\ b_4 & b_3 \end{pmatrix},\quad \underline{C}=\begin{pmatrix} c_1 & c_2 \\ c_4 & c_3 \end{pmatrix},\text{ and } \underline{Q}=\begin{pmatrix} q_1 & q_2 \\ q_4 & q_3 \end{pmatrix} \tag{6}$$

where the elements of the above matrices are formed from all parameters of the problem.
It is worthwhile to note that in the absence of the imaginary coefficients from Eq. (4), one finds the well-known damping coupled Mathieu's equation.

Since the amplitudes $\gamma_1(t)$ and $\gamma_2(t)$ that appear in the characteristic Eq. (5) are real, one may separate the real and imaginary parts to get

$$\ddot{\underline{Y}}(t)+\underline{R}\dot{\underline{Y}}(t)+(\underline{A}+4\underline{Q}\cos^2\Omega t)\underline{Y}(t)=0 \tag{7}$$

and

$$\underline{P}\cos\Omega t\dot{\underline{Y}}(t)+\left(\underline{B}\cos\Omega t+\underline{C}\sin\Omega t\right)\underline{Y}(t)=0 \tag{8}$$

Differentiating Eq. (8) with respect to time t, and combining the result with Eq. (7), one finds

$$(1+2\underline{P}\cos\Omega t)\ddot{\underline{Y}}(t)+[\,\underline{R}+2\underline{B}\cos\Omega t+2(\underline{C}-\Omega\underline{P})\sin\Omega t]\dot{\underline{Y}}(t).$$

$$+\left[\underline{A}+2\Omega\underline{C}\cos\Omega t-2\Omega\underline{B}\sin\Omega t+4\underline{Q}\cos^2\Omega t\right]\underline{Y}(t)=0 \qquad (9)$$

Eq. (9) may be considered as a modified coupled Ince's equation. In the special case, as the matrices $\underline{P}, \underline{B}$ and $\underline{C}$ becomes zero, it is reduced to the well-known classical Mathieu system.

2.2 Multiple-Scales Homotopy Perturbation Approach

In accordance with He' approach, Eq. (9) may be divided into two parts as $\underline{L}(\underline{Y})$ and $\underline{N}(\underline{Y})$, where

$$\underline{L}(\underline{Y})=\ddot{\underline{Y}}(t)+\underline{A}\underline{Y}(t) \qquad (10)$$

and

$$\underline{N}(\underline{Y})=2\underline{P}\cos\Omega t\ddot{\underline{Y}}(t)+\left[\underline{R}+2\underline{B}\cos\Omega t+2(\underline{C}-\Omega\underline{P})\sin\Omega t\right]\dot{Y}(t)$$
$$+\left[2\Omega\underline{C}\cos\Omega t-2\Omega\underline{B}\sin\Omega t+4\underline{Q}\cos^2\Omega t\right]Y(t) \qquad (11)$$

Now, the Homotopy equation may be constructed as

$$\underline{H}(\underline{Y},\delta)=\underline{L}(\underline{Y})+\delta\underline{N}(\underline{Y})=0\ , \qquad \delta\in[0,1] \qquad (12)$$

where δ is a non-zero small parameter. As in the He's Homotopy perturbation method [19], it is obvious that when $\delta=0$ Eq. (12) gives the harmonic system $L(Y)=0$.

When $\delta\rightarrow 1$, Eq. (12) becomes like the Ince's system (10). For $0<\delta<1$, the solution of Eq. (12) can be sought in terms of δ, so that the function$\underline{Y}(t)$ could become $\underline{Y}(t,\delta)$ Accordingly, Eq. (12) can be rewritten as

$$\left(\frac{d^2}{dt^2}+\underline{A}\right)\underline{Y}(t,\delta)+\delta\left\{\ 2\underline{P}\cos\Omega t\frac{d^2}{dt^2}+\left[\underline{R}+2\underline{B}\cos\Omega t+2(\underline{C}-\Omega\underline{P})\sin\Omega t\right]\frac{d}{dt}\right.$$
$$\left.+\left[2\Omega\underline{C}\cos\Omega t-2\Omega\underline{B}\sin\Omega t+4\underline{Q}\cos^2\Omega t\right]\right\}\underline{Y}(t,\delta)=0 \qquad (13)$$

On applying the homotopy-multiple-scales method [21], we may use the two scales t_o and τ such that $\tau=\delta t$ Therefore, the second-order differential operator may be expressed in terms of the partial derivatives as follows

$$\frac{d^2 ..}{dt^2}\equiv\frac{\partial^2 ..}{\partial t_0^2}+2\delta\frac{\partial^2 ..}{\partial t_0\partial\tau} \qquad (14)$$

Also, the dependent vector variable $\underline{Y}(t,\delta)$ may be expanded in the form

$$\underline{Y}(t,\delta)=\underline{Y}_0(t_0,\tau)+\delta\underline{Y}_1(t_0,\tau)+\ldots \qquad (15)$$

where the vector $\underline{Y}_n(t_0,\tau)$ is an unknown complex vector to be determined. On substituting from expansions (14) and (15) into the homotopy Eq. (13), then equate the coefficients of like powers of δ to zero, one gets the following set of equations

$$\delta^0 : \frac{\partial^2}{\partial t_0^2} \underline{Y}_0(t_0, \tau) + \underline{A}\,\underline{Y}_0(t_0, \tau) = \underline{0}, \tag{16}$$

and

$$\delta : \frac{\partial^2}{\partial t_0^2} \underline{Y}_1(t_0, \tau) + \underline{A}\,\underline{Y}_1(t_0, \tau) = -2\frac{\partial^2}{\partial t_0 \partial \tau} \underline{Y}_0(t_0, \tau) - \left(\underline{R} + 2\underline{B}\cos\Omega t_0 + 2(\underline{C} - \Omega \underline{P})\sin\Omega t_0\right)\frac{\partial}{\partial t_0}\underline{Y}_0(t_0, \tau)$$
$$- \left(2(\Omega \underline{C} - \underline{P}\,\underline{A})\cos\Omega t_0 - 2\Omega \underline{B}\sin\Omega t_0 + 4\underline{Q}\cos^2\Omega t_0\right)\underline{Y}_0(t_0, \tau) \tag{17}$$

The solution of the system (16) may be written in the following form

$$\underline{Y}_0(t_0, \tau) = \underline{\pi}_j\left(\alpha_j(\tau)e^{i\omega_j t_0} + \overline{\alpha}_j(\tau)e^{-i\omega_j t_0}\right) \tag{18}$$

where $\alpha_j(t_0, \tau)$ and its complex conjugate $\overline{\alpha}_j(t_0, \tau)$ are unknown as constant functions of integration. The constant vector $\underline{\pi}_j$ is given by

$$\underline{\pi}_j = \begin{pmatrix} a_2 \\ \omega_j^2 - a_1 \end{pmatrix}, \tag{19}$$

where a_1 and a_2 are the first two elements in the first row of the second-order matrix $\underline{A}$. $\omega_j^2; j = 1,2$ are the eigenvalues of the characteristic equation det $\left(\underline{A} - \omega^2 \underline{I}\right)$, which gives

$$\omega^4 - Tr\left(\underline{A}\right)\omega^2 + det\left(\underline{A}\right) = 0. \tag{20}$$

The substitution from Eq. (18) into Eq. (17) yields

$$\begin{aligned} \frac{\partial^2 \underline{Y}_1}{\partial t_0^2} + \underline{A}\,\underline{Y}_1 = & -[2i\omega_j \frac{\partial}{\partial \tau}\alpha_j \underline{\pi}_j + (i\omega_j \underline{R} + 2\underline{Q})\,\underline{\pi}_j\,\alpha_j]e^{i\omega_j t_0} + [2i\omega_j \frac{\partial}{\partial \tau}\overline{\alpha}_j \underline{\pi}_j + (i\omega_j \underline{R} - 2\underline{Q})\,\underline{\pi}_j\,\overline{\alpha}_j]e^{-i\omega_j t_0} + \\ & (\omega_j + \Omega)\,[(\omega_j \underline{P} - \underline{C}) - i\underline{B}]\,\underline{\pi}_j\,\alpha_j\,e^{i(\omega_j + \Omega)t_0} + (\omega_j + \Omega)\,[(\omega_j \underline{P} - \underline{C}) + i\underline{B}]\,\underline{\pi}_j\,\overline{\alpha}_j\,e^{-i(\omega_j + \Omega)t_0} + \\ & (\omega_j - \Omega)\,[(\omega_j \underline{P} + \underline{C}) - i\underline{B}]\,\underline{\pi}_j\,\alpha_j\,e^{i(\omega_j - \Omega)t_0} + (\omega_j - \Omega)\,[(\omega_j \underline{P} + \underline{C}) + i\underline{B}]\,\underline{\pi}_j\,\overline{\alpha}_j\,e^{-i(\omega_j - \Omega)t_0} - \\ & \underline{Q}\,\underline{\pi}_j\,\alpha_j\,[e^{i(\omega_j + 2\Omega)t_0} + e^{i(\omega_j - 2\Omega)t_0}] - \underline{Q}\,\underline{\pi}_j\,\overline{\alpha}_j\,[e^{-i(\omega_j + 2\Omega)t_0} + e^{-i(\omega_j - 2\Omega)t_0}]. \end{aligned} \tag{21}$$

At this stage, the analysis reveals two categories; the first category is concerned with the non-resonance case. Meanwhile, the second one deals with the resonant cases. The second case depends on the nature of the external frequency Ω when it approaches the natural frequency $\left[\omega_j, 2\omega_j, (\omega_1 \pm \omega_2)/2 \; and \; (\omega_1 \pm \omega_2)\right]$. The following sections are devoted to illustrate these cases in more details.

3. The Non-Resonance Case

There are many perturbation problems which may properly be singular, but they are not of the boundary-layer type. A large family of such problems has a secular nature. The presence of the secular term resulted in an unbounded solution. Therefore, to obtain a uniformly-valid expansion of the perturbed solution, the secular terms must be avoided. The foregoing Eq. (21) reveals the secular term. Obviously, this term is represented as a coefficient of the exponential $e^{\pm i\omega_j t}$. The elimination of this secular term leads to the following solvability conditions

$$2i\omega_j \alpha_j'(\tau)\underline{\pi}_j + (i\omega_j \underline{R} + 2\underline{Q})\underline{\pi}_j \alpha_j(\tau) = 0 \tag{22}$$

In addition, the complex conjugate of Eq. (22) must be omitted. Let us operate from the right on Eq. (22) by the transpose vector $\underline{\pi}_j^T$ and then use the following normalization condition

$$\frac{\underline{\pi}_j \underline{\pi}_j^T}{|\underline{\pi}_j|\,|\underline{\pi}_j^T|} = 1 \tag{23}$$

Eq. (22) is, therefore, then reduced to

$$\alpha_j'(\tau) + (\hat{R}_j - i\hat{Q}_j)\alpha_j(\tau) = 0 \tag{24}$$

where $\hat{R}_j$ and $\hat{Q}_j$ are real constants defined as follows

$$\hat{R}_j = \frac{1}{2}\underline{S}_j \underline{R}\, \underline{\pi}_j, \ \hat{Q}_j = \frac{1}{\omega_j}\underline{S}_j \ \underline{Q}\ \underline{\pi}_j, \text{ where } \underline{S}_j = \frac{\underline{\pi}_j^T}{|\underline{\pi}_j|\,|\underline{\pi}_j^T|} \tag{25}$$

The solution of the system of Eq. (24) may be written as

$$\alpha_j(\tau) = \alpha_0 \ e^{-(\hat{R}_j - i\,\hat{Q}_j)\tau} \tag{26}$$

where α_0 is a real constant that represents the amplitude. The nature of the stability behavior of this system requires that the exponential in Eq. (26) should have a negative real part. This can be occurring when

$$\hat{R}_j > 0, \text{ which implies that } \underline{S}_j \underline{R}_j \underline{\pi}_j > 0 \tag{27}$$

Consequently, the solution at the first-order problem may be formulated as follows

$$\begin{aligned}\underline{Y}_1(t,\tau) = 2\alpha_0 e^{-\hat{R}_j \tau}[&\underline{F}_j \cos(\hat{\underline{Q}}_j + \omega_j + \Omega)t_0 + \underline{K}_j \sin(\hat{\underline{Q}}_j + \omega_j + \Omega)t_0 + \underline{G}_j \cos(\hat{\underline{Q}}_j + \omega_j - \Omega)t_0 \\ &+ \underline{E}_j \sin(\hat{\underline{Q}}_j + \omega_j - \Omega)t_0 - \underline{L}_j \cos(\hat{\underline{Q}}_j + \omega_j + 2\Omega)t_0 - \underline{H}_j \cos(\hat{\underline{Q}}_j + \omega_j - 2\Omega)t_0\,]\end{aligned} \tag{28}$$

where $\underline{F}_j, \underline{K}_j, \underline{G}_j, \underline{E}_j, \underline{L}_j$ and $\underline{H}_j$ are real constant matrices of order (2×1). They may be written as follows

$$\underline{F}_j = \left(-(\omega_j+\Omega)^2 \underline{I} + \underline{A}\right)^{-1}(\omega_j+\Omega)(\omega_j \underline{P} - \underline{C})\,\underline{\pi}_j, \quad \underline{K}_j = \left(-(\omega_j+\Omega)^2 \underline{I} + \underline{A}\right)^{-1}(\omega_j+\Omega)\,\underline{B}\,\underline{\pi}_j,$$
$$\underline{G}_j = \left(-(\omega_j-\Omega)^2 \underline{I} + \underline{A}\right)^{-1}(\omega_j-\Omega)\,(\omega_j \underline{P} + \underline{C})\,\underline{\pi}_j, \quad \underline{E}_j = \left(-(\omega_j-\Omega)^2 \underline{I} + \underline{A}\right)^{-1}(\omega_j-\Omega)\,\underline{B}\,\underline{\pi}_j, \quad (29)$$
$$\underline{L}_j = \left(-(\omega_j+2\Omega)^2 \underline{I} + \underline{A}\right)^{-1}\underline{Q}\,\underline{\pi}_j, \quad \underline{H}_j = \left(-(\omega_j-2\Omega)^2 \underline{I} + \underline{A}\right)^{-1}\underline{Q}\,\underline{\pi}_j$$

The complete solution of the zero-order problem is modulated when $\delta \to 1$. Consequently, $\tau \to t$, so, we have

$$\underline{Y}_0(t) = 2\alpha_0 \underline{\pi}_j\, e^{-\hat{R}_j t} \cos(\omega_j + \hat{Q}_j)t \quad (30)$$

Therefore, the approximate solution of the non-resonance case is formulated by substituting from Eq. (28) and (30) into Eq. (16), and setting $\delta = 1$ to obtain

$$\underline{Y}(t) = 2\alpha_0 e^{-\hat{R}_j t}[\underline{\pi}_j \cos(\hat{Q}_j+\omega_j)t + \underline{F}_j \cos(\hat{\underline{Q}}_j+\omega_j+\Omega)t + \underline{K}_j \sin(\hat{\underline{Q}}_j+\omega_j+\Omega)t + \underline{G}_j \cos(\hat{\underline{Q}}_j+\omega_j-\Omega)t$$
$$+\underline{E}_j \sin(\hat{\underline{Q}}_j+\omega_j-\Omega)t - \underline{L}_j \cos(\hat{\underline{Q}}_j+\omega_j+2\Omega) - \underline{H}_j \cos(\hat{\underline{Q}}_j+\omega_j-2\Omega)t] \quad (31)$$

It should be noted that, as given from the calculations, $\hat{R}_j$ is independent of the external frequency Ω. Therefore, the condition that $\hat{R}_j > 0$ is necessary for stability requirements. At this point of the numerical calculations of the sample of the chosen data, when we have plotted $Log\Omega$ versus the wave number k, this condition must be satisfied.

4. Harmonic- Resonance Cases

The inspection of the right-hand side of Eq. (21) reveals other secular terms. These terms depend on the nature of the external frequency Ω. It approaches the natural frequency $[\omega_j, 2\omega_j, (\omega_1 \pm \omega_2)/2\ and\ (\omega_1 \pm \omega_2)]$. The following subsections are devoted to investigating these cases.

4.1 Harmonic-Resonance in Case of Ω Near ω_j

Now, the first level of perturbation system as given in Eq. (21) reveals that there is another term that may be produced as a secular term. On introducing a detuning parameter σ, such that the frequency of the streaming field intensity may approach the frequency of the surface wave as follows

$$\Omega = \omega_j + \delta\sigma \quad (32)$$

therefore, we may write

$$-i(\omega_j - 2\Omega)t_0 = i\omega_j t_0 + 2i\sigma\tau \quad (33)$$

The implication of Eq. (33) into the right-hand side of Eq. (21) will convert the coefficient of the exponential $exp[\pm i(\omega - 2\Omega)t_0]$ to $exp[\pm(i\omega t_0 + 2i\sigma\tau)]$. Therefore, the secular terms will be rising again. At this end, the solvability condition given by Eq. (22) will be modified to become

$$\alpha_j' + (\hat{R}_j - i\hat{Q}_j)\alpha_j - i\frac{1}{2}\hat{Q}_j\,\bar{\alpha}_j\,e^{2i\sigma\tau} = 0 \tag{34}$$

In addition, a similar behavior for the complex conjugate of Eq. (34) is obtained. This equation admits non-trivial solutions in the following forms

$$\alpha_j(\tau) = [\alpha_{jr}(\tau) + i\,\alpha_{ji}(\tau)]e^{(-\hat{R}_j + i\sigma)\tau} \tag{35}$$

where the functions $\alpha_{jr}(\tau)$ and $\alpha_{ji}(\tau)$ are real ones. Substituting from Eq. (35) into Eq. (34) and separating the real and imaginary parts, we obtain a pair of governing equations in the following forms

$$\alpha_{jr}'(\tau) + \left(-\sigma + \frac{1}{2}\hat{Q}_j\right)\alpha_{ji}(\tau) = 0 \tag{36}$$

and

$$\alpha_{ji}'(\tau) + \left(\sigma - \frac{3}{2}\hat{Q}_j\right)\alpha_{jr}(\tau) = 0 \tag{37}$$

The solutions of the coupled Eq. (36) and (37) may be written as

$$\alpha_{jr}(\tau) = \left(\sigma - \frac{1}{2}\hat{Q}_j\right)\sin\theta\tau \tag{38}$$

and

$$\alpha_{ji}(\tau) = \theta\cos\theta\tau \tag{39}$$

The amplitude functions $\alpha_j(\tau)$ and $\bar{\alpha}_j(\tau)$, at this resonance case, may be formulated by substituting from Eq. (38) and (39) into Eq. (35) and its complex conjugate, Therefore, one finds

$$\alpha_j(\tau) = \left\{\left(\sigma - \frac{1}{2}\hat{Q}_j\right)\sin\theta\tau + i\,\theta\cos\theta\tau\right\}e^{(-\hat{R}_j + i\sigma)\tau} \tag{40}$$

and

$$\bar{\alpha}_j(\tau) = \left\{\left(\sigma - \frac{1}{2}\hat{Q}_j\right)\sin\theta\tau - i\,\theta\cos\theta\tau\right\}e^{-(\hat{R}_j + i\sigma)\tau} \tag{41}$$

Substituting from Eq. (38) and (39) into Eq. (37), one obtains

$$\theta^2 - \left(\sigma - \frac{3}{2}\hat{Q}_j\right)\left(\sigma - \frac{1}{2}\hat{Q}_j\right) = 0 \tag{42}$$

By using the relations that are given by Eq. (38), (39) and Eq. (32), the zero-order solution is given by

$$\underline{Y}_0(t_0,\tau) = e^{-\hat{R}_j\tau}\left\{\left(2\sigma - \hat{Q}_j\right)\sin(\theta\tau)\cos\omega_j t_0 - 2\,\theta\cos(\theta\tau)\sin\omega_j t_0\right\}\underline{\pi}_j \tag{43}$$

Finally, the complete first-order solution in this case is given by

$$\begin{aligned}\underline{Y}_1(t_0,\tau) = e^{-\hat{R}_j\tau}\{\,&(2\sigma - \hat{Q}_j)\sin\theta\tau\left(\underline{G}_j + \underline{F}_j\cos 2\Omega t_0 + \underline{K}_j\sin 2\Omega t_0 - \underline{L}_j\cos 3\Omega t_0\right)\\ &+ 2\theta\cos\theta\tau\left(\underline{E}_j - \underline{F}_j\sin 2\Omega t_0 + \underline{K}_j\cos 2\Omega t_0 + \underline{L}_j\sin 3\Omega t_0\right)\}\end{aligned} \tag{44}$$

Therefore, the approximate solution, at this resonance case, is formulated by substituting from Eq. (43) and (44) into Eq. (15), setting $\delta \to 1$ and using (32), to get

$$\begin{aligned}\underline{Y}(t) = &\left(2\Omega - 2\omega_j - \hat{Q}_j\right)\sin\theta t\; e^{-\hat{R}_j t}\left[\underline{G}_j + \underline{\pi}_j\cos\omega_j t + \underline{F}_j\cos 2\Omega t + \underline{K}_j\sin 2\Omega t - \underline{L}_j\cos 3\Omega t\right]\\ &+ 2\theta\cos\theta t\, e^{-\hat{R}_j t}\left[\underline{E}_j - \underline{\pi}_j\sin\omega_j t - \underline{F}_j\sin 2\Omega t + \underline{K}_j\cos 2\Omega t + \underline{L}_j\sin 3\Omega t\right]\end{aligned} \tag{45}$$

The final results that are given in Eq. (45) tell us that the stability occurs when $\hat{R}_j > 0$. Eq. (42) is a quadratic equation. Clearly, the stability criterion requires that the right-hand-side of it must be a positive, which implies that

$$\sigma > \frac{3}{2}\hat{Q}_j \quad \text{and} \quad \sigma < \frac{1}{2}\hat{Q}_j \tag{46}$$

In view of Eq. (32), we conclude that the bounded solution requires

$$\hat{R}_j > 0,\; \Omega > \omega_j + \frac{3}{2}\hat{Q}_j \quad \text{and} \quad \Omega < \omega_j + \frac{1}{2}\hat{Q}_j \tag{47}$$

4.2 The Sub-Harmonic Resonance in Case of Ω Near $2\omega_j$

The investigation of the first level of perturbation system that is given by Eq. (21) reveals another term that may produce a secular one. Introducing another detuning parameterσ_a, when Ω approaches $2\omega_j$, we may write

$$\Omega = 2\omega_j + 2\delta\sigma_a \tag{48}$$

Consequently, one finds

$$-i(\omega_j - \Omega)t_0 = i\omega_j t_0 + 2i\sigma_a\tau \tag{49}$$

The implication of Eq. (49) into the right-hand side of Eq. (21) will convert the coefficient of the exponential $exp[\pm i(\omega - \Omega)t_0]$ into $exp[\pm(i\omega t_0 - 2i\sigma_a \tau)]$. Therefore, the secular term will be rising again. At this point, the solvability condition is given by

$$\alpha'_j + (\hat{R}_j - i\hat{Q}_j)\alpha_j + (-\hat{B}_j + i\hat{P}_j)\overline{\alpha}_j e^{2i\sigma_a \tau} = 0 \tag{50}$$

Similar techniques may be used for the complex conjugate of Eq. (50). The matrices $\hat{B}_j$ and $\hat{P}_j$ that are given in Eq. (50) are defined as

$$\hat{P}_j = \frac{(\omega_j - \Omega)}{2} \underline{S}_j (\underline{P} + \frac{1}{\omega_j}\underline{C})\underline{\pi}_j \text{ and } \hat{B}_j = \frac{(\omega_j - \Omega)}{2\omega_j}\underline{S}_j \underline{B}\underline{\pi}_j \tag{51}$$

To investigate the stability analysis, at the resonance case of Ω as near $2\omega_j$, we proceed as in the previous section. This analysis leads to obtain a solution of (50) in the form

$$\alpha_j(\tau) = \left[(\sigma_a - \hat{Q}_j)\sin\hat{\theta}\tau + i\,(\hat{\theta}\cos\hat{\theta}\tau - \hat{B}\sin\hat{\theta}\tau)\right]e^{(-\hat{R}_j + i\,\sigma_a)\tau} \tag{52}$$

where $\hat{\theta}$ is given by the following characteristic equation

$$\hat{\theta}^2 + \hat{B}_j^2 - (\sigma_a - \hat{Q}_j)^2 = 0 \tag{53}$$

Clearly, the stability criterion requires that $\hat{R}_j > 0$ and $\hat{\theta}$ should be real. The first condition is the same condition as in the non-resonance case and the second condition can be satisfied, whence

$$\sigma_a > \hat{Q}_j + \hat{B}_j \text{ and } \sigma_a < \hat{Q}_j - \hat{B}_j \tag{54}$$

On using the (52), the complete zero order and the first-order solutions are

$$\underline{Y}_0(t_0,\tau) = 2e^{-\hat{R}\tau}[(\sigma_a - \hat{Q}_j)\sin\,\hat{\theta}\tau\,\cos\tfrac{1}{2}\Omega t_0 - (\hat{\theta}\cos\hat{\theta}\tau - \hat{B}\sin\,\hat{\theta}\,\tau)\sin\tfrac{1}{2}\Omega t_0]\,\underline{\pi}_j \tag{55}$$

and

$$\begin{aligned}\underline{Y}_1(t_0,\tau) = {}& 2e^{-\hat{R}\tau}(\sigma_a - \hat{Q}_j)\sin\,\hat{\theta}\tau\left[\underline{F}_j\cos\tfrac{3}{2}\Omega t_0 + \underline{K}_j\sin\tfrac{3}{2}\Omega t_0 - \underline{L}_j\cos\tfrac{5}{2}\Omega\,t_0 - \underline{H}_j\cos\tfrac{3}{2}\Omega\,t_0\right] + \\ & 2e^{-\hat{R}\tau}(\hat{\theta}\cos\hat{\theta}\tau - \hat{B}\sin\,\hat{\theta}\,\tau)\left[-\underline{F}_j\sin\tfrac{3}{2}\Omega\,t_0 + \underline{K}_j\cos\tfrac{3}{2}\Omega\,t_0 + \underline{L}_j\sin\tfrac{5}{2}\Omega t_0 - \underline{H}_j\sin\tfrac{3}{2}\Omega\,t_0\right]\end{aligned} \tag{56}$$

Finally, the complete approximate solution, may be formulated in the following form

$$\begin{aligned}\underline{Y}(t) = {}& 2e^{-\hat{R}t}[(\tfrac{1}{2}\Omega - \omega_j - \hat{Q}_j)\sin\hat{\theta}t\cos\tfrac{1}{2}\Omega t - (\hat{\theta}\cos\hat{\theta}t - \hat{B}\sin\hat{\theta}t)\sin\tfrac{1}{2}\Omega t]\,\underline{\pi}_j + \\ & 2e^{-\hat{R}t}(\tfrac{1}{2}\Omega - \omega_j - \hat{Q}_j)\sin\hat{\theta}t\left[\underline{F}_j\cos\tfrac{3}{2}\Omega t + \underline{K}_j\sin\tfrac{3}{2}\Omega t - \underline{L}_j\cos\tfrac{5}{2}\Omega t - \underline{H}_j\cos\tfrac{3}{2}\Omega t\right] + \\ & 2e^{-\hat{R}t}(\hat{\theta}\cos\hat{\theta}t - \hat{B}\sin\hat{\theta}t)\left[-\underline{F}_j\sin\tfrac{3}{2}\Omega t + \underline{K}_j\cos\tfrac{3}{2}\Omega t + \underline{L}_j\sin\tfrac{5}{2}\Omega t - \underline{H}_j\sin\tfrac{3}{2}\Omega t\right]\end{aligned} \tag{57}$$

This solution behaves as a damping oscillator, provided that following conditions hold

$$\hat{R}_j > 0,\ \ \Omega < 2(\omega_j + \hat{Q}_j - \hat{B}_j) \ \text{ and } \Omega > 2(\omega_j + \hat{Q}_j + \hat{B}_j) \tag{58}$$

where the relation (48) has been employed with (54).

4.3 The Combination of the Super-Harmonic Resonance in Case Of Ω Near $(\omega_1 \pm \omega_2)/2$

The following calculation considers only the positive sign of ω_2. Obviously, in case of the negative sign, it may be obtained for replacing the sign of ω_2 in the final results. We express the nearness of Ω to $(\omega_1 \pm \omega_2)/2$ by introducing another detuning parameter σ_b such that

$$\Omega = (\omega_1 + \omega_2)/2 + \delta\,\sigma_b \tag{59}$$

Hence, we may write

$$-i(\omega_2 - 2\Omega)t_0 = i\omega_1 t_0 + 2i\sigma_b\tau \tag{60}$$

and

$$-i(\omega_1 - 2\Omega)t_0 = i\omega_2 t_0 + 2i\sigma_b\tau \tag{61}$$

At this end, the secular term that appears in Eq. (21) can be avoided by introducing the following two solvability conditions

$$\left[D + (\hat{R}_j - i\hat{Q}_j)\right] \alpha_j(\tau) - iU_{3-j}\,\bar{\alpha}_{3-j}(\tau)e^{2i\sigma_b\tau} = 0 \tag{62}$$

and

$$\left[D + (\hat{R}_{3-j} + i\hat{Q}_{3-j})\right] \bar{\alpha}_{3-j}(\tau) + iU_j\,\alpha_j(\tau)e^{-2i\sigma_b\tau} = 0 \tag{63}$$

where $D = \partial/\partial\tau$, $U_j = \underline{S}_j\underline{Q}_{3-j}\underline{\pi}_{3-j}$ and $U_{3-j} = \underline{S}_{3-j}\underline{Q}_j\underline{\pi}_j$. These equations admit a non-trivial solution of the following form

$$\alpha_j(\tau) = \left[M_j(\tau) + i\,\Gamma_j(\tau)\right]e^{i\sigma_b\tau} \tag{64}$$

where the functions $M_j(\tau)$and $\Gamma_j(\tau)$ are real ones. By substituting from Eq. (64) into Eq. (62) and (63) and then separating the real and imaginary parts, one obtains the following governing equations

$$(D + \hat{R}_1)M_1(\tau) + (\hat{Q}_1 - \sigma_b)\Gamma_1(\tau) - U_2\,\Gamma_2(\tau) = 0 \tag{65}$$

$$(D + \hat{R}_1)\Gamma_1(\tau) - (\hat{Q}_1 - \sigma_b)M_1(\tau) - U_2 M_2(\tau) = 0 \tag{66}$$

$$(D + \hat{R}_2)M_2(\tau) + (\hat{Q}_2 - \sigma_b)\Gamma_2(\tau) - U_1\Gamma_1(\tau) = 0 \tag{67}$$

and

$$(D+\hat{R}_2)\Gamma_2(\tau)-(\widehat{Q}_2-\sigma_b)M_2(\tau)-U_1M_1(\tau)=0 \tag{68}$$

The above system consists of four homogenous first-order differential equations in four unknown functions. These functions are $M_1(\tau)$,$\Gamma_1(\tau)$, $M_2(\tau)$ and $\Gamma_2(\tau)$. For a non-trivial solution, there exists at least one repeated equation. Therefore, we have at least two depended functions. Suppose that these functions are: $\Gamma_1(\tau)$ and $\Gamma_2(\tau)$, therefore, one finds

$$\Gamma_1(\tau)=\lambda\left(\Theta^2+\hat{R}_1^2\right)\left(\Theta^2+\hat{R}_2^2\right)\sin\Theta\tau \tag{69}$$

and

$$\Gamma_2(\tau)=\left(\Theta^2+\hat{R}_1^2\right)\left(\Theta^2+\hat{R}_2^2\right)\sin\Theta\tau \tag{70}$$

Substituting from Eq. (66) and (67) into Eq. (62) and (64), from the elementary calculus, one finds

$$M_1(\tau)=\left(\lambda(\sigma_b-\widehat{Q}_1)+U_2\right)\left(\Theta^2+\hat{R}_2^2\right)\left(-\Theta\cos\Theta\tau+\hat{R}_1\sin\Theta\tau\right) \tag{71}$$

and

$$M_2(\tau)=\left((\sigma_b-\widehat{Q}_2)+\lambda U_1\right)\left(\Theta^2+\hat{R}_1^2\right)\left(-\Theta\cos\Theta\tau+\hat{R}_2\sin\Theta\tau\right) \tag{72}$$

It should be noted that the functions $M_1(\tau)$ and $M_2(\tau)$ are two independent functions. Now, substituting from Eq. (69) to (72) into Eq. (66) and (68), one finds

$$\begin{aligned}&\left[\lambda-\frac{(\sigma_b-\widehat{Q}_1)(\lambda(\sigma_b-\widehat{Q}_1)+U_2)}{\Theta^2+\hat{R}_1^2}+\frac{U_2(\sigma_b-\widehat{Q}_2+\lambda U_1)}{\Theta^2+\hat{R}_2^2}\right]\Theta\cos\Theta\tau+\\&\left[\lambda\hat{R}_1+\frac{\hat{R}_1(\sigma_b-\widehat{Q}_1)(\lambda(\sigma_b-\widehat{Q}_1)+U_2)}{\Theta^2+\hat{R}_1^2}-\frac{U_2\hat{R}_2(\sigma_b-\widehat{Q}_2+\lambda U_1)}{\Theta^2+\hat{R}_2^2}\right]\sin\Theta\tau=0\end{aligned} \tag{73}$$

and

$$\begin{aligned}&\left[1+\frac{U_1(\lambda(\sigma_b-\widehat{Q}_1)+U_2)}{\theta^2+\hat{R}_1^2}-\frac{(\sigma_b-\widehat{Q}_2+\lambda U_1)(\sigma_b-\widehat{Q}_2)}{\Theta^2+\hat{R}_2^2}\right]\Theta\cos\Theta\tau+\\&\left[\hat{R}_2+\frac{\hat{R}_2(\sigma_b-\widehat{Q}_2)(\sigma_b-\widehat{Q}_2+\lambda U_1)}{\Theta^2+\hat{R}_2^2}-\frac{U_1\hat{R}_1(\lambda(\sigma_b-\widehat{Q}_1)+U_2)}{\Theta^2+\hat{R}_1^2}\right]\sin\Theta\tau=0\end{aligned} \tag{74}$$

Since $sin\Theta\tau$ and $cos\Theta\tau$ are two independent functions, then it follows that their coefficient must vanish. This leads to

$$\lambda - \frac{(\sigma_b - \widehat{Q}_1)(\lambda(\sigma_b - \widehat{Q}_1) + U_2)}{\Theta^2 + \hat{R}_1^2} + \frac{U_2(\sigma_b - \widehat{Q}_2 + \lambda U_1)}{\Theta^2 + \hat{R}_2^2} = 0 \tag{75}$$

$$\lambda \hat{R}_1 + \frac{\hat{R}_1(\sigma_b - \widehat{Q}_1)(\lambda(\sigma_b - \widehat{Q}_1) + U_2)}{\theta^2 + \hat{R}_1^2} - \frac{U_2 \hat{R}_2(\sigma_b - \widehat{Q}_2 + \lambda U_1)}{\theta^2 + \hat{R}_2^2} = 0 \tag{76}$$

$$1 + \frac{U_1(\lambda(\sigma_b - \widehat{Q}_1) + U_2)}{\Theta^2 + \hat{R}_1^2} - \frac{(\sigma_b - \widehat{Q}_2)(\sigma_b - \widehat{Q}_2 + \lambda U_1)}{\Theta^2 + \hat{R}_2^2} = 0 \tag{77}$$

and

$$\hat{R}_2 - \frac{U_1 \hat{R}_1(\lambda(\sigma_b - \widehat{Q}_1) + U_2)}{\Theta^2 + \hat{R}_1^2} + \frac{\hat{R}_2(\sigma_b - \widehat{Q}_2)(\sigma_b - \widehat{Q}_2 + \lambda U_1)}{\Theta^2 + \hat{R}_2^2} = 0 \tag{78}$$

The above four equations may be reduced to

$$2\hat{R}_1 \lambda + \frac{U_2(\hat{R}_1 - \hat{R}_2)(\sigma_b - \widehat{Q}_2 + \lambda U_1)}{\Theta^2 + \hat{R}_2^2} = 0 \tag{79}$$

and

$$2\hat{R}_2 + \frac{U_1(\hat{R}_2 - \hat{R}_1)(\lambda(\sigma_b - \widehat{Q}_1) + U_2)}{\Theta^2 + \hat{R}_1^2} = 0 \tag{80}$$

The elimination of the parameter Θ between Eq. (79) and (80) yields

$$\lambda^2 \hat{R}_1 U_1(\sigma_b - \widehat{Q}_1) + \lambda(\hat{R}_1 + \hat{R}_2)\left(U_1 U_2 - 2\hat{R}_1 \hat{R}_2\right) + \hat{R}_2 U_2(\sigma_b - \widehat{Q}_2) = 0 \tag{81}$$

Eq. (81) is a quadratic equation, in λ. On the other side, the elimination of the parameter λ between Eq. (79) and (78) yields

$$\hat{a}_0 \Theta^4 + \hat{b}_0 \Theta^2 + \hat{c}_0 = 0 \tag{82}$$

where

$\hat{a}_0 = 4\hat{R}_1 \hat{R}_2,$

$\hat{b}_0 = \left(4\hat{R}_1 \hat{R}_2(\hat{R}_1^2 + \hat{R}_2^2) - 2(\hat{R}_1 - \hat{R}_2)^2 U_1 U_2\right),$

$\hat{c}_0 = 4\hat{R}_1^3 \hat{R}_2^3 + U_1 U_2(\hat{R}_1 - \hat{R}_2)^2\left((\sigma_b - \widehat{Q}_1)(\sigma_b - \widehat{Q}_2) + 2\hat{R}_1 \hat{R}_2\right) - (\hat{R}_1 - \hat{R}_2)^2 U_1^2 U_2^2$

The stability criterion requires that Θ^2 must be of real and positive. This requires that

$$\hat{b}_0 / \hat{a}_0 < 0, \quad \hat{c}_0 / \hat{a}_0 > 0 \text{ and } \hat{b}_0^2 - 4\hat{a}_0 \hat{c}_0 > 0 \tag{83}$$

At the end, the approximate solution, at the present resonance, may be formulated as

$$\begin{aligned}\underline{Y}(t)=2M_1(t)\ \{\ &\underline{\pi}_1\cos(\Omega+\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{F}_1\cos(2\Omega+\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{K}_1\sin(2\Omega+\tfrac{1}{2}(\omega_1-\omega_2))t\\ &+\underline{G}_1\cos(\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{E}_1\sin(\tfrac{1}{2}(\omega_1-\omega_2))t-\underline{L}_1\cos(3\Omega+\tfrac{1}{2}(\omega_1-\omega_2))t\ \}\\ +2\Gamma_1(t)\ \{&-\underline{\pi}_1\sin(\Omega+\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{K}_1\cos(2\Omega+\tfrac{1}{2}(\omega_1-\omega_2))t-\underline{F}_1\sin(2\Omega+\tfrac{1}{2}(\omega_1-\omega_2))t\\ &+\underline{E}_1\cos(\tfrac{1}{2}(\omega_1-\omega_2))t-\underline{G}_1\sin(\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{L}_1\sin(3\Omega+\tfrac{1}{2}(\omega_1-\omega_2))t\ \}\\ +2M_2(t)\ \{\ &\underline{\pi}_2\cos(\Omega-\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{F}_2\cos(2\Omega-\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{K}_2\sin(2\Omega-\tfrac{1}{2}(\omega_1-\omega_2))t\\ &+\underline{G}_2\cos(\tfrac{1}{2}(\omega_1-\omega_2))t-\underline{E}_2\sin(\tfrac{1}{2}(\omega_1-\omega_2))t-\underline{L}_2\cos(3\Omega-\tfrac{1}{2}(\omega_1-\omega_2))t\ \}\\ +2\Gamma_2(t)\ \{&-\underline{\pi}_2\sin(\Omega-\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{K}_2\cos(2\Omega-\tfrac{1}{2}(\omega_1-\omega_2))t-\underline{F}_2\sin(2\Omega-\tfrac{1}{2}(\omega_1-\omega_2))t\\ &+\underline{E}_2\cos(\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{G}_2\sin(\tfrac{1}{2}(\omega_1-\omega_2))t+\underline{L}_2\sin(3\Omega-\tfrac{1}{2}(\omega_1-\omega_2))t\ \}\end{aligned} \tag{84}$$

where the functions $M_j(\tau)$and $\Gamma_j(\tau)$ are the final form of Eq. (69) to (72). The final resonance case will be presented in the next subsection.

4.4 The Combination of the Harmonic Resonance Case When Ω Near $(\omega_1 \pm \omega_2)$

This is the final resonance's case. As before, we will consider only the positive sign. To do this, we express the nearness of Ω to $(\omega_1 \pm \omega_2)$along a detuning parameter σ_c, such that

$$\Omega=\omega_1+\omega_2+2\delta\,\sigma_c \tag{85}$$

hence, we may write

$$-i(\omega_2-\Omega)t_0=i\omega_1 t_0+2i\sigma_c\tau \tag{86}$$

and

$$-i(\omega_1-\Omega)t_0=i\omega_2 t_0+2i\sigma_c\tau \tag{87}$$

On using Eq. (86) and (87) in the right-hand side of Eq. (21), the secular term will be rising again. To avoid this secular term, we must have the following solvability conditions

$$[D+(\hat{R}_j-i\hat{Q}_j)]\alpha_j+(-u_{3-j}+i\,v_{3-j})\overline{\alpha}_{3-j}\,e^{2i\sigma_c\tau}=0 \tag{88}$$

and

$$[D+(\hat{R}_j+i\hat{Q}_j)]\overline{\alpha}_j+(-u_{3-j}-i\,v_{3-j})\alpha_{3-j}\,e^{-2i\sigma_c\tau}=0 \tag{89}$$

where $u_{3-j}=\dfrac{-(\omega_{3-j}-\Omega)}{2\omega_j}\underline{S}_j\underline{B}\,\underline{\pi}_{3-j}$ and $v_{3-j}=\dfrac{(\omega_{3-j}-\Omega)}{2\omega_j}\underline{S}_j\left(\omega_{3-j}\underline{P}+\underline{C}\right)\underline{\pi}_{3-j}$. Now, we may consider

$$\alpha_j(\tau)=\left(\varpi_j(\tau)+i\,\chi_j(\tau)\right)e^{i\sigma_c\tau} \tag{90}$$

and

$$\bar{\alpha}_j(\tau) = \left(\varpi_j(\tau) - i\,\chi_j(\tau)\right)e^{-i\sigma_c \tau} \tag{91}$$

On substituting from Eq. (90) and (91) into Eq. (88) and (89), and then separating the real and imaginary parts, one gets

$$(D + R_1)\varpi_1 + (\hat{Q}_1 - \sigma_c)\chi_1 - u_2\,\varpi_2 + v_2\,\chi_2 = 0 \tag{92}$$

$$(D + R_1)\chi_1 - (\hat{Q}_1 - \sigma_c)\varpi_1 + v_2\,\varpi_2 + u_2\,\chi_2 = 0 \tag{93}$$

$$(D + R_2)\varpi_2 + (\hat{Q}_2 - \sigma_c)\chi_2 - u_1\,\varpi_1 + v_1\,\chi_1 = 0 \tag{94}$$

and

$$(D + R_2)\chi_2 - (\hat{Q}_2 - \sigma_c)\varpi_2 + v_1\,\varpi_1 + u_1\,\chi_1 = 0 \tag{95}$$

The above system consists of four first-order homogeneous differential equations in the four unknown functions. These functions are $\bar{\omega}_1(\tau), X_1(\tau), \bar{\omega}_2(\tau)$ and $X_2(\tau)$. For a non-trivial solution, there exists at least one repeated equation. So, we must have at least two depended functions. From this point of view, one may proceed as in the previous section to get the following solutions

$$\chi_1(\tau) = \left(u_2^2 + v_2^2\right)\sin\ \varphi\tau \tag{96}$$

$$\varpi_1(\tau) = \left(u_2^2 + v_2^2\right)\varphi\cos\varphi\tau \tag{97}$$

$$\chi_2(\tau) = \sin\varphi\tau\,\left[(\varphi^2 + \sigma_c - \hat{Q}_1)v_2 - \hat{R}_1 u_2\right] - \varphi\cos\varphi\tau\,\left[(\sigma_c - \hat{Q}_1 + 1)u_2 + \hat{R}_1 v_2\right] \tag{98}$$

and

$$\varpi_2(\tau) = \sin\varphi\tau\left[-(\varphi^2 + \sigma_c - \hat{Q}_1)u_2 - \hat{R}_1 v_2\right] + \varphi\cos\varphi\tau\left[-(\sigma_c - \hat{Q}_1 + 1)v_2 + \hat{R}_1 u_2\right] \tag{99}$$

where the constant argument φ is given by

$$\hat{a}_1\varphi^4 + \hat{b}_1\varphi^2 + \hat{c}_1 = 0 \tag{100}$$

where, the coefficients are listed as in the following

$$\hat{a}_1 = (1 - \hat{Q}_1 + \hat{Q}_2)$$

$$\hat{b}_1 = \Big[-2\sigma_c^2 + (1+\sigma_c)\hat{Q}_2^2 + \hat{Q}_1^2(1-\sigma_c+\hat{Q}_2) + \hat{R}_1^2 + \hat{R}_2^2 + 2u_1u_2 + v_2\big(u_1(\hat{R}_1+\hat{R}_2)+2v_1\big)$$
$$-(\hat{R}_1+\hat{R}_2)\big(\sigma_c(\hat{R}_1-\hat{R}_2)+u_2v_1\big) + \hat{Q}_1\big(-1+\sigma_c(2+\sigma_c) - \hat{Q}_2(4+\hat{Q}_2) - \hat{R}_2^2 - u_1u_2 - v_1v_2\big)$$
$$+\hat{Q}_2\big(1-\sigma_c(-2+\sigma_c)+\hat{R}_1^2+u_1u_2+v_1v_2\big)\Big],$$

and

$$\hat{c}_1 = \Big[\sigma_c^4 - \sigma_c\hat{R}_1^2 + \sigma_c^2\hat{R}_1^2 + \hat{Q}_2^2(\sigma_c+\sigma_c^2+\hat{R}_1^2) + \sigma_c\hat{R}_2^2 + \sigma_c^2\hat{R}_2^2 + \hat{R}_1^2\hat{R}_2^2 - 2\sigma_c^2u_1u_2 - 2\hat{R}_1\hat{R}_2u_1u_2 + u_1^2u_2^2 +$$
$$Q_1^2\big((-1+\sigma_c)\sigma_c + \hat{Q}_2(1-2\sigma_c+\hat{Q}_2)+\hat{R}_2^2\big) + (2\sigma_c-1)\hat{R}_1u_2v_1 - (2\sigma_c+1)\hat{R}_2u_2v_1 + u_2^2v_1^2 -$$
$$\hat{Q}_1\big((2\sigma_c+1)(\hat{Q}_2^2+\hat{R}_2^2) + (2\sigma_c-1)(\sigma_c^2-u_1u_2) + \hat{Q}_2(-4\sigma_c^2+2u_1u_2) - 2\hat{R}_2u_2v_1\big) -$$
$$\hat{Q}_1v_2\big(2\hat{R}_2u_1 + (1-2\sigma_c+2\hat{Q}_2)v_1\big) + v_2\big(u_1\big((1-2\sigma_c)\hat{R}_1 + (2\sigma_c+1)\hat{R}_2\big) - 2v_1(\sigma_c^2+\hat{R}_1\hat{R}_2)\big) +$$
$$(u_1^2+v_1^2)v_2^2 + \hat{Q}_2\big((1-2\sigma_c)\hat{R}_1^2 + 2\hat{R}_1(u_1v_2-u_2v_1) - (2\sigma_c+1)(\sigma_c^2-u_1u_2-v_1v_2)\big)\Big].$$

The stability criteria of the transition curve as given by Eq. (100) are the same as that given before by the conditions (83). The final approximate solution, at this resonance case, is formulated as

$$\begin{aligned}
\underline{Y}(t) = 2\varpi_1(t)[&\underline{\pi}_1\cos(\tfrac{1}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t + \underline{F}_1\cos(\tfrac{3}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t + \underline{K}_1\sin(\tfrac{3}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t \\
&-\underline{L}_j\cos(\tfrac{5}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t - \underline{H}_j\cos(\tfrac{3}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t] + \\
2\chi_1(t)[&-\underline{\pi}_1\sin(\tfrac{1}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t - \underline{F}_1\sin(\tfrac{3}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t + \underline{K}_1\cos(\tfrac{3}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t \\
&+\underline{L}_1\sin(\tfrac{5}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t - \underline{H}_1\sin(\tfrac{3}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t] + \\
2\varpi_2(t)[&\underline{\pi}_2\cos(\tfrac{1}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t + \underline{F}_2\cos(\tfrac{3}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t + \underline{K}_2\sin(\tfrac{3}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t \\
&-\underline{L}_2\cos(\tfrac{5}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t - \underline{H}_2\cos(\tfrac{3}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t] + \\
2\chi_2(t)[&-\underline{\pi}_2\sin(\tfrac{1}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t - \underline{F}_2\sin(\tfrac{3}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t + \underline{K}_2\cos(\tfrac{3}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t \\
&+\underline{L}_j\sin(\tfrac{5}{2}\Omega-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)t - \underline{H}_j\sin(\tfrac{3}{2}\Omega+\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_2)t]
\end{aligned} \tag{101}$$

Finally, the theoretical approach has been completed. The remaining part of this paper is concerned with the numerical calculations of some of the transition curves as well as some of the perturbed solutions.

5. Results

This section is devoted to depicting a set of figures to illustrate the stability picture. Therefore, the influence of some parameters of the problem at hand with the stability diagram is plotted. As given in the previous analysis, and in contrast to the mechanisms of the symmetric and antisymmetric modes of perturbations, the present study has adapted a general case of the amplitudes of the interface surface waves. Therefore, in accordance with the non-resonance as well as the resonance cases, some of the perturbed solutions of the surface waves between the three fluids will be graphed.

Before dealing with the numerical calculations, it is convenient to write the coefficients of coupled Ince's equations that are given in Eq. (4) in an appropriate non-dimension form. This can be done in several ways depending mainly on the choice of the characteristics. For this purpose, consider that the parameters: $h, \sqrt{h/g}$ and $\eta_2 h, \sqrt{h/g}$ refer to the characteristic length, time and mass, respectively.

In what follows, we shall make a numerical estimation on the stability picture of the surface waves that propagate between the three fluids throughout the porous media. The numerical calculations show that it is better to compute $Log\Omega$ versus the wave number k. These calculations are made for the non-resonance as well as the resonance cases. As it is previously shown, the stability condition for the non-resonance case, and in all resonance cases requires $\hat{R}_j > 0$. The given sample of choosing data indicated that this condition is automatically satisfied. This means that the stability of the system is automatically satisfied only through the non-resonant case. In the following figures, the stable regions are characterized by the letter S, meanwhile, the letter U stands for the unstable ones.

Now, the numerical calculations are made for the perturbed solutions where the matrix solution $\underline{Y}(t)$ is plotted versus the independent time t for choosing sample systems throughout the following curves as follows

Figure 2 is depicted for the system having the following particulars

$$a = 2.5, \rho_1 = 0.05, \rho_2 = 0.2, \rho_3 = 1.5, \zeta_1 = 0.4, \zeta_2 = 0.05, \zeta_3 = 0.01, \alpha = 1, V_1 = 0.3, V_2 = 10, V_3 = 7, \mu_1 = 0.7, \mu_2 = 0.8,$$
$$\mu_3 = 0.5, \eta_1 = 7, \eta_3 = 3, H_0^{(1)} = 7, H_0^{(2)} = 2, H_0^{(3)} = 5, T_{12} = 0.5, T_{23} = 7, \nu_1 = 0.15, \nu_2 = 0.5, \nu_3 = 1.9, \omega = 0.4, \Omega = 0.1$$

This figure shows the perturbed solution in the non-resonance case that is presented in Eq. (31). The figure plotted for the two-perturbed solutions that are given by the distributions $\gamma_1(t)$ and $\gamma_2(t)$. This figure indicates that the present analysis ignores the symmetric and anti-symmetric perturbations of the surface deflections.

Similar arguments are depicted in Figure 3 for the perturbed solution in the first resonance case as Ω approaches ω_j that is shown in Eq. (45). Simultaneously, similar arguments are drawn in Figure 4, for the perturbed solution in the second resonance case as Ω approaches $2\omega_j$ as shown in Eq. (57). In addition, other arguments are plotted in Figure 5, for the perturbed solution in the third resonance case as Ω approaches $(\omega_1 \pm \omega_2)/2$ that shown in Eq. (84).

Figure 6, displays the perturbed solution in the fourth resonance case as Ω approaches $(\omega_1 \pm \omega_2)$, as shown in Eq. (101). Now, it is convenient to graph the influences of some parameters on the perturbed solutions. To do this, in Figure 7, we plot a single solution $\gamma_1(t)$, in the resonance case, where Ω approaches ω_j for various values of the magnetic field intensity $H_0^{(1)}$. As seen, the increase of the values of $H_0^{(1)}$, yields a small amplitude of the resulted wave which shows a stabilizing influence of $H_0^{(1)}$. This mechanism is already obtained in the tangential magnetic field in the absence of the surface currents as given previously by Zelazo, and Melcher [2], and also, by Cowely and Rosensweig [3].

The influence of the porosity ζ_2 is depicted in Figure 8 This figure is graphed for the same resonances as in Figure 7. It is shown that the increase in the parameter ζ_2, yields a large amplitude of the resulted wave which shows a destabilizing influence for this parameter.

The influence of streaming V_2 is depicted in Figure 9 for the same previous resonances as in Figure 8. It is seen that the increasing value of V_2, yields a large amplitude of the resulted wave, which shows a destabilizing influence of V_2.

Figure 10 is plotted a single solution in the resonance case, where Ω approaches $2\omega_j$ for a single solution $\gamma_1(t)$ for various values of the magnetic permeability parameter μ_2. As stated before, it is found that this parameter has a stabilizing influence. The influence porosity parameter ζ_3 in the previous resonance, as given in Figure 10, is depicted in Figure 11 for a single solution $\gamma_2(t)$. It is found that this parameter has a stabilizing effect. Figure 12 plots a single solution in the resonance case where Ω approaches $(\omega_1 \pm \omega_2)/2$ for a single solution, $\gamma_2(t)$ for various values of the magnetic field intensity parameter $H_0^{(2)}$. As seen before, the parameter $H_0^{(2)}$ has a stabilizing influence.

It is more convenient to graph the transition curves in the resonance cases. For simplicity, the two resonances as Ω approaches ω_j and Ω approaches $2\omega_j$ are only depicted. These two transition curves are given in Eq. (47) and (58).

Figure 13 is plotted to indicate these two transition curves. As well-known by Nayfeh [18] and others, the region between the two curves is an unstable region while the region outside the two curves is a stable one.

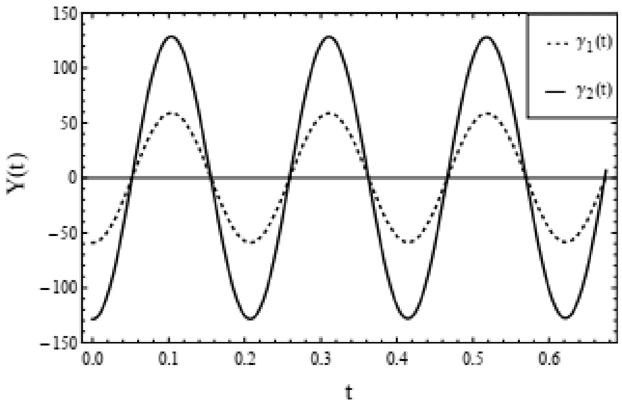


Fig. 2. The perturbed solutions as given in Eq. (31), for the non-resonance case

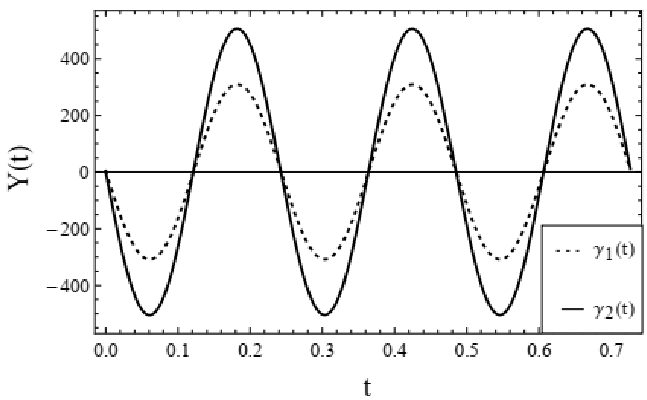


Fig. 3. The perturbed solutions as given in Eq. (45) for the first resonance case as Ω approaches ω_j

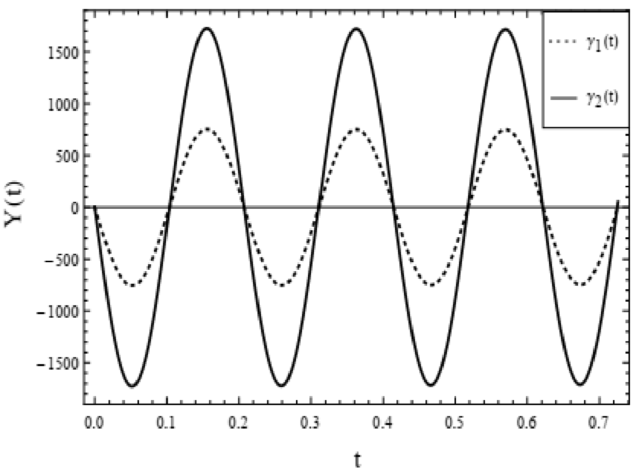


Fig. 4. The perturbed solutions as given in Eq. (57) for the second resonance case as Ω approaches $2\omega_j$

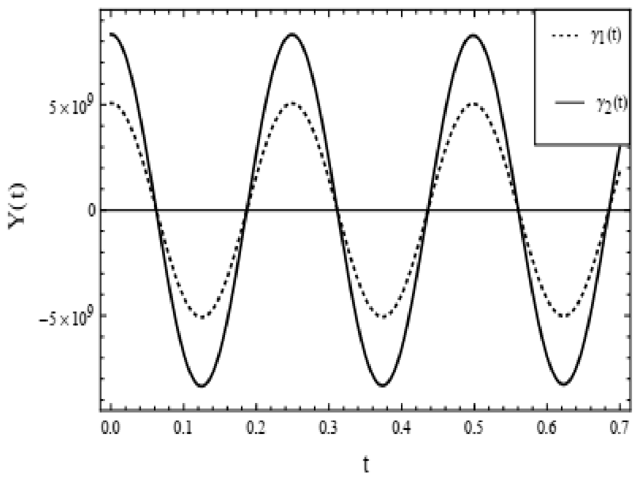
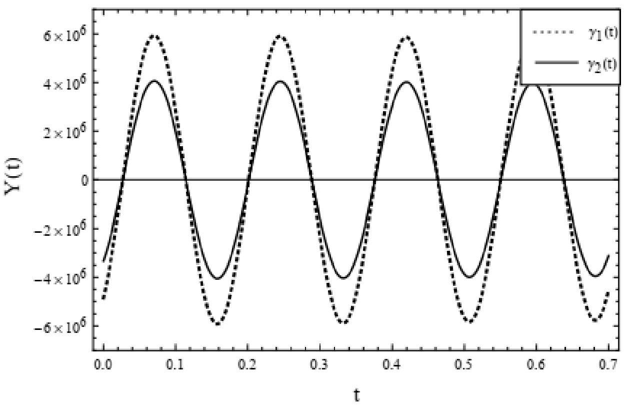


Fig. 5. The perturbed solutions as given in Eq. (84) for the third resonance case as Ω approaches $(\omega_1 \pm \omega_2)/2$



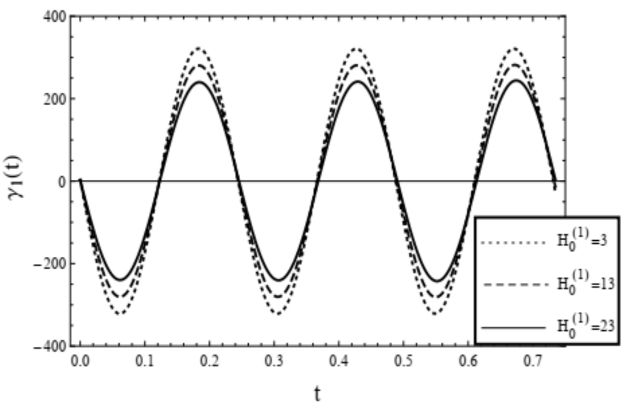


Fig. 6. The perturbed solutions as given in Eq. (101) for the fourth resonance case as Ω approaches $(\omega_1 \pm \omega_2)$

Fig. 7. The variation of the magnetic field intensity $H_0^{(1)}$ in the resonance case, where Ω approaches ω_j for a single solution, $\gamma_1(t)$, as given in Eq. (45)

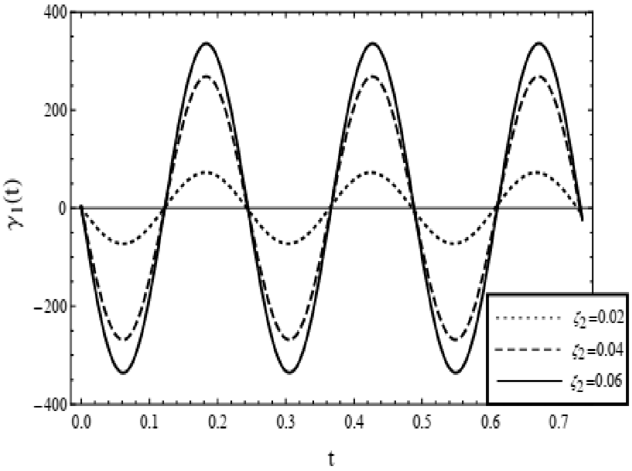


Fig. 8. The variation of the porosity ζ_2 in the resonance case, where Ω approaches ω_j for a single solution, $\gamma_1(t)$, as given in Eq. (45)

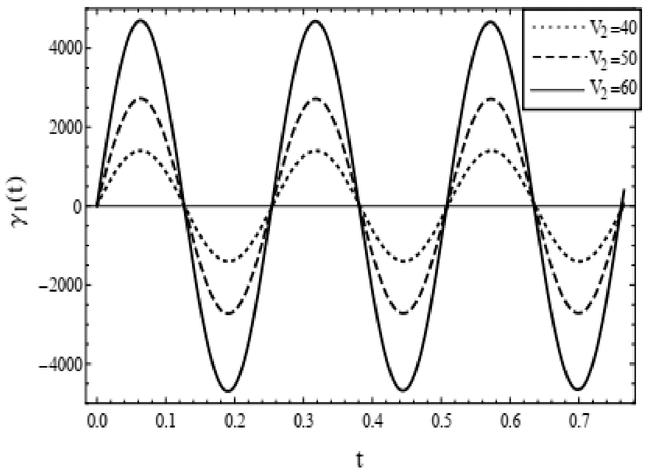


Fig. 9. The variation of the streaming V_2 in the resonance case, where Ω approaches ω_j for a single solution, $\gamma_1(t)$, as given in Eq. (45)

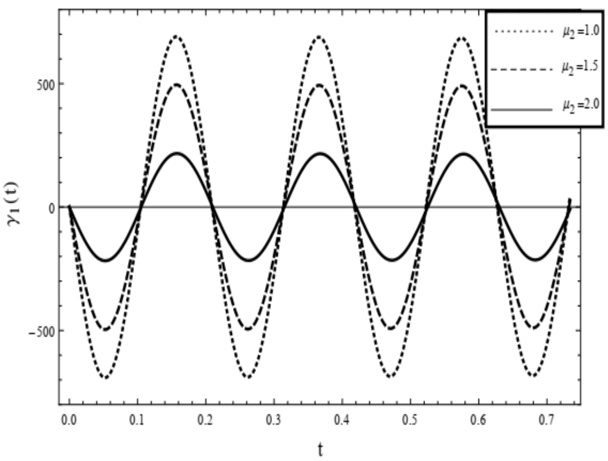


Fig. 10. The magnetic permeability parameter μ_2 in the resonance case, where Ω approaches $2\omega_j$for a single solution, $\gamma_1(t)$,as given in Eq. (57)

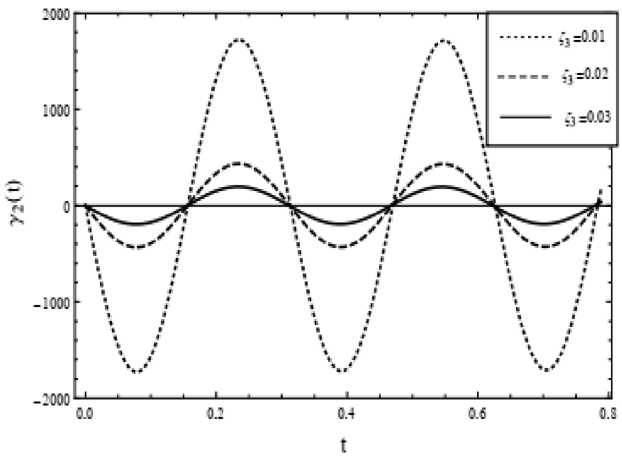
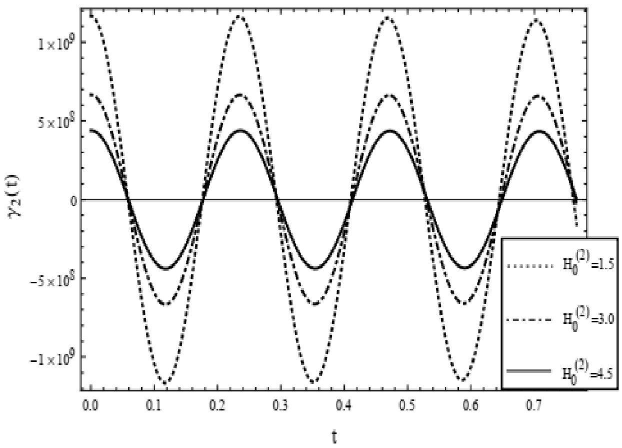


Fig. 11. The porosity parameter ζ_3 in the resonance case, where Ω approaches $2\omega_j$ for a single solution, $\gamma_2(t)$, as given in Eq.(57)



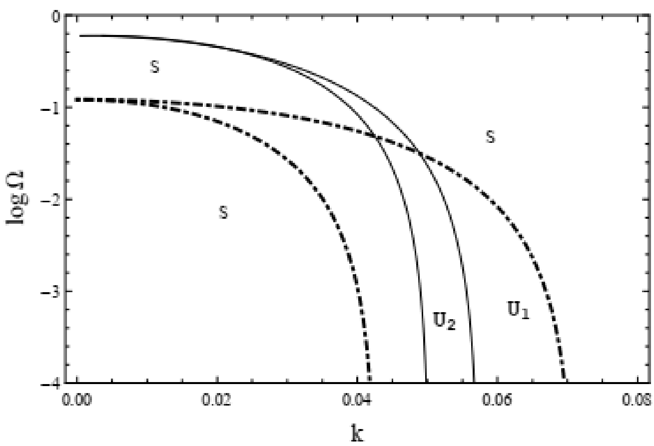


Fig. 12. The magnetic field intensity parameter $H_0^{(2)}$ in the resonance case, where Ω approaches $(\omega_1 \pm \omega_2)/2$ for a single solution, $\gamma_2(t)$as given in Eq. (84)

Fig. 13. The two transition curves are given in Eq. (47) and (58)

6. Conclusions

The present study investigates the stability problem of a horizontal infinite magnetic liquid sheet and embedded between two different magnetic fluids. The three magnetic fluids are acted upon by a uniform tangential magnetic field. Simultaneously, the system is influenced by a horizontal periodic basic streaming. Because of instability in porous media of a plane interface between three fluids, may be of great interest in geophysics and bio-mechanics therefore, the three fluids are saturated throughout porous media where the porosity parameters are, also, included. In accordance with the wide applications of the viscous forces, the current study considers these forces. The viscous potential theory is adopted to relax the mathematical manipulation of the analysis. Therefore, the viscosity enters, only, through the normal stress conditions. Meanwhile, the fluids are considered as the perfect ones elsewhere. Away from the special cases of the symmetric and antisymmetric modes of perturbations, the current work provides a general form of the surface wave deflections.

The boundary-value problem leads to two coupled Ince's equations. These equations are presented throughout an approach of the matrices. The aim of this work is focused on the stability analysis of the Ince's equations. Therefore, to avoid the unjustified length of the paper, all the appendices are removed, but they are available under the request of the reader. In this direction, the purpose of this work is focused on the Homotopy perturbation and the multiple-time scales techniques. Therefore, the analysis does not need a presence of any small parameter in the governing equations. For simplicity, the perturbed solutions are obtained up to the first-order.

The stability analysis reveals the resonance as well as the non-resonance cases, along the following concluding remarks

i. The zero-order perturbed solution is presented in Eq. (30).
ii. The theoretical calculations showed that the stability conditions in the non-resonant case $\hat{R}_j > 0$ is independent of the external frequency Ω. In accordance of the chosen sample, the numerical calculations reveal that this condition is automatically satisfied.
iii. The harmonic resonance as Ω approaches ω_j, yields the transition curves that are represented in Eq. (47). The perturbed solution in this case is formulated in Eq. (45).
iv. The harmonic resonance as Ω approaches $2\omega_j$, yields the transition curves that are represented in Eq. (58). The perturbed solution in this case is formulated in Eq. (57).
v. Again, the combination super-harmonic resonance as Ω approaches $(\omega_1 \pm \omega_2)/2$ resulting in the perturbed solution in this case is formulated in Eq. (84), and it is the same for the transition curves that are represented in Eq. (83)
vi. The last harmonic resonance as Ω approaches $(\omega_1 \pm \omega_2)$ giving the perturbed solution in this case is formulated in Eq. (100).

Finally, numerical calculations are made for some resonance as well as some perturbed solutions. The influences of some parameters are indicated numerically. For instance, the parameter $\mu_2, \zeta_3, H_0^{(1)}$ and $H_0^{(2)}$ have a stabilizing effect. In contrast, the parameter ζ_2 and V_2 have a destabilizing influence.

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