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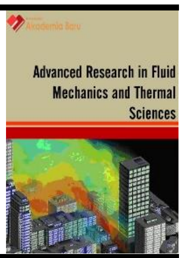

New Analytical Solution Formula for Heat Transfer of Unsteady Two-Dimensional Squeezing Flow of a Casson Fluid between Parallel Circular Plates

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ABSTRACT

In this paper, the effect of physical parameters on the velocity and temperature distributions for heat transfer of unsteady two-dimensional squeezing flow of a Casson fluid between parallel circular plates is studied. Thrive similarity transforms is used to reduce the equations of problem into highly nonlinear ordinary differential equations. The resulting equations are solved by a new analytical technique and obtained new analytical approximate solution. This new analytical technique essentially depends on the coefficients of powers series that result from integration nth order of a differential equation. Fourth order Runge-Kutta method is also using to obtain numerical solution. The influence of involved physical parameters on the velocity and temperature distributions is discussed with the help of tables and graphics. Also, as novel idea in this work, some theorems to prove the convergence of a new analytical technique theoretically, and the verifications of these theorems computationally are introduced. The results of new analytical approximate technique are verified an excellent agreement by comparing it with Runge-Kutta method and homotopy perturbation method (HPM).

Keywords:
Squeezing flow; parallel plates; analytical approximate solution; heat transfer; converge analysis

1. Introduction

Heat transfer occurs in many physical situation. Moving objects are heated not only because of some external sources but their movement against other surfaces may also produce hypothermia of great importance. Mechanical systems consisting of fast moving pistons or parts can be sustained by proper understanding of heat transfer occurring in these systems. For proper operation of these machines lubricants are used to reduce friction between the parts. Rheology is a branch of physics in which we study how material is distorted or flowing response to applied forces or pressures. The properties of the substances that control the specific way in which these deformations or flow behaviors are called rheological requirements. The rheological properties of these lubricants under

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the thermal conditions for assembling highly efficient mechanical heat transfer analysis are very important. The squeezing flow between circular plates moves orthogonal in many practical situations. Special applications have in polymer processing, modeling of synthetics transportation inside living bodies, hydro-mechanical machinery and compression injection processes. Many researches have considered these flows and contributed to their effort to make these typesof flow better. Stefan's [1] pioneering efforts have opened new doors for researchers, and many studies have been carried out following him. As an example, Homotopy perturbation solution for two dimensional MHD squeezing flow between parallel plates has been determined by Siddiqui *et al.,* [2]. Domairry and Aziz [3] investigated the same problem for the flow between parallel disks. Jafari *et al.,* [4] studied analytically the heat and mass transfer in a viscous fluid which squeeze between parallel plates by using homotopy perturbation method. Syad *et al.,* [5] proposed and applied for semi-analytic solution of heat transfer analysis for the squeezing flow of a Casson fluid between parallel circular plates by using differential transform method. Mustafa *et al.,* [6] tested heat and mass transfer for squeezing flow between parallel plates using homotopy analysis method (HAM). Khan *et al.,* [7] showed that heat transfer analysis for unsteady squeezing flow of a Casson fluid between parallel plates by using homotopy perturbation method (HPM). Rashidi *et al.,* [8] used a homotopy simulation of nanofluid dynamics from a nonlinearly stretching isothermal permeable sheet with transpiration. The homotopy analysis of transient magneto-bio-fluid dynamics of micropolar squeeze -film in a porous medium was studied by Anwar *et al.,* [9]. Rashidi and Erfani [10] an analytical method for solving steady MHD convective and slip flow due to a rotating disk with viscous dissipation and ohmic heating. Recently, Al-saif and Harfash [11] used similarity transformation to transform the problem of unsteady squeezing flow between parallel plates into ordinary differential equations and then solved it to obtain analytical approximate solutions by perturbation-iteration algorithm. All above contributions for authors focus on analytical approximate methods. These methods are alternative to numerical method and are useful for finding analytical approximate solutions of highly non-linear differential equations. This encouraged us to adopted the new analytical method that is proposed in [12, 13] to solve the squeezing flow between two parallel plats and give good results with less computational workload compared with other methods. Moreover, to the best of our simple knowledge, a comparative study on heat transfer unsteady two-dimensional squeezing flow of a Casson fluid between parallel circular plates using a new analytical technique has not been implemented in literature. Therefore, in this paper, a comparative study and analytical investigations are presented using new proposed technique that is to consider modified integral method and presented in our previous works [12, 13], depending on its structure on power series coefficients that are resulting from integration processes. The new analytical technique has been successfully applied to find analytical approximate solution. Also, the effect of physical parameters on the velocity and temperature distributions for heat transfer of unsteady two-dimensional squeezing flow of a Casson fluid between parallel circular plates is discussed and established. From current work, one can see that the obtained analytical approximate solution shows excellent compatibility with numerical solution obtained by Runge-Kutta fourth order and other methods in literatures. The organization of this paper is as follows: The governing equations are derived in section (2). Details of derivation of the new analytical technique have been written as steps in the section(3). The performance of the new analytical technique for the squeezing flow is applied in section (4). In section (5) the convergence analysis is presented. Results and discussions are given in section (6). Finally, the conclusions are indicated in section (7).

2. Mathematical Formulation

The governing equations for the heat transfer in an the unsteady two-dimensioned incompressible flow of a Casson fluid between two parallel plates can be expressed :

$$\frac{\partial \hat{u}}{\partial \hat{x}}+\frac{\partial \hat{v}}{\partial \hat{y}}=0, \tag{1}$$

$$\begin{aligned}&\frac{\partial \hat{u}}{\partial t}+\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{u}}{\partial \hat{y}}=-\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{x}}+\upsilon(1+\frac{1}{\beta})(2\frac{\partial^2 \hat{u}}{\partial \hat{x}^2}+\frac{\partial^2 \hat{u}}{\partial \hat{y}^2}+\frac{\partial^2 \hat{v}}{\partial \hat{x}\partial \hat{x}}),\\ &\frac{\partial \hat{v}}{\partial t}+\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{v}}{\partial \hat{y}}=-\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{y}}+\upsilon(1+\frac{1}{\beta})(2\frac{\partial^2 \hat{v}}{\partial \hat{x}^2}+\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}+\frac{\partial^2 \hat{u}}{\partial \hat{x}\partial \hat{x}}),\end{aligned} \tag{2}$$

$$\begin{aligned}&\frac{\partial T}{\partial t}+\hat{u}\frac{\partial T}{\partial \hat{x}}+\hat{v}\frac{\partial T}{\partial \hat{y}}=\frac{k}{\rho C_p}(\frac{\partial^2 T}{\partial \hat{x}^2}+\frac{\partial^2 T}{\partial \hat{y}^2})+\frac{\upsilon}{C_p}(1+\frac{1}{\beta})\\ &(2(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2})^2+(\frac{\partial^2 \hat{u}}{\partial \hat{y}^2}+\frac{\partial^2 \hat{v}}{\partial \hat{x}^2})^2+2(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2})^2)\end{aligned} \tag{3}$$

Boundary conditions for the flow problem are

$$\begin{aligned}&\hat{u}=0,\quad \hat{v}=v_w=\frac{dh}{dt},\quad T=T_H,\quad at\quad \hat{y}=h(t),\\ &\frac{\partial \hat{u}}{\partial \hat{y}},\quad \frac{\partial T}{\partial \hat{y}},\quad \hat{v}=0,\quad at\quad \hat{y}=0,\end{aligned} \tag{4}$$

In the above equation $\hat{u}$ and $\hat{v}$ are the velocity components in $\hat{x}$ and $\hat{y}$ -direction respectively, $\hat{p}$ is the pressure, T is the temperature, υ is the kinematic viscosity of the fluid, ρ is density of the fluid, C_p is the specific heat and k is thermal conductivity of the fluid and $\beta=\mu_B\frac{\sqrt{2\pi}}{p_y}$ is the Casson fluid parameter. Where μ_B is the dynamic viscosity of the non-Newtonian fluid, p_y is yield a stress of fluid and π is the product of the component of deformation rate with itself , i-e. $\pi=e_{ij}e_{ij}$, where eij is the (i, j) the component of the deformation rate. Rheological equation of Casson fluid is defined as under [14-17].

$$\tau_{ij}=[\mu_B+(\frac{p_y}{2\pi})]2e_{ij}, \tag{4}$$

The effects of viscosity dissipation are maintained for the study of heat generation due to the friction caused by shear flow. The viscosity of the fluid is taken as constant, and it does not depend on temperature. Distance $y=\pm l(1-\alpha t)^{1/2}=\pm h(t)$ apart, where l is the initial position (at $t=0$). Further, $\alpha>0$ corresponds to squeezing motion of both plats until they touch each other at $t=\frac{1}{\alpha}$, for $\alpha < 0$ the plates leave each other and dilate. The flow geometry and the coordinate system are shown in Figure 1.

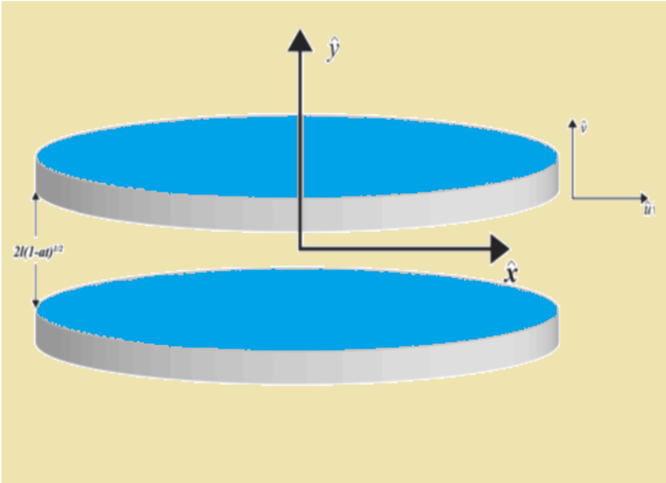


Fig. 1. Schematic diagram for the flow problem

We can simplify this system of equation by eliminating pressure terms from Eq. (2)-(4). After cross differentiation and introducing vorticity Ω, we get

$$\frac{\partial\omega}{\partial t}+\hat{u}\frac{\partial\omega}{\partial\hat{x}}+\hat{v}\frac{\partial\omega}{\partial\hat{y}}=\upsilon(1+\frac{1}{\beta})(\frac{\partial^2\omega}{\partial\hat{x}^2})+(\frac{\partial^2\omega}{\partial\hat{y}^2}), \quad (1)$$

where,

$$\omega=(\frac{\partial\hat{v}}{\partial\hat{x}}+\frac{\partial\hat{u}}{\partial\hat{y}}), \quad (2)$$

By using similarty transformation [18,19]

$$\hat{u}=\frac{\alpha\hat{x}}{[2(1-\alpha t)]}f'(\eta), \quad (3)$$

$$\hat{v}=\frac{-\alpha l}{[2(1-\alpha t)^{1/2}]}f(\eta), \quad (4)$$

$$\theta=\frac{T}{T_H}, \quad (5)$$

$$\eta=\frac{\hat{y}}{[l(1-\alpha)^{1/2}]}. \quad (6)$$

Substituting Eq. (8)-(11) in Eq. (2) and Eq. (3) , we obtain a nonlinear ordinary differential equations for Casson fluid flow as

$$(1+\frac{1}{\beta})f^{iv}-S(\eta f'''(\eta)+3f'(\eta)+f'(\eta)f''(\eta)-f(\eta)f'''(\eta))=0,$$

$$\theta''(\eta)+P_rS(f(\eta)\theta'(\eta)-\eta\theta'(\eta))+P_rE_c(1+\frac{1}{\beta})((f''(\eta))^2+4\delta^2(f'(\eta^2))=0, \tag{7}$$

The boundary conditions of Eq. (4) are reduced to

$$f(0)=0,\quad f''(0)=0,\quad f(1)=1,\quad f'(1)=0, \tag{8}$$

$$\theta'(0)=0,\quad \theta(1)=1, \tag{9}$$

where S denotes the non-dimensioned squeeze number. P_r is Prandtl number, E_c is Eckert number and δ is the dimensionless length. Which are defined as

$$S=\frac{\alpha l^2}{2\upsilon},\qquad P_r=\frac{\mu C_p}{k},\qquad E_c=\frac{1}{C_p}(\frac{\alpha\hat{x}}{2(1-\alpha t)})^2,\qquad \delta=\frac{1}{\hat{x}},$$

It should be noted that squeezing number S describes the movement of the plates $S>0$ corresponding to the plates moving apart, while $S<0$ corresponds to collapsing movement of the plates. Also $E_c=0$ corresponds to the case when viscous dissipation effects are neglected. Physical quantities of interest are skin friction coefficient and Nusselt number defined as

$$\frac{l^2}{\hat{x}^2(1-\alpha t)}Re_xC_f=(1+\frac{1}{\beta})f''(1), \tag{10}$$

$$\sqrt{(1-\alpha t)}Nu=-\theta'(\eta), \tag{11}$$

where,

$$C_f=\upsilon(1+\frac{1}{\beta})\frac{(\frac{\partial\hat{u}}{\partial t})_{\hat{y}=h(t)}}{v_w^2}, \tag{12}$$

$$Re_x=\frac{2lv_w^2}{v\hat{x}(1-\alpha t)^{1/2}}, \tag{13}$$

$$Nu=\frac{-lk(\frac{\partial T}{\partial\hat{y}})_{\hat{y}=h(t)}}{kT_H}, \tag{14}$$

3. The Basic Steps of The New Analytical Technique

This section describes how to obtain a new analytical technique to calculate the coefficients of the power series solution resulting from solving nonlinear ordinary differential equations resulting in using transforms (8)-(11) to find analytical-approximate solution.These coefficients are important bases to construct the solution formula, therefore they can be computed recursively by differentiation ways. To illustrate the computation of these coefficients and the derivation of the new analytical technique, we summarized the new outlook detailed in the following steps.

Step (1): Consider the non-linear ordinary differential equation as follows.

$$f^{(n)}(\eta) = H(f(\eta), f'(\eta), f''(\eta),..., f^{(n-1)}(\eta)), \tag{20}$$

where H is a function of η and the derivatives of $f(\eta)$, $f(\eta)$ is an unknown function and η denotes spatial independent variable. Integrating Eq. (20) n times with respect to η on $[0,\eta]$ yield

$$f(\eta) = f(0) + f'(0)\eta + f''(0)\frac{\eta^2}{2!},... + f^{(n-1)}(0)\frac{\eta^{n-1}}{(n-1)!} + L^{-1}G[f(\eta)], \tag{15}$$

where,

$$G[f(\eta)] = H(f(\eta), f'(\eta), f''(\eta),..., f^{(n-1)}(\eta)), L^{-1} = \int_0^\eta\int_0^\eta ... \int_0^\eta (d\eta)^n, \tag{16}$$

Step (2): Assume that

$$G[f(\eta)] = \sum_{n=1}^{\infty} \frac{d^{n-1}G(f_0(\eta))}{d\eta^{n-1}}, \tag{17}$$

rewriting the Eq. (23)

$$G[f(\eta)] = G[f_0(\eta)] + G'[f_0(\eta)] + G''[f_0(\eta)] + G'''[f_0(\eta)] + G''''[f_0(\eta)] + ..., \tag{18}$$

substituting Eq. (24) in Eq. (21), we obtain

$$f(\eta) = f_0 + f_1 + f_2 + f_3 + f_4 + ..., \tag{19}$$

where,

$$f_0 = f(0) + f'(0)\eta + f''(0)\frac{\eta^2}{2!} ... + f^{(n-1)}(0)\frac{\eta^{(n-1)}}{(n-1)!}, \quad f_1 = L^{-1}G[f_0(\eta)],$$
$$f_2 = L^{-1}G'[f_0(\eta)], f_3 = L^{-1}G''[f_0(\eta)], \quad f_4 = L^{-1}G'''[f_0(\eta)],... \tag{20}$$

Step (3): We focus on computing the derivatives of G with respect to η which is the crucial part of the proposed method. Let start calculating $G[f(\eta)], G'[f(\eta)], G''[f(\eta)], G'''[f(\eta)],....$

$$G[f(\eta)] = H(f(\eta), f'(\eta), f''(\eta), f'''(\eta), f''''(\eta),..., f^{(n-1)}(\eta)), \tag{21}$$

$$G'[f(\eta)] = \frac{dG[f(\eta)]}{d\eta} = G_f.f_\eta + G_{f'}.(f_\eta)' + ... + G_{f^{(n-1)}}.(f_\eta)^{(n-1)}, \tag{22}$$

$$\begin{aligned}G''[f(\eta)] &= \frac{d^2G[f(\eta)]}{d\eta^2} = G_{ff}.(f_\eta)^2 + G_{ff'}.(f_\eta)' f_\eta + G_{ff''}.f_\eta(f_\eta)'' \\ &+ ... + G_{ff^{(n-1)}}.(f_\eta)(f_\eta)^{(n-1)} + G_f.f_{\eta\eta} + G_{ff}.(f_\eta)'.f_\eta + G_{ff'}.(f_\eta)'^2 + ... + \\ &G_{ff^{(n-1)}}.(f_\eta)'(f_\eta)^{(n-1)} + G_{f'}.(f_{\eta\eta})' + G_{f'f}.(f_\eta)''.f_\eta + G_{f'f'}.(f_\eta)'(f_\eta)'' + G_{f'f''}.(f_\eta)''^2 \\ &+ G_{f'f^{(n-1)}}.(f_\eta)''(f_\eta)^{(n-1)} + Gf''.(f_{\eta\eta})'' + ... + G_{f^{(n-1)}f}.(f_\eta)^{(n-1)}.f_\eta + G_{f^{(n-1)}f'}. \\ &(f_\eta)^{(n-1)}.(f_\eta)' + ... + G_{f^{(n-1)}f^{(n-1)}}.(f_\eta)^{(n-1)2} + G_{f^{(n-1)}}.(f_{\eta\eta})^{(n-1)},\end{aligned} \tag{23}$$

$$\begin{aligned}G'''[f(\eta)] &= \frac{d^3G[f(\eta)]}{d\eta^3} = G_{fff}.(f_\eta)^3 + G_{fff'}.(f_\eta)^2(f_\eta)' + ... + G_{fff^{(n-1)}}.(f_\eta)^2. \\ &(f_\eta)^{(n-1)} + G_{ff}.2(f_\eta).f_{\eta\eta} + G_{fff}.(f_\eta)'(f_\eta)^2 + G_{fff'}.(f_\eta)'^2(f_\eta) + ... + G_{fff^{(n-1)}} \\ &.(f_\eta)'(f_\eta).(f_\eta)^{(n-1)} + G_{ff'}.[(f_{\eta\eta})'.f_\eta + (f_\eta)'.f_{\eta\eta}] + G_{ff'f}.(f_\eta)''(f_\eta)^2 + G_{ff'f'}.(f_\eta)'' \\ &(f_\eta).(f_\eta)' + ... + G_{ff'f^{(n-1)}}.(f_\eta)''(f_\eta).(f_\eta)^{(n-1)} + G_{ff''}.[f_{\eta\eta}.(f_\eta)'' + f_\eta.(f_{\eta\eta})''] + ... \\ &+ G_{ff^{(n-1)}f}.(f_\eta)^2.(f_\eta)^{(n-1)} + G_{ff^{(n-1)}f'}.(f_\eta).(f_\eta)'.(f_\eta)^{(n-1)} + ... + G_{ff^{(n-1)}f^{(n-1)}}.(f_\eta) \\ &.(f_\eta)^{(n-1)2} + G_{ff^{(n-1)}}.[(f_{\eta\eta}).(f_\eta)^{(n-1)} + (f_\eta)(f_{\eta\eta})^{(n-1)}] + G_{ff}.f_{\eta\eta}.(f_\eta) + G_{ff'}.f_{\eta\eta}.(f_\eta)' \\ &+ .. + G_{ff^{(n-1)}}.f_{\eta\eta}.(f_\eta)^{(n-1)} + G_\eta.f_{\eta\eta\eta} + G_{fff}.(f_\eta)'(f_\eta)^2 + G_{fff}.(f_\eta)'^2(f_\eta) + ... + \\ &G_{fff^{(n-1)}}.(f_\eta)'(f_\eta).(f_\eta)^{(n-1)} + G_{ff}.[(f_{\eta\eta})'.f_\eta + (f_\eta)'.f_{\eta\eta}] + G_{fff}.(f_\eta)'^2.f_z + G_{fff'} \\ &.(f_\eta)'^3 + ... + G_{fff^{(n-1)}}.(f_\eta)'^2.(f_\eta)^{(n-1)} + G_{ff'}.2(f_\eta)'.(f_{\eta\eta})' + ... + G_{f^{(n-1)}f^{(n-1)}f} \\ &.(f_\eta)^{(n-1)2}.f_\eta + G_{f^{(n-1)}f^{(n-1)}f'}.(f_\eta)^{(n-1)2}.(f_\eta)' + ... + G_{f^{(n-1)}f^{(n-1)}f^{(n-1)}}.(f_\eta)^{(n-1)3} \\ &+ G_{f^{(n-1)}f^{(n-1)}}.2.(f_\eta)^{(n-1)}.(f_{\eta\eta})^{(n-1)} + G_{f^{(n-1)}f}.(f_{\eta\eta})^{(n-1)}.f_\eta + G_{f^{(n-1)}f'}.(f_{\eta\eta})^{(n-1)}.(f_\eta)' \\ &+ ... + G_{f^{(n-1)}f^{(n-1)}}.(f_{\eta\eta})^{(n-1)}.(f_\eta)^{(n-1)} + G_{f^{(n-1)}}.(f_{\eta\eta\eta})^{(n-1)}.\end{aligned} \tag{24}$$

We see that the calculations become more complicated in the second and third derivatives because of the numerous calculations. Consequently, the systematic structure on calculation is extremely important. Fortunately, due to the assumption that the operator G and the solution f are analytic functions, then the mixed derivatives are equivalent. We note that the function derivatives of f unknown, so we suggest the following hypothesis

$$f_\eta = f_1 = L^{-1}G[f_0(\eta)], \quad f_{\eta\eta} = f_2 = L^{-1}G'[f_0(\eta)], \quad f_{\eta\eta\eta} = f_3 = L^{-1}G''[f_0(\eta)],$$

$$f_{\eta\eta\eta\eta} = f_4 = L^{-}1G'''[f_0(\eta)], \quad f_{\eta\eta\eta\eta\eta} = f_5 = L^{-1}G''''[f_0(\eta)],... \tag{25}$$

Therefore, Eq. (27)-(30) are evaluated by

$$G[f_0(\eta)] = H(f_0(\eta), f_0'(\eta), f_0''(\eta), \ldots, f_0^{(n-1)}(\eta)), \tag{26}$$

$$G'[f_0(\eta)] = G_{f_0}.f_1 + G_{f_{0'}}.(f_1)' + G_{f_{0''}}.(f_1)'' + \ldots + G_{f_0^{(n-1)}}.(f_1)^{(n-1)}, \tag{27}$$

$$\begin{aligned} G''[f_0(\eta)] &= G_{f_0 f_0}.(f_1)^2 + G_{f_0 f_{0'}}.(f_1)' f_1 + G_{f_0 f_{0''}}.f_1(f_1)'' + \ldots + G_{f_0 f_0^{(n-1)}}.(f_1) \\ & (f_1)^{(n-1)} + G_{f_0}.f_2 + G_{f_{0'} f_0}.(f_1)'.f(_1) + G_{f_{0'} f_{0'}}.(f_1)'^2 + \ldots + G_{f_{0'} f_0^{(n-1)}}.(f_1)'(f_1)^{(n-1)} \\ & + G_{f_{0'}}.(f_2)' + G_{f_{0''} f_0}.(f_1)''.f_1 + G_{f_{0''} f_{0'}}.(f_1)'(f_1)'' + G_{f_{0''} f_{0''}}.(f_1)''^2 + \ldots + + G_{f_{0''} f_0^{(n-1)}} \\ & .(f_1)''(f_1)^{(n-1)} + G_{f_{0''}}.(f_2)'' + G_{f^{(n-1)} f}.(f_1)^{(n-1)}.f_1 + G_{f_0^{(n-1)} f_{0'}}.(f_1)^{(n-1)}.(f_1)' \\ & + \ldots + G_{f_0^{(n-1)} f_0^{(n-1)}}.(f_1)^{(n-1)2} + + \ldots + G_{f_0^{(n-1)}}.(f_2)^{(n-1)}, \end{aligned} \tag{28}$$

$$\begin{aligned} G'''[f_0(\eta)] &= G_{f_0 f_0 f_0}.(f_1)^3 + G_{f_0 f_0 f_{0'}}.(f_1)^2(f_1)' + \ldots + G_{f_0 f_0 f_0^{(n-1)}}.(f_1)^2. \\ & (f_1)^{(n-1)} + G_{f_0 f_0}.2(f_1).f_2 + G_{f_0 f_{0'} f_0}.(f_1)'(f_1)^2 + G_{f_0 f_{0'} f_{0'}}.(f_1)'^2(f_1) + \ldots + G_{f_0 f_{0'} f_0^{(n-1)}} \\ & .(f_1)'(f_1).(f_1)^{(n-1)} + G_{f_0 f_{0'}}.[(f_2)'.f_z + (f_1)'.f_2] + G_{f_0 f_{0''} f_0}.(f_1)''(f_1)^2 + G_{f_0 f_{0''} f_{0'}}.(f_1)'' \\ & (f_1).(f_1)' + \ldots + G_{f_0 f_{0''} f_0^{(n-1)}}.(f_1)''(f_1).(f_1)^{(n-1)} + G_{f_0 f_{0''}}.[f_2.(f_1)'' + f_1.(f_2)''] + \ldots \\ & + G_{f_0 f_0^{(n-1)} f_0}.(f_1)^2.(f_1)^{(n-1)} + G_{f_0 f_0^{(n-1)} f_{0'}}.(f_1).(f_1)'.(f_1)^{(n-1)} + \ldots + G_{f_0 f_0^{(n-1)} f_0^{(n-1)}}.(f_1) \\ & .(f_1)^{2(n-1)} + G_{f_0 f_0^{(n-1)}}.[(f_2).(f_2)^{(n-1)} + (f_1)(f_2)^{(n-1)}] + G_{f_0 f_0}.f_2.(f_1) + G_{f_0 f_{0'}}.f_2.(f_1)' \\ & + .. + G_{f_0 f_0^{(n-1)}}.f_2.(f_1)^{(n-1)} + G_{f_0}.f_3 + G_{f_{0'} f_0 f_0}.(f_1)'(f_1)^2 + G_{f_{0'} f_{0'} f_0}.(f_1)'^2(f_1) + \ldots + \\ & G_{f_{0'} f_0 f_0^{(n-1)}}.(f_1)'(f_1).(f_1)^{(n-1)} + G_{f_{0'} f_0}.[(f_2)'.f_1 + (f_1)'.f_1] + G_{f_{0'} f_{0'} f_0}.(f_1)'^2.f_1 + G_{f_{0'} f_{0'} f_{0'}} \\ & .(f_1)'^3 + \ldots + G_{f_{0'} f_{0'} f_0^{(n-1)}}.(f_1)'^2.(f_1)^{(n-1)} + G_{f_{0'} f_{0'}}.2(f_1)'.(f_2)' + \ldots + G_{f_0^{(n-1)} f_0^{(n-1)} f_0} \\ & .(f_1)^{(n-1)2}.f_1 + G_{f_0^{(n-1)} f_0^{(n-1)} f_{0'}}.(f_1)^{(n-1)2}.(f_1)' + \ldots + G_{f_0^{(n-1)} f_0^{(n-1)} f_0^{(n-1)}}.(f_1)^{(n-1)3} \\ & + G_{f_0^{(n-1)} f_0^{(n-1)}}.2.(f_1)^{(n-1)}.(f_2)^{(n-1)} + G_{f_0^{(n-1)} f_0}.(f_2^{(n-1)}.f_1 + G_{f_0^{(n-1)} f_{0'}}.(f_2)^{(n-1)}.(f_1)' \\ & + \ldots + G_{f_0^{(n-1)} f_0^{(n-1)}}.(f_2)^{(n-1)}.(f_1)^{(n-1)} + G_{f_0^{(n-1)}}.(f_3)^{(n-1)}, \end{aligned} \tag{29}$$

Step (4): Substituting Eq. (32)-(35) in Eq. (25) we will get the required analytical-approximate solution for the Eq. (20).

4. The Application of the New Analytical Technique To Heat Transfer Analysis Unsteady Squeezing Flow of a Casson Fluid Between Parallel Plates

The new algorithm described in the previous section can be used as a powerful solver to the nonlinear differential Eq. (12)-(14) and to find new analytical-approximate solution. From Step (1) we have

$$f(\eta) = f(0) + f'(0)\eta + f''(0)\frac{\eta^2}{2!} + f'''(0)\frac{\eta^3}{3!} + L_1^{-1}[\frac{\beta S}{1+\beta}(\eta f'''(\eta) + 3f'(\eta)$$
$$+ f'(\eta)f''(\eta) - f(\eta)f'''(\eta))],$$
$$\theta(\eta) = \theta(0) + \theta'(0)\eta + L_2^{-1}[P_r S(f(\eta)\theta'(\eta) - \eta\theta'(\eta)) + P_r E_c(1+\frac{1}{\beta})$$
$$((f''(\eta))^2 + 4\delta^2(f'(\eta^2))], \tag{30}$$

rewrite the Eq. (36) as follows

$$f(\eta) = A_1 + A_2\eta + A_3\frac{\eta^2}{2!} + A_4\frac{\eta^3}{3!} + L^{-1}G_1[f(\eta)],$$
$$\theta(\eta) = B_1 + B_2\eta + L^{-1}G_2[\theta(\eta)], \tag{31}$$

where,

$$A_1 = f(0), \quad A_2 = f'(0), \quad A_3 = f''(0), \quad A_4 = f'''(0),$$
$$B_1 = \theta(0), \quad B_2 = \theta'(0),$$
$$G_1[f] = \frac{\beta S}{1+\beta}(\eta f'''(\eta) + 3f'(\eta) + f'(\eta)f''(\eta) - f(\eta)f'''(\eta)),$$
$$G_2[\theta] = P_r S(f(\eta)\theta'(\eta) - \eta\theta'(\eta)) + P_r E_c(1+\frac{1}{\beta})((f''(\eta))^2 + 4\delta^2(f'(\eta^2)),$$
$$and \quad L_1^{-1}(.) = \int_0^\eta\int_0^\eta\int_0^\eta\int_0^\eta(d\eta)^4, \quad L_2^{-1}(.) = \int_0^\eta\int_0^\eta(d\eta)^2. \tag{32}$$

From the boundary conditions Eq. (37) becomes

$$f(\eta) = A_2\eta + A_4\frac{\eta^3}{3!} + L^{-1}[G_1[f(\eta)],$$
$$\theta(\eta) = B_1 + L^{-1}G_2[\beta(\eta)], \tag{33}$$

From Step (2) we have $f_0 = 1 + A_3\frac{\eta^2}{2!} \quad f_1 = L_1^{-1}G_1[f_0(\eta)], \quad f_2 = L_2^{-1}G_{1'}[f_0(\eta)],...,$

$$f_0 = B_1, \quad \theta_1 = L_3^{-1}G_3[\theta_0(\eta)], \quad \theta_2 = L_3^{-1}G_{3'}[\theta_0(\eta)],..., \tag{34}$$

and the analytical-approximate solutions are

$$f(\eta) = f_0 + f_1 + f_2 + f_3 + ...,$$

$$\theta(\eta)=\theta_0+\theta_1+\theta_2+\theta_3+\ldots, \tag{35}$$

From Step (3) yields

$$G_1[f(\eta)]=\frac{\beta S}{1+\beta}(\eta f'''(\eta)+3f'(\eta)+f'(\eta)f''(\eta)-f(\eta)f'''(\eta)),$$
$$G_2[\theta(\eta)]=P_r S(f(\eta)\theta'(\eta)-\eta\theta'(\eta))+P_r E_c(1+\frac{1}{\beta})((f''(\eta))^2+4\delta^2(f'(\eta)^2), \tag{36}$$

$$G_1^{'}[f(\eta)]=\frac{dG_1[f(\eta)]}{d\eta}=G_{1f}.f_\eta+G_{1f'}.(f_\eta)'+G_{1f''}.(f_\eta)''+G_{1f'''}.(f_\eta)''',$$
$$G_2^{'}[\theta(\eta)]=\frac{dG_2[\theta(\eta)]}{d\eta}=G_{2f}.f_\eta+G_{2f'}.(f_\eta)'+G_{2f''}.(f_\eta)''+G_{2\theta'}.(\theta_\eta)', \tag{37}$$

$$G_1^{''}[f(\eta)]=\frac{d^2G_1[f(\eta)]}{d\eta^2}=G_{1ff}.(f_\eta)^2+2.G_{1ff'}.f_\eta(f_\eta)'+2.G_{1ff''}.f_\eta(f_\eta)''$$
$$+2.G_{1ff'''}.f_\eta(f_\eta)'''+G_{1f'f'}.(f_\eta)^{'2}+2.G_{1f'f''}.(f_\eta)'(f_\eta)''+2.G_{1f'f'''}.(f_\eta)'(f_\eta)'''$$
$$+G_{1f''f''}.(f_\eta)^{''2}+2.G_{1f''f'''}.(f_\eta)''(f_\eta)'''+G_{1f'''f'''}.(f_\eta)^{'''2}+G_{1f}.f_{\eta\eta}+G_{1f'}.(f_{\eta\eta})'$$
$$+G_{1f''}.(f_{\eta\eta})''+G_{1f'''}.(f_{\eta\eta})''',$$
$$G_2^{''}[\theta(\eta)]=\frac{d^2G_2[\theta(\eta)]}{d\eta^2}=G_{2ff}.(f_\eta)^2+2.G_{2ff'}.f_\eta(f_\eta)'+2.G_{2ff''}.f_\eta(f_\eta)''$$
$$+2.G_{2f\theta'}.f_\eta(\theta_\eta)'+G_{2f'f'}.(f_\eta)^{'2}+2.G_{2f'f''}.(f_\eta)'(f_\eta)''+2.G_{2f'\theta'}.(f_\eta)'(\theta_\eta)'$$
$$+G_{2f''f''}.(f_\eta)^{''2}+2.G_{2f''\theta'}.(f_\eta)''(\theta_\eta)'+G_{2\theta'\theta'}.(\theta_\eta)^{'2}+G_{2f}.f_{\eta\eta}+G_{2f'}.(f_{\eta\eta})'$$
$$+G_{2f''}.(f_{\eta\eta})''+G_{2\theta'}.(\theta_{\eta\eta})', \tag{38}$$

$$G_1^{'''}[f(\eta)]=\frac{d^3G_1[f(\eta)]}{d\eta^3}=G_{1fff}(f_\eta)^3+3.G_{1fff'}(f_\eta)^2.(f_\eta)'+3.G_{1fff''}(f_\eta)^2.(f_\eta)''$$
$$+3.G_{1fff'''}(f_\eta)^2.(f_\eta)'''+3.G_{1ff'f'}.(f_\eta)(f_\eta)^{'2}+4.G_{1ff'f''}.(f_\eta).(f_\eta)'.(f_\eta)''$$
$$+4.G_{1ff'f'''}.(f_\eta).(f_\eta)'.(f_\eta)'''++3.G_{1ff''f''}.(f_\eta)(f_\eta)^{''2}+4.G_{1ff''f'''}.(f_\eta).(f_\eta)''.(f\eta)'''$$
$$+3.G_{1ff'''f'''}.(f_\eta).(f_\eta)^{'''2}+G_{1f'f'f'}.(f_\eta)^{'3}+3.G_{1f'f'f''}.(f_\eta)^{'2}.(f_\eta)''+G_{1f'f'f'''}.(f_\eta)^{'2}.(f_\eta)'''$$
$$+4.G_{1f'f''f}.(f_\eta)'.f_\eta.(f_\eta)''+3.G_{1f'f''f''}.(f_\eta)^{'}.(f_\eta)^{''2}+4.G_{1f'f''f'''}.(f_\eta)^{'}.(f_\eta)^{''}.(f_\eta)^{'''}$$
$$+4.G_{1f'f'''f}.(f_\eta)'.f_\eta.(f_\eta)'''+4.G_{1f'f'''f''}.(f_\eta)'.(f_\eta)''.(f_\eta)'''+3.G_{1f'f'''f'''}.(f_\eta)'.(f_\eta)^{'''2}$$
$$+3.G_{1f''f''}.(f_{\eta\eta})'''.(f_\eta)''+3.G_{1f''f'''}.(f_{\eta\eta})'''.(f_\eta)'''+G_{1f'''}.(f_{\eta\eta\eta})''',$$
$$G_2^{'''}[\theta(\eta)]=\frac{d^3G_2[f(\eta)]}{d\eta^3}=G_{2fff}(f_\eta)^3+3.G_{2fff'}(f_\eta)^2.(f_\eta)'+3.G_{2fff''}(f_\eta)^2.(f_\eta)''$$
$$+3.G_{ff\theta'}(f_\eta)^2.(\theta_\eta)'+3.G_{ff'f'}.(f_\eta)(f_\eta)^{'2}+4.G_{ff'f''}.(f_\eta).(f_\eta)'.(f_\eta)''$$
$$+4.G_{ff'\theta'}.(f_\eta).(f_\eta)'.(\theta_\eta)'++3.G_{ff''f''}.(f_\eta)(f_\eta)^{''2}+4.G_{ff''\theta'}.(f_\eta).(f_\eta)''.(\theta_\eta)'$$
$$+3.G_{2ff}.(f_{\eta\eta})'(f_\eta)+3.G_{ff'}.(f_{\eta\eta})'(f_\eta)'+3.G_{2ff''}.(f_{\eta\eta})'(f_\eta)''+3.G_{2f'\theta'}.(f_{\eta\eta})'(f_\eta)''$$
$$+G_{2f'}.(f_{\eta\eta\eta})'+3.G_{2f'f}.(f_{\eta\eta})''.(f_\eta)+3.G_{2f'f'}.(f_{\eta\eta})''.(f_\eta)'+3.G_{2f'f''}.(f_{\eta\eta})''.(f_\eta)''$$

$$+3.G_{2f''\theta'}.(f_{\eta\eta})''.(\theta_\eta)'+G_{2f'''}.(f_{\eta\eta\eta})''+3.G_{2\theta f}.(\theta_{\eta\eta})'.(f_\eta)+3.G_{2\theta f'}.(\theta_{\eta\eta})'.(f_\eta)'$$
$$+3.G_{2\theta f''}.(\theta_{\eta\eta})'.(f_\eta)''+3.G_{2\theta\theta'}.(\theta_{\eta\eta})'.(\theta_\eta)'+G_{2\theta'}.(\theta_{\eta\eta\eta})', \quad (39)$$

We note that the derivatives of f with respect to η that are given in (31) can be computed by Eq. (42)-(45) as

$$G_1[f_0(\eta)]=\frac{\beta S}{1+\beta}((\eta.f_{0'''}(\eta)+3.f_{0'}(\eta)+f_{0'}(\eta).f_{0''}(\eta)-f_0(\eta).f_{0'''}(\eta)),$$
$$G_2[\theta_0(\eta)]=P_rS(f_0(\eta)\theta_{0'}(\eta)-\eta\theta_{0'}(\eta))+P_rE_c(1+\frac{1}{\beta})((f_{0''}(\eta))^2+4\delta^2(f_{0'}(\eta))^2), \quad (40)$$

$$G_{1'}[f_0(\eta)]=G_{1f_0}.f_1+G_{f_{0'}}.(f_1)'+G_{1f_{0''}}.(f_1)''+G_{1f_{0'''}}.(f_1)''',$$
$$G_{2'}[\theta_0(\eta)]=G_{2f_0}.f_1+G_{2f_{0'}}.(f_1)'+G_{2f_{0''}}.(f_1)''+G_{2\theta_{0'}}.(\theta_1)', \quad (41)$$

$$G_{1''}[f_0(\eta)]=G_{1f_0f_0}.(f_1)^2+2.G_{1f_0f_{0'}}.f_1(f_1)'+2.G_{1f_0f_{0''}}.f_1(f_1)''+2.G_{1f_0f_{0'''}}.f_1(f_1)'''$$
$$+G_{1f_{0'}f_{0'}}.(f_1)^{'2}+2.G_{1f_{0'}f_{0''}}.(f_1)'(f_1)''+2.G_{1f_{0'}f_{0'''}}.(f_1)'(f_1)'''+G_{1f_{0''}f_{0''}}.(f_1)^{''2}+2.G_{1f_{0''}f_{0'''}}$$
$$.(f_1)''(f_1)'''+G_{1f_{0'''}f_{0'''}}.(f_1)^{'''2}+G_{1f_0}.f_2+G_{1f_{0'}}.(f_2)'+G_{1f_{0''}}.(f_2)''+G_{1f_{0'''}}.(f_2)''',$$
$$G_{2''}[\theta_0(\eta)]=G_{2f_0f_0}.(f_1)^2+2.G_{2f_0f_{0'}}.f_1(f_1)'+2.G_{2f_0f_{0''}}.f_1(f_1)''+2.G_{2f_0\theta_{0'}}.f_1(\theta_1)'$$
$$+G_{2f_{0'}f_{0'}}.(f_1)^{'2}+2.G_{2f_{0'}f_{0''}}.(f_1)'(f_1)''+2.G_{2f_{0'}\theta_{0'}}.(f_1)'(\theta_1)'+G_{2f_{0''}f_{0''}}.(f_1)^{''2}+2.G_{2f_{0''}\theta_{0'}}$$
$$.(f_1)''(\theta_1)'+G_{2\theta_{0'}\theta_{0'}}.(\theta_1)^{'2}+G_{2f_0}.f_2+G_{2f_{0'}}.(f_2)'+G_{2f_{0''}}.(f_2)''+G_{2\theta_{0'}}.(\theta_2)', \quad (42)$$

$$G_{1'''}[f_0(\eta)]=G_{1f_0f_0f_0}(f_1)^3+3.G_{1f_0f_0f_{0'}}(f_1)^2.(f_1)'+3.G_{1f_0f_0f_{0''}}(f_1)^2.(f_1)''$$
$$+3.G_{1f_0f_0f_{0'''}}(f_1)^2.(f_1)'''+3.G_{1f_0f_{0'}f_{0'}}.(f_1)(f_1)^{'2}+4.G_{1f_0f_{0'}f_{0''}}.(f_1).(f_1)'.(f_1)''$$
$$+4.G_{1f_0f_{0'}f_{0'''}}.(f_1).(f_1)'.(f_1)'''++3.G_{1f_0f_{0''}f_{0''}}.(f_1)(f_1)^{''2}+4.G_{1f_0f_{0''}f_{0'''}}.(f_1).(f_1)''.(f1)'''$$
$$+G_{1f_{0''}f_{0''}f_{0''}}.(f_1)^{''3}+3.G_{1f_{0''}f_{0'''}f_{0''}}.(f_1)^{''2}.(f_1)'''+3.G_{1f_{0'''}f_{0'''}f_{0''}}.(f_1)^{'''2}.(f_1)''+G_{1f_{0'''}f_{0'''}f_{0'''}}.(f_1)^{'''3}$$
$$+3.G_{1f_0f_0}.f_2.f_1+3.G_{1f_0f_{0'}}.f_2.(f_1)'+3.G_{1f_0f_{0''}}.f_2.(f_1)''+3.G_{1f_0f_{0'''}}.f_2.(f_1)'''+G_{1f_0}.f_3$$
$$+3.G_{1f_{0'}f_0}.(f_2)'(f_1)+3.G_{1f_{0'}f_{0'}}.(f_2)'(f_1)'+3.G_{1f_{0'}f_{0''}}.(f_2)'(f_1)''+3.G_{1f_{0'}f_{0'''}}.(f_2)'(f_1)''$$
$$+G_{1f_{0'}}.(f_3)'+3.G_{1f_{0''}f_0}.(f_2)''.(f_1)+3.G_{1f_{0''}f_{0'}}.(f_2)''.(f_1)'+3.G_{1f_{0''}f_{0''}}.(f_2)''.(f_1)''$$
$$+3.G_{1f_{0''}f_{0'''}}.(f_2)''.(f_1)'''+G_{1f_{0''}}.(f_3)''+3.G_{1f_{0'''}f_0}.(f_2)'''.(f_1)+3.G_{1f_{0'''}f_{0'}}.(f_2)'''.(f_1)'$$
$$+3.G_{1f_{0'''}f_{0''}}.(f_2)'''.(f_1)''+3.G_{1f_{0'''}f_{0'''}}.(f_2)'''.(f_1)'''+G_{1f_{0'''}}.(f_3)''',$$
$$G'''[\theta_0(\eta)]=G_{2f_0f_0f_0}(f_1)^3+3.G_{2f_0f_0f_{0'}}(f_1)^2.(f_1)'+3.G_{2f_0f_0f_{0''}}(f_1)^2.(f_1)''$$
$$+3.G_{2f_0f_0\theta_{0'}}(f_1)^2.(\theta_1)'+3.G_{2f_0f_{0'}f_{0'}}.(f_1)(f_1)^{'2}+4.G_{2f_0f_{0'}f_{0''}}.(f_1).(f_1)'.(f_1)''$$

$$
\begin{aligned}
&+4.G_{2f_0 f_{0'}\theta_{0'}}.(f_1).(f_1)'.(\theta_1)'++3.G_{2f_0 f_{0''} f_{0''}}.(f_1)(f_1)^{''2}+4.G_{2f_0 f_{0''}\theta_{0'}}.(f_1).(f_1)''.(\theta_1)' \\
&+3.G_{2f_0\theta_{0'} f_{0'''}}.(f_1).(\theta_1)^{'2}+G_{2f_{0'} f_{0'} f_{0'}}.(f_1)^{'3}+3.G_{2f_{0'} f_{0'} f_{0''}}.(f_1)^{'2}.(f_1)''+G_{2f_{0'} f_{0'}\theta_{0'}}.(f_1)^{'2}.(\theta_1)' \\
&+4.G_{2f_{0'} f_{0''} f_0}.(f_1)'.f_1.(f_1)''+3.G_{2f_{0'} f_{0''} f_{0''}}.(f_1)^{'}.(f_1)^{''2}+4.G_{2f_{0'} f_{0''}\theta_{0'}}.(f_1)^{'}.(f_1)^{''}.(\theta_1)^{'} \\
&+3.G_{2f_0 f_0}.f_2.f_1+3.G_{2f_0 f_{0'}}.f_2.(f_1)'+3.G_{2f_0 f_{0''}}.f_2.(f_1)''+3.G_{2f_0\theta_{0'}}.f_2.(\theta_1)'+G_{f_0}.f_3 \\
&+3.G_{2f_{0'} f_0}.(f_2)'(f_1)+3.G_{2f_{0'} f_{0'}}.(f_2)'(f_1)'+3.G_{f_{0'} f_{0''}}.(f_2)'(f_1)''+3.G_{f_{0'}\theta_{0'}}.(f_2)'(f_1)'' \\
&+G_{f_{0'}}.(f_3)'+3.G_{f_{0''} f_0}.(f_2)''.(f_1)+3.G_{f_{0''} f_{0'}}.(f_2)''.(f_1)'+3.G_{f_{0''} f_{0''}}.(f_2)''.(f_1)'' \\
&+3.G_{2f_{0''} f_{0'''}}.(f_2)''.(f_1)'''+G_{f_{0''}}.(f_3)''+3.G_{2f_{0'''} f_0}.(f_2)'''.(f_1)+3.G_{2f_{0'''} f_{0'}}.(\theta_2)'.(f_1)' \\
&+3.G_{2\theta_{0'} f_{0''}}.(\theta_2)'.(f_1)''+3.G_{2f_{0'''} f_{0'''}}.(\theta_2)'.(f_1)'''+G_{2f_{0'''}}.(\theta_3)', \qquad (43)
\end{aligned}
$$

Now, we need to extract the first derivatives of G as follows

$$
\begin{aligned}
&G_{1f_0}=-\frac{\beta S}{1+\beta}f_{0'''}(\eta), \quad G_{1f_0 f_0}=G_{1f_0 f_{0'}}=G_{1f_0 f_{0''}}=0, \quad G_{1f_0 f_{0'''}}=-\frac{\beta S}{1+\beta}, \\
&G_{1f_0 f_0 f_0}=G_{1f_0 f_{0'} f_0}=G_{1f_0 f_{0'} f_{0'}}=G_{1f_0 f_{0''} f_{0'}}=G_{1f_0 f_0 f_{0'}}=G_{1f_0 f_{0'''} f_{0'''}}=G_{1f_0 f_{0''} f_{0''}}=0, \\
&G_{1f_{0'}}=\frac{\beta S}{1+\beta}=(3+f_0''(\eta)), G_{1f_{0'} f_0}=G_{1f_{0'} f_{0'}}=G_{1f_{0'} f_{0'''}}=0, \quad G_{1f_{0'} f_{0''}}=\frac{\beta S}{1+\beta}, \\
&G_{1f_{0'} f_0 f_0}=G_{1f_{0'} f_{0'} f_0}=G_{1f_{0'} f_{0'} f_{0'}}=G_{1f_{0'} f_{0''} f_{0'}}=G_{1f_{0'} f_{0'''} f_{0'}}=G_{1f_{0'} f_{0'''} f_{0'''}}=G_{1f_{0'} f_{0''} f_{0''}}=0, \\
&G_{1f_{0''}}=\frac{\beta S}{1+\beta}f'(\eta), \quad G_{1f_{0''} f_0}=G_{1f_{0''} f_{0''}}=G_{1f_{0''} f_{0'''}}=0, G_{1f_{0''} f_{0'}}=\frac{\beta S}{1+\beta}, \\
&G_{1f_{0''} f_0 f_0}=G_{1f_{0''} f_{0'} f_0}=G_{1f_{0''} f_{0'} f_{0'}}=G_{1f_{0''} f_{0''} f_{0'}}=G_{1f_{0''} f_{0''} f_{0'}}=G_{1f_{0''} f_{0'''} f_{0'''}}=G_{1f_{0''} f_{0''} f_{0''}}=0, \\
&G_{1f_{0'''}}=\frac{\beta S\eta}{1+\beta}-\frac{\beta S}{1+\beta}f_0(\eta), \quad G_{1f_{0'''} f_{0'}}=G_{1f_{0'''} f_{0''}}=G_{1f_{0'''} f_{0'''}}=0, G_{1f_{0'''} f_0}=-\frac{\beta S}{1+\beta}, \\
&G_{1f_{0'''} f_0 f_0}=G_{1f_{0'''} f_{0'} f_0}=G_{1f_{0'''} f_{0'} f_{0'}}=G_{1f_{0'''} f_{0''} f_{0'}}=G_{1f_{0'''} f_{0'''} f_{0'}}=G_{1f_{0'''} f_{0'''} f_{0'''}}=0, \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
&G_{2f_0}=P_r S\theta_{0'}(\eta), \quad G_{2f_0 f_0}=G_{2f_0 f_{0'}}=G_{2f_0 f_{0''}}=0, \quad G_{2f_0\theta_{0'}}=P_r S, \\
&G_{2f_0 f_0 f_0}=G_{2f_0 f_{0'} f_0}=G_{2f_0 f_{0'} f_{0'}}=G_{2f_0 f_{0''} f_{0'}}=G_{2f_0 f_0 f_{0'}}=G_{2f_0 f_{0'''} f_{0'''}}=G_{2f_0 f_{0''} f_{0''}}=0, \\
&G_{2f_{0'}}=8\delta^2 P_r E_c(1+\frac{1}{\beta})f'(\eta), G_{2f_{0'} f_0}=G_{2f_{0'} f_{0''}}=G_{2f_{0'}\theta_{0'}}=0, \quad G_{2f_{0'} f_{0'}}=8\delta^2 P_r E_c(1+\frac{1}{\beta}), \\
&G_{2f_{0'} f_0 f_0}=G_{2f_{0'} f_{0'} f_0}=G_{2f_{0'} f_{0'} f_{0'}}=G_{2f_{0'} f_{0''} f_{0'}}=G_{2f_{0'}\theta_{0'} f_{0'}}=G_{2f_{0'}\theta_{0'}\theta_{0'}}=G_{2f_{0'} f_{0''} f_{0''}}=0, \\
&G_{2f_{0''}}=2P_r E_c(1+\frac{1}{\beta})f''(\eta), \quad G_{f_{0''} f_0}=G_{f_{0''} f_{0''}}=G_{f_{0''} f_{0'}}=0, G_{f_{0''} f_{0''}}=2P_r E_c(1+\frac{1}{\beta}), \\
&G_{2f_{0''} f_0 f_0}=G_{2f_{0''} f_{0'} f_0}=G_{2f_{0''} f_{0'} f_{0'}}=G_{2f_{0''} f_{0''} f_{0'}}=G_{2f_{0''} f_{0''} f_{0'}}=G_{2f_{0''}\theta_{0'}\theta_{0'}}=G_{2f_{0''} f_{0''} f_{0''}}=0, \\
&G_{2\theta_{0'}}=P_r S(f(\eta)-\eta), \quad G_{2\theta_{0'} f_0}=P_r S, G_{2\theta_{0'} f_{0''}}=G_{2\theta_{0'} f_{0'}}=0, G_{2\theta_{0'}\theta_{0'}}=G_{\theta_{0'} f_0 f_0}=0, \\
&G_{\theta_{0'} f_{0'} f_0}=G_{\theta_{0'} f_{0'} f_{0'}}=G_{\theta_{0'} f_{0''} f_{0'}}=G_{\theta_{0'}\theta_{0'} f_{0'}}=G_{\theta_{0'}\theta_{0'}\theta_{0'}}=0, \qquad (45)
\end{aligned}
$$

from Eq. (31) by using Eq. (46)-(49), we obtain

$$f_0 = A_2\eta + \frac{1}{6}A_4\eta^3,\ \theta_0 = B_1, \tag{46}$$

$$f_1 = \frac{1}{30}(\frac{\beta}{1+\beta})SA_4\eta^5 + \frac{1}{2520}(\frac{\beta}{1+\beta})SA_4^2\eta^7,$$

$$\theta_1 = 2P_rE_c(1+\frac{1}{\beta})\delta^2A_2^2\eta^2 - \frac{1}{12}P_rE_c(1+\frac{1}{\beta})(4A_2A_4\delta^2 + A_4^2)\eta^4$$

$$-\frac{1}{30}P_rE_c(1+\frac{1}{\beta})\delta^2A_4^2\eta^6, \tag{47}$$

$$f_2 = \frac{1}{840}(\frac{\beta}{1+\beta})^2S^2(4A_4 - \frac{4}{3}A_4A_2)\eta^7 + \frac{1}{3024}(\frac{\beta}{1+\beta})^2S^2(\frac{4}{15}A_4^2 - \frac{1}{15}A_4^2A_2)\eta^9$$

$$-\frac{1}{2494800}(\frac{\beta}{1+\beta})^2S^2A_4^3\eta^{11},$$

$$\theta_2 = (\frac{1}{3}A_2^3E_cS(1+\frac{1}{\beta})P_r^2\delta^2 - \frac{1}{3}A_2^2E_cS(1+\frac{1}{\beta})P_r^2\delta^2)\eta^4 \tag{48}$$

$$\vdots$$

From Step (4) substituting Eq. (52)-(54) in Eq. (25), we get the analytical-approximate solution

$$f(\eta) = A_2\eta + \frac{1}{6}A_4\eta^3 + \frac{1}{30}(\frac{\beta}{1+\beta})SA_4\eta^5 + \frac{1}{2520}(\frac{\beta}{1+\beta})SA_4^2\eta^7 +$$

$$\frac{1}{840}(\frac{\beta}{1+\beta})^2S^2(4A_4 - \frac{4}{3}A_4A_2)\eta^7 + \frac{1}{3024}(\frac{\beta}{1+\beta})^2S^2(\frac{4}{15}A_4^2 - \frac{1}{15}A_4^2A_2)\eta^9$$

$$-\frac{1}{2494800}(\frac{\beta}{1+\beta})^2S^2A_4^3\eta^{11} + \dots,$$

$$\theta(\eta) = B_1 + 2P_rE_c(1+\frac{1}{\beta})\delta^2A_2^2\eta^2 + (\frac{1}{12}P_rE_c(1+\frac{1}{\beta})(-4A_2A_4\delta^2 - A_4^2) +$$

$$\frac{1}{3}A_2^3E_cS(1+\frac{1}{\beta})P_r^2\delta^2 - \frac{1}{3}A_2^2E_cS(1+\frac{1}{\beta})P_r^2\delta^2)\eta^4 + \tag{49}$$

5. The Analysis of Convergence

Here, the analysis of convergence for the analytical-approximate solution (55) that was resulted from the application of new power series of algorithm for solving the problem has been extensively studied.

Definition 5.1: Suppose that H is Banach space, R is the real numbers and $G[F,D] = (G_1[F], G_2[D])$ is a nonlinear operators defined by $G[F,D]: H^2 \to R^2$. Then the sequence of the solutions generated from a new analytical can be written as

$$F_{n+1} = G_1[F_n], \quad F_n = \sum_{k=0}^{n} f_k, \quad n = 0,1,2,3,...$$
$$D_{n+1} = G_2[D_n], \quad D_n = \sum_{k=0}^{n} \theta_k, \quad n = 0,1,2,3,... \quad n = 0,1,2,3,... \tag{50}$$

Definition 5.2: Suppose that $G[F, D]$ satisfies Lipschitz condition such that for $0\Box\gamma_1,\gamma_2 < 1$, $\gamma_1,\gamma_2 \in R$, we have

$$\| G_1[F_n] - G_1[F_{n-1}] \| \le \Box\gamma_1 \| F_n - F_{n-1} \|,$$
$$\| G_2[D_n] - G_2[D_{n-1}] \| \leqq \gamma_2 \| D_n - D_{n-1} \|, \tag{51}$$

Now, we assume that $G[F_n, D_n] = G(n)$ for simplify with $\gamma = \gamma_1 + \gamma_2$, $0\Box\gamma < 1$ yield,

$$\| G(n) - G(n-1)] \| \leqq \gamma \| (F_n, D_n) - (F_{n-1}, D_{n-1}) \|. \tag{52}$$

The sufficient condition for the convergent of the series of analytical-approximate solutions F_n, D_n *is* given in the following theorems.

Theorem 5.1 The series of the analytical-approximate solution $\{S_n = (F_n, D_n)\}_0^\infty$ generated from new algorithm converge if the following condition satisfied

$$\| S_n - S_m \| \to 0, \quad as \quad m \to \infty, \quad for \quad 0\Box\gamma < 1, \tag{53}$$

Proof. From the above definition, the next equation can be written as

$$\begin{aligned}
\| S_n - S_m \| &= \| (F_n, D_n) - (F_m, D_m) \| \\
&= \| (\sum_{k=0}^{n} f_k, \sum_{k=0}^{n} \theta_k) - (\sum_{k=0}^{m} f_k, \sum_{k=0}^{m} \theta_k) \|, \\
&= \| (f_0 + L^{-1} \sum_{k=0}^{n} \frac{d^{(k)} G_1[f_0(\eta)]}{d\eta^{(k)}}, \theta_0 + L^{-1} \sum_{k=0}^{n} \frac{d^{(k)} G_2[\theta_0(\eta)]}{d\eta^{(k)}} \\
&\quad - (f_0 + L^{-1} \sum_{k=0}^{m} \frac{d^{(k)} G_1[f_0(\eta)]}{d\eta^{(k)}}, \theta_0 + L^{-1} \sum_{k=0}^{m} \frac{d^{(k)} G_2[\theta_0(\eta)]}{d\eta^{(k)}}) \| \\
&= \| L^{-1} G[\sum_{k=0}^{n-1} f_k, \sum_{k=0}^{n-1} \theta_k] - L^{-1} G[\sum_{k=0}^{m-1} f_k, \sum_{k=0}^{m-1} \theta_k] \|, \\
&\leqq |Ł^{-1}| \| G[\sum_{k=0}^{n-1} f_k, \sum_{k=0}^{n-1} \theta_k] - G[\sum_{k=0}^{m-1} f_k, \sum_{k=0}^{m-1} \theta_k] \|, \\
&\leqq |L^{-1}| \| G[F_{n-1}, D_{n-1}] - G[F_{m-1}, D_{m-1}] \|, \\
&\leqq \gamma \| (F_{n-1}, D_{n-1}) - (F_{m-1}, D_{m-1}) \|, \\
&= \gamma \| S_{n-1} - S_{m-1} \|,
\end{aligned} \tag{54}$$

since $G[F, D]$ satisfies Lipschitz condition. Let $n = m+1$, then

$$\| F_{m+1} - F_m \| \leq \gamma_1 \| F_m - F_{m-1} \|, \quad \| D_{m+1} - D_m \| \leq \gamma_2 \| D_m - D_{m-1} \|, \tag{55}$$

hence,

$$\| F_m - F_{m-1} \| \leq \gamma_1 \| F_{m-1} - F_{m-2} \| \leq \ldots \leq \gamma_1^{m-1} \| F_1 - F_0 \|, \quad \| D_m - D_{m-1} \| \leq \gamma_2 \| D_{m-1} - D_{m-2} \| \leq \ldots \leq \gamma_2^{m-1} \| D_1 - D_0 \|, \tag{56}$$

from Eq. (62) we get

$$\begin{aligned}
&\| F_2 - F_1 \| \leq \gamma_1 \| F_1 - F_0 \|, \| D_2 - D_1 \| \leq \square \gamma_2 \| D_1 - D_0 \|, \\
&\| F_3 - F_2 \| \leq \gamma_1^2 \| F_1 - F_0 \|, \| D_3 - D_2 \| \leq \gamma_2^2 \| D_1 - D_0 \|, \\
&\| F_4 - F_3 \| \leq \gamma_1^3 \| F_1 - F_0 \|, \| D_4 - D_3 \| \leq \square \gamma_2^3 \| D_1 - D_0 \|, \\
&\vdots \\
&\| F_{m-1} - F_{m-2} \| \leq \gamma_1^{m-2} \| F_1 - F_0 \|, \| D_{m-1} - D_{m-2} \| \leq \gamma_2^{m-2} \| D_1 - D_0 \|, \\
&\| F_m - F_{m-1} \| \leq \gamma_1^{m-1} \| F_1 - F_0 \|, \| D_m - D_{m-1} \| \leq \gamma_2^{m-1} \| D_1 - D_0 \|,
\end{aligned} \tag{57}$$

By using triangle inequality, we find that

$$\begin{aligned}
&\| S_n - S_m \| = \| S_n - S_{n-1} - S_{n-2} - \ldots - S_{m+1} - S_m \|, \\
&\square \| S_n - S_{n-1} \| + \| S_{n-1} - S_{n-2} \| + \ldots + \| S_{m+1} - S_m \|, \\
&\square \| (F_n, D_n) - (F_{n-1}, D_{n-1}) \| + \| (F_{n-1}, D_{n-1}) - (F_{n-2}, D_{n-2}) \| \\
&+ \ldots + \| (F_{m+1}, D_{m+1}) - (F_m, D_m) \| \\
&\square \| F_n - F_{n-1} \| + \| D_n - D_{n-1} \| + \| F_{n-1} - F_{n-2} \| + \| D_{n-1} \\
&- D_{n-2} \| + \ldots + \| F_{m+1} - F_m \| + \| D_{m+1} - D_m \| \\
&\square [\gamma_1^{n-1} + \gamma_1^{n-2} + \ldots + \gamma_1^m] \| F_1 - F_0 \| + [\gamma_2^{n-1} + \gamma_2^{n-2} + \ldots + \gamma_2^m] \| H_1 - H_0 \| \\
&\square [\gamma^{n-1} + \gamma^{n-2} + \ldots + \gamma^m] (\| F_1 - F_0 \| + \| D_1 - D_0 \|), \\
&\square [\gamma^{n-1} + \gamma^{n-2} + \ldots + \gamma^m] (\| (F_1, D_1) - (F_0, D_0) \|), \\
&= \gamma^m [\gamma^{n-m-1} + \gamma^{n-m-2} + \ldots + 1] \| S_1 - S_0 \|, \\
&\square \frac{\gamma^m}{1 - \gamma} \| S_1 - S_0 \|,
\end{aligned}$$

as $m \to \infty$, we have $\| S_n - S_m \| \to 0$, then S_n is a Cauchy sequence in Banach space H^3.

Theorem 5.2 Let $G = (G_1, G_2)$ be a nonlinear operator satisfies Lipschitz condition from H^2 to H^2. If the series of analytical-approximate solution $\{S_n\}$ converges, then it is converged to the solution of the problem (12)-(14).

Proof.

$$
\begin{aligned}
& \| G[S_2]-G[S_1] \| \\
= & \| (G_1[F_2],G_2[H_2])-(G_1[F_1],G_2[H_1]) \| \\
= & \| (G_1[F_2]-(G_1[F_1],G_2[H_2]-G_2[H_1] \| \\
\leq & \| G_1[F_2]-(G_1[F_1] \| + \| G_2[H_2]-G_2[H_1] \| \\
\leq & \gamma_1 \| F_2-F_1 \| + \gamma_2 \| H_2-H_1 \| \\
\leq & (\gamma_1+\gamma_2) \| (F_2,D_2)-(F_1,D_1) \| \\
\leq & \gamma \| S_2-S_1 \|
\end{aligned}
$$

Therefore, from the Banach fixed-point theorem, there is a unique solution of the problem (12)-(14). we will prove that $\{S_n\}_0^\infty$ converges to S.

$$G[S]=G[\sum_{k=0}^{\infty}S_k]=lim_{n\to\infty}G[\sum_{k=0}^{n}S_k]=lim_{n\to\infty}G[S_n]=lim_{n\to\infty}S_{n+1}=S,$$

In practice, the theorems 5.1. and 5.2 suggest to compute the value of γ_1,γ_2, as described in the following definition

Definition 5.3 For $k=1,2,3,...$

$$
\gamma_1^k=\begin{cases}\dfrac{\| F_{k+1}-F_k \|}{\| F_1-F_0 \|}=\dfrac{\| f_{k+1} \|}{\| f_1 \|}, & \| f_1 \| \neq 0,\\ 0, & \| f_1 \| = 0,\end{cases}
\qquad
\gamma_2^k=\begin{cases}\dfrac{\| D_{k+1}-D_k \|}{\| D_1-D_0 \|}=\dfrac{\| \theta_{k+1} \|}{\| \theta_1 \|}, & \| \theta_1 \| \neq 0,\\ 0, & \| \theta_1 \| = 0,\end{cases}
\tag{58}
$$

Now, the definition (5.3) can be applied to heat transfer analysis unsteady squeezing flow of a Casson fluid between parallel plates to find convergence, then to obtain for examples as below: if we choose then obtain $S=0.5, P_r=E_c=0.1, P_r=0.1, \delta=0.2, \beta=0.3$

$$
\begin{aligned}
& \| F_2-F_1 \|_2 \leq \gamma \| F_1-F_0 \|_2 \Rightarrow \gamma = 0.008312597066<1,\\
& \| F_3-F_2 \|_2 \leq \gamma^2 \| F_1-F_0 \|_2 \Rightarrow \gamma^2 = 0.00002917586591<1,\\
& \| F_4-F_3 \|_2 \leq \gamma^3 \| F_1-F_0 \|_2 \Rightarrow \gamma^3 = 0.0000001312994049<1,\\
& \quad \vdots\\
& \| D_2-D_1 \|_2 \leq \gamma \| D_1-D_0 \|_2 \Rightarrow \gamma = 0.003236547163<1,\\
& \| D_3-D_2 \|_2 \leq \gamma^2 \| D_1-D_0 \|_2 \Rightarrow \gamma^2 = 0.00003108379718<1,\\
& \| D_4-D_3 \|_2 \leq \gamma^3 \| D_1-D_0 \|_2 \Rightarrow \gamma^3 = 0.0000008193778328<1,\\
& \quad \vdots
\end{aligned}
$$

$$\| F_2 - F_1 \|_{+\infty} \leq \gamma \| F_1 - F_0 \|_{+\infty} \Rightarrow \gamma = 0.008299220881 < 1,$$
$$\| F_3 - F_2 \|_{+\infty} \leq \gamma^2 \| F_1 - F_0 \|_{+\infty} \Rightarrow \gamma^2 = 0.00002715644457 < 1,$$
$$\| F_4 - F_3 \|_{+\infty} \leq \gamma^3 \| F_1 - F_0 \|_{+\infty} \Rightarrow \gamma^3 = 0.00008511439187 < 1,$$
$$\vdots$$
$$\| D_2 - D_1 \|_{+\infty} \leq \gamma \| D_1 - D_0 \|_{+\infty} \Rightarrow \gamma = 0.00003355398388 < 1,$$
$$\| D_3 - D_2 \|_{+\infty} \leq \gamma^2 \| D_1 - D_0 \|_{+\infty} \Rightarrow \gamma^2 = 0.00003227408681 < 1,$$
$$\| D_4 - D_3 \|_{+\infty} \leq \gamma^3 \| D_1 - D_0 \|_{+\infty} \Rightarrow \gamma^3 = 0.0000001276586797 < 1,$$
$$\vdots$$

Also, if we get $S = -0.5, P_r = E_c = 0.1, P_r = 0.1, \delta = 0.2, \beta = 0.3$

$$\| F_2 - F_1 \|_2 \leq \gamma \| F_1 - F_0 \|_2 \Rightarrow \gamma = 0.008197358426 < 1,$$
$$\| F_3 - F_2 \|_2 \leq \gamma^2 \| F_1 - F_0 \|_2 \Rightarrow \gamma^2 = 0.00002814781756 < 1,$$
$$\| F_4 - F_3 \|_2 \leq \gamma^3 \| F_1 - F_0 \|_2 \Rightarrow \gamma^3 = 0.0000001392452234 < 1,$$
$$\vdots$$
$$\| D_2 - D_1 \|_2 \leq \gamma \| D_1 - D_0 \|_2 \Rightarrow \gamma = 0.002934898481 < 1,$$
$$\| D3 - D_2 \|_2 \leq \gamma^2 \| D_1 - D_0 \|_2 \Rightarrow \gamma^2 = 0.00003104191667 < 1,$$
$$\| D_4 - D_3 \|_2 \leq \gamma^3 \| D_1 - D_0 \|_2 \Rightarrow \gamma^3 = 0.00008060325784 < 1,$$
$$\vdots$$
$$\| F_2 - F_1 \|_{+\infty} \leq \gamma \| F_1 - F_0 \|_{+\infty} \Rightarrow \gamma = 0.008181412469 < 1,$$
$$\| F_3 - F_2 \|_{+\infty} \leq \gamma^2 F_1 - F_0 \|_{+\infty} \Rightarrow \gamma^2 = 0.00002564647652 < 1,$$
$$\| F_4 - F_3 \|_{+\infty} \leq \gamma^3 \| F_1 - F_0 \|_{+\infty} \Rightarrow \gamma^3 = 0.000001368147437 < 1,$$
$$\vdots$$
$$\| D_2 - D_1 \|_{+\infty} \leq \gamma \| D_1 - D_0 \|_{+\infty} \Rightarrow \gamma = 0.000030118247 < 1,$$
$$\| D_3 - D_2 \|_{+\infty} \leq \gamma^2 \| D_1 - D_0 \|_{+\infty} \Rightarrow \gamma^2 = 0.00003191317638 < 1,$$
$$\| D_4 - D_3 \|_{+\infty} \leq \gamma^3 \| D_1 - D_0 \|_{+\infty} \Rightarrow \gamma^3 = 0.00008291352364 < 1,$$
$$\vdots$$

6. Results and Discussion

The solution of physical problem is effectively obtained in the section(4) using a new analytical technique. This part is dedicated to explore the effects of various values of non-dimensional physical parameters under dimensionless velocity and dimensionless temperature. Especially, this section highlights the behavior of squeeze number S, Casson fluid parameter β, Prandtl number P_r, Eckert number E_c and δ on the curves of axial velocity $f(\eta)$, radial velocity $f'(\eta)$ and temperature profile $\theta(\eta)$. For this purpose, graphics in Figure 2 to Figure 7 are plotted with disussion. Figure 2 depicts the influence of increasing values for squeeze number S on axial velocity, radial velocity and temperature profile. This Figure shows no significant change on $f(\eta)$ by increasing squeeze number S. $f'(\eta)$ is decreasing with the increase of number S until it touches the neighbourhood point of

$\eta \leq 0.4$ and after this, it becomes reverse for increasing values of the squeeze number S. This means, there is an increase in the curve of $f'(\eta)$ when $0.4 < \eta \leq 1$. Furthermore, the decrease is obvious in temperature profile for increasing values of squeeze number S. The Temperature profile reaches to high level when $\eta = 0$ with a thickness boundary layer while the opposite situation happens when $\eta = 1$, that is thiner boundary layer. On the other hand, the radial velocity increases near the power plate and enhances near the upper plate. It can also be obesraved that for fixed values of physical parameters the axial velocity increases while the radial velocity monotonically and temperature profile decrease**.** The behaviors of velocity and temperature profiles for two different cases with increasing values of Casson fluid parameter β. Figure 3 shows increasing Casson fluid parameter that leads to increase axial velocity $f(\eta)$ and the behavior of radial velocity $f'(\eta)$ near the upper plat is increasing and is slowed down in $0.4 < \eta \leq 1$. This figure signifies that the temperture profile is gradually decreasing when the values of the Casson fluid parameter β increased in the case the plates are moving apart coming together($S < 0$) . As for the second case, where the plates are moving apart $S > 0$ are shown in Figure 4 as increasing Casson fluid parameter lead to decrease axial velocity $f(\eta)$. This figure reveals that the radial velocity $f'(\eta)$ decreased when the Casson fluid parameter β is increased, this happen when $\eta \leq 0.4$ and occur opposite situation when $0.4 < \eta \leq 1$. Moreover, the decreasing behavior of the temperature is achieved by increasing Casson fluid parameter β. The effect of the Prandtl number P_r, Eckert number E_c and δ on the non-dimensionl temparaure profile is displayed in Figure 5 to Figure 7. It is evident from the figures that the temparure profile is gradually increasing when the Prandtl number P_r, Eckert number E_c and δ are increased.

Physically the squeeze number S describes the movement of the plates (S > 0 corresponds to the plates moving apart, while S < 0 corresponds to the plates moving together). It can be easily seen that the value of velocity $f(\eta)$ near the lower plate surface decreases regularly with the increase in the value of S, and as we move away from lower plate surface, this value increases . In addition increasing value of squeeze number S decreases the temperature (η). By studying the effects of the squeeze, Prandtl and Eckert numbers on the temperature (η). We can note that the decrease in temperature values is observed with the increasing of S values, since any increase in S can be related to the increase in the distance between the plates, the increase in the plate movement and the decrease in the kinematic viscosity. The values of Pr are lesser than one that characterizes liquid materials with high thermal diffusion and low viscosity while the values of Pr are greater than one that represents high viscosity liquids, so it is observed that the increase in the values of Pr leads to increase temperature value. Eckert number Ec expresses the relationship between a flow kinetic energy and the thermal boundary layer thickness, so we note that the increase in E_c leads to an increase in (η). Eckert number Ec is the coefficient of dissipation of viscosity and joule heat and thus an increase in heat dissipation is due to the friction force and joule heating process. The dispersed heat increases as a result of friction and the joule heating is added to the fluid and hence its temperature. To check the convergence of the solution obtained by new analytical techinqe, Table 1 shows it can be observed that only 3 iterations of the analytical solution are enough for a convergent solution in situation $S > 0$. In Table 2 that can be made 4 iterations only of the analytical solution when $S < 0$. Table 3 to Table 8 show a comparison of all solutions are shown. Table 3 to Table 6 compare the resulting solutions with solutions for HAM [7] and numerical solutions. These tables observed that the solution agrees well with each other. Table 7 and Table 8 shows compare the resulting solutions with numerical solutions. These tables observed that the solution agree well with each other. Numerical values for Nusselt number are presented in Table 9. Observations show that magnitude of Nusselt is decreasing function for increasing values of S and β. On the other hand, the magnitude of Nusselt number is observed as an increasing function for increasing values of P_r, E_c and δ.

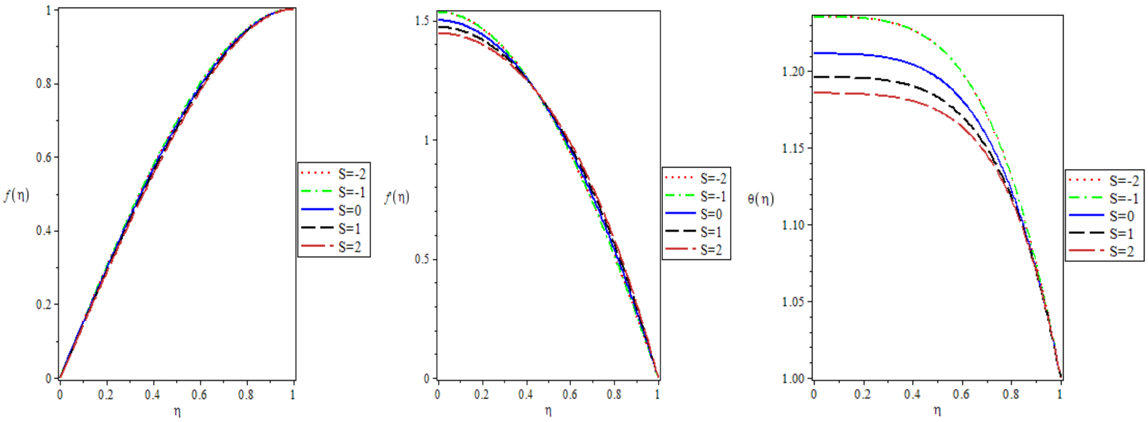


Fig. 2. $f(\eta)$, $f'(\eta)$, $\theta(\eta)$ for the value $Pr = Ec = 0.5, \beta = 0.1, \delta = 0.1$ when the Squeezing number S is varied

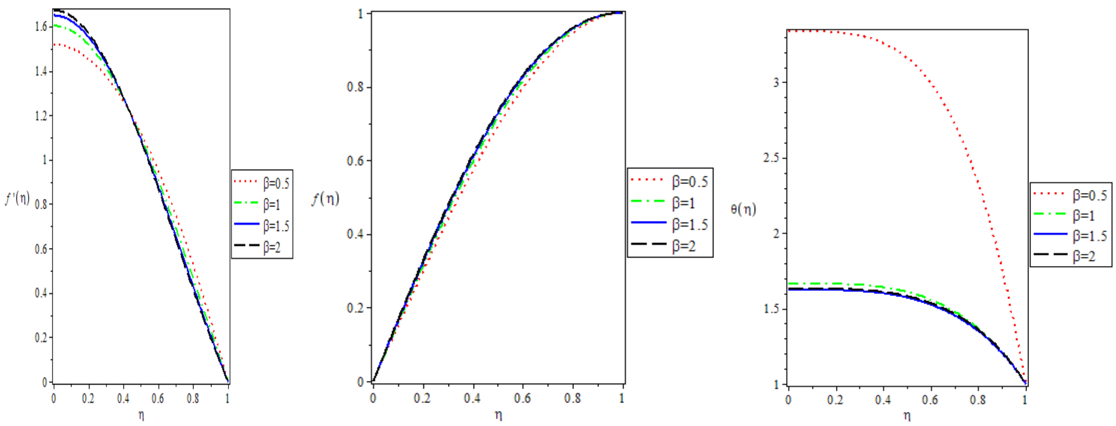


Fig. 3. $f(\eta)$, $f'(\eta)$, $\theta(\eta)$ for the value $Pr = Ec = 0.5, \delta = 0.1, S = -2$ when β is varied

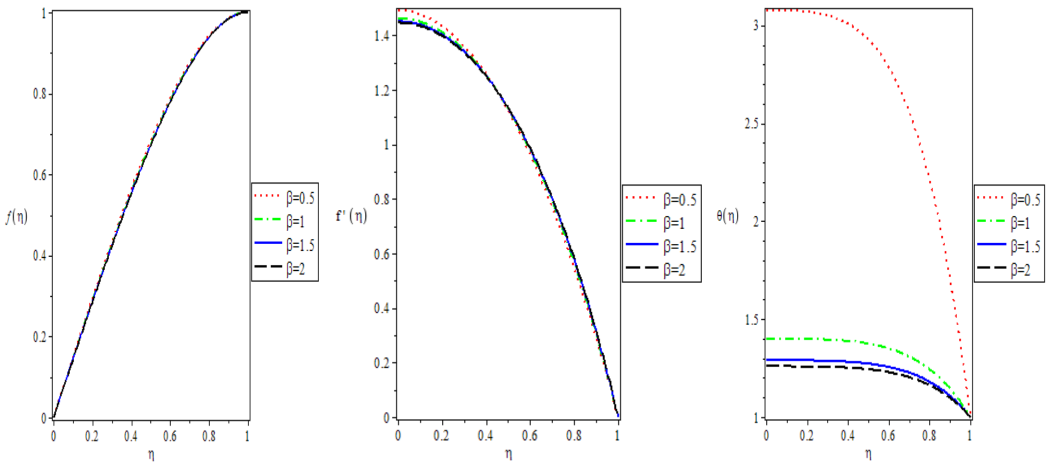
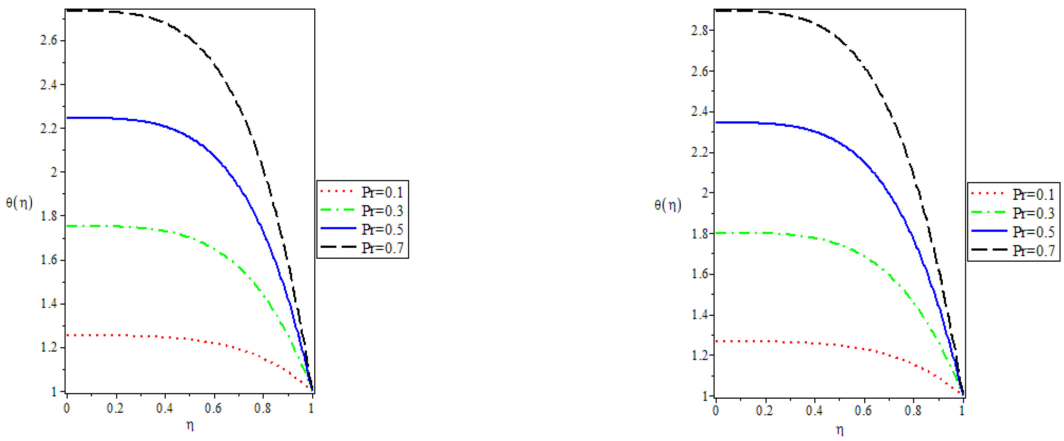
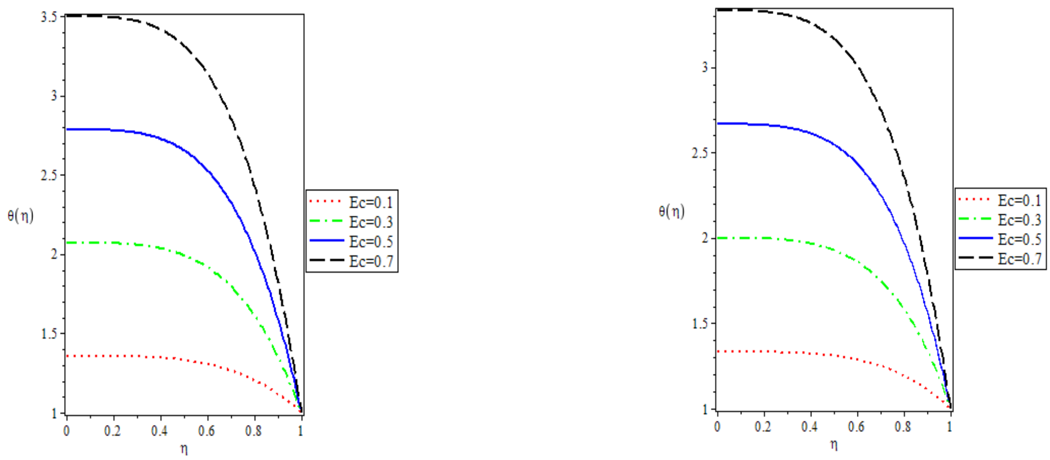


Fig. 4. $f(\eta)$, $f'(\eta)$, $\theta(\eta)$ for the value $Pr = Ec = 0.5, \delta = 0.1, S = 1$ when β is varied



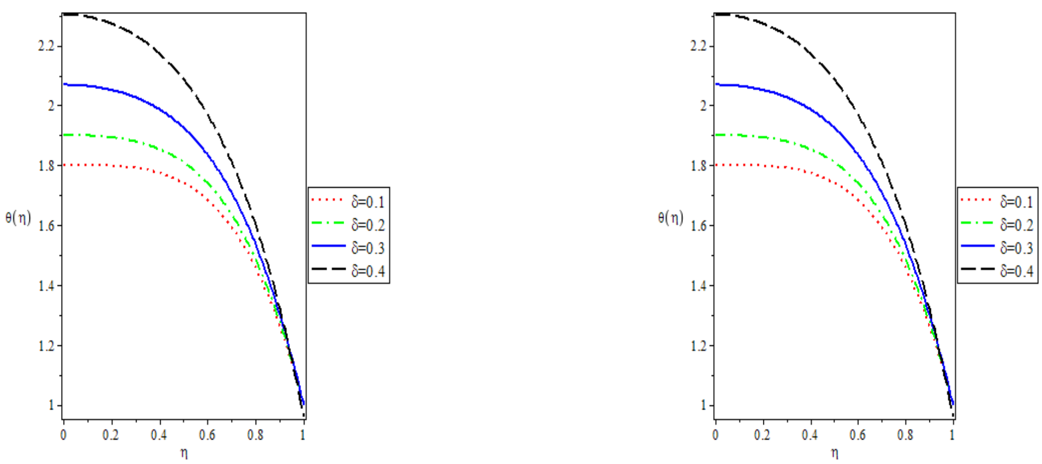
(a) (b)

Fig. 5. The behavior of the temperature profile $\theta(\eta)$ when $E_c = 0.3, \beta = 0.1, \delta = 0.1$ and $(a) S < 0$, $(b) S > 0$



(a) (b)

Fig. 6. The behavior of the temperature profile $\theta(\eta)$ when $P_r = 0.4, \beta = 0.1, \delta = 0.1$ and $(a) S < 0$, $(b) S > 0$



(a) (b)

Fig. 7. The behavior of the temperature profile $\theta(\eta)$ when $P_r = 0.3, E_c = 0.3, \beta = 0.1$ and $(a) S > 0$, (b) S< 0

Table 1
$S = 0.5, P_r = E_c = 0.1, P_r = 0.1, \delta = 0.2, \beta = 0.3$

Approximation	A_2	A_4	B_1
1 term	1.4897020	-2.8741555	1.0355278
2 term	1.4895422	-2.8727139	1.0338137
3 term	1.4895418	-2.8727104	1.0372680
4 term	1.4895418	-2.8727104	1.0372680
5 term	1.4895418	-2.8727104	1.0372680
6 term	1.4895418	-2.8727104	1.0372680
7 term	1.4895418	-2.8727104	1.0372680
8 term	1.4895418	-2.8727104	1.0372680

Table 2
$S = -0.5, P_r = E_c = 0.1, P_r = 0.1, \delta = 0.2, \beta = 0.3$

Approximation	A_2	A_4	B_1
1 term	1.511163104	3.136660164	1.041274725
2 term	1.510982550	3.134993427	1.043239377
3 term	1.510982924	3.134996751	1.039333714
4 term	1.510982932	3.134996808	1.039336384
5 term	1.510982932	3.134996808	1.039336384
6 term	1.510982932	3.134996808	1.039336384
7 term	1.510982932	3.134996808	1.039336384

Table 3
Comparison between new scheme and $(R-K4)$ scheme for the analytical solutions $f'(\eta)$ when $S = 1, P_r = E_c = 0.5, \beta = 0.5, \delta = 0.2$

η	Present results	HAM [7]	$(R-K4)$
0.0	1.471094	1.471088	1.471094
0.1	1.457822	1.457816	1.457822
0.2	1.417831	1.417826	1.417831
0.3	1.350591	1.358507	1.350591
0.4	1.255222	1.255219	1.255222
0.5	1.130495	1.130494	1.130495
0.6	0.974839	0.974841	0.974839
0.7	0.786347	0.786351	0.786347
0.8	0.562787	0.562793	0.562787
0.9	0.301615	0.301621	0.301615
1.0	1.000000	1.000000	1.000000

Table 4
Comparison between new scheme and $(R-K4)$ scheme for the analytical solutions $\theta(\eta)$ when $S=1, P_r=E_c=0.5, \beta=0.5, \delta=0.2$

η	Present results	HAM [7]	$(R-K4)$
0.0	1.611826	1.608405	1.611826
0.1	1.610488	1.607067	1.610488
0.2	1.605995	1.602574	1.605995
0.3	1.596884	1.593464	1.596884
0.4	1.580624	1.577207	1.580624
0.5	1.553467	1.550065	1.553467
0.6	1.510220	1.506874	1.510220
0.7	1.443918	1.440742	1.443918
0.8	1.345375	1.342629	1.345375
0.9	1.202583	1.200787	1.202583
1.0	0.000000	0.000000	0.000000

Table 5
Comparison between new scheme and $(R-K4)$ scheme for the analytical solutions $f'(\eta)$ when $S=-1, P_r=E_c=0.5, \beta=0.5, \delta=0.2$

η	Present results	HAM [7]	$(R-K4)$
0.0	1.533303	1.533312	1.533303
0.1	1.516216	1.516264	1.516216
0.2	1.464734	1.465347	1.464734
0.3	1.378172	1.381234	1.378172
0.4	1.255405	1.265029	1.255405
0.5	1.118232	1.118232	1.118232
0.6	0.942707	0.942704	0.942707
0.7	0.740622	0.740614	0.740622
0.8	0.514391	0.514381	0.514391
0.9	0.266614	0.266604	0.266614
1.0	0.000000	0.000000	0.000000

Table 6
Comparison between new scheme and $(R-K4)$ scheme for the analytical solutions $\theta(\eta)$ when $S=-1, P_r=E_c=0.5, \beta=0.5, \delta=0.2$

η	Present results	HAM [7]	$(R-K4)$
0.0	1.737069	1.729214	1.737069
0.1	1.737069	1.727735	1.737069
0.2	1.731825	1.722486	1.731825
0.3	1.720341	1.711061	1.720340
0.4	1.698339	1.689567	1.698339
0.5	1.662089	1.652786	1.662089
0.6	1.603619	1.594420	1.603619
0.7	1.516245	1.507393	1.516245
0.8	1.392097	1.384217	1.392096
0.9	1.222867	1.217417	1.222867
1.0	1.000000	1.000000	1.000000

Table 7
Comparison between new scheme and $(R-K4)$ scheme for the analytical solutions $f(\eta)$, $f'(\eta)$, $\theta(\eta)$ when $S=0.5, P_r=E_c=0.1, \beta=0.3, \delta=0.2$

η	$f(\eta)$	$(R-K4)$	$f'(\eta)$	$(R-K4)$	$\theta(\eta)$	$(R-K4)$
0.00	0.000000	0.000000	1.489542	1.489542	1.037263	1.037263
0.05	0.074417	0.074417	1.485951	1.485951	1.037243	1.037243
0.10	0.148475	0.148475	1.475173	1.475173	1.037183	1.037183
0.15	0.221815	0.221815	1.457196	1.457196	1.037076	1.037076
0.20	0.294074	0.294074	1.431999	1.431999	1.036911	1.036911
0.25	0.364894	0.364894	1.399554	1.399554	1.036675	1.036675
0.30	0.433909	0.433909	1.359824	1.359824	1.036348	1.036348
0.35	0.500754	0.500754	1.312763	1.312763	1.035907	1.035907
0.40	0.565062	0.565062	1.258319	1.258319	1.035324	1.035324
0.45	0.626462	0.626462	1.196431	1.196431	1.034567	1.034567
0.50	0.684579	0.684579	1.127032	1.127032	1.033599	1.033599
0.55	0.739038	0.739038	1.050044	1.050044	1.032377	1.032377
0.60	0.789456	0.789456	0.965388	0.965388	1.030852	1.030852
0.65	0.835448	0.835448	0.872973	0.872973	1.028972	1.028972
0.70	0.876623	0.876623	0.772702	0.772702	1.026677	1.026677
0.75	0.912585	0.912585	0.664474	0.664474	1.023900	1.023900
0.80	0.942936	0.942936	0.548180	0.548180	1.020570	1.020570
0.85	0.967267	0.967267	0.423707	0.423707	1.016606	1.016606
0.90	0.985168	0.985168	0.290936	0.290936	1.011922	1.011922
0.95	0.996221	0.996220	0.149743	0.149743	1.006421	1.006421
1.00	1.000000	1.000000	0.000000	0.000000	1.000000	1.000000

Table 8
Comparison between new scheme and $(R-K4)$ scheme for the analytical solutions $f(\eta)$, $f'(\eta)$, $\theta(\eta)$ when $S=-0.5, P_r=E_c=0.1, \beta=0.3, \delta=0.2$

η	$f(\eta)$	$(R-K4)$	$f'(\eta)$	$(R-K4)$	$\theta(\eta)$	$(R-K4)$
0.00	0.000000	0.000000	1.510983	1.510983	1.039336	1.039336
0.05	0.075484	0.075484	0.149743	0.149743	1.006421	1.006421
0.10	0.150575	0.150575	1.495313	1.495313	1.039253	1.039253
0.15	0.224885	0.224885	1.475744	1.475744	1.039142	1.039142
0.20	0.298020	0.298020	1.448379	1.448379	1.038967	1.038967
0.25	0.369593	0.369593	1.413249	1.413249	1.038713	1.038713
0.30	0.439216	0.439216	1.370394	1.370394	1.038359	1.038359
0.35	0.506505	0.506505	1.319862	1.319862	1.037877	1.037877
0.40	0.571075	0.571075	1.261710	1.261710	1.037236	1.037236
0.45	0.632550	0.632550	1.196004	1.196004	1.036401	1.036401
0.50	0.690551	0.690551	1.122816	1.122816	1.035331	1.035331
0.55	0.744708	0.744708	1.042226	1.042226	1.033981	1.033981
0.60	0.794652	0.794652	0.954318	0.954318	1.032305	1.032304
0.65	0.840011	0.840011	0.859189	0.859189	1.030247	1.030247
0.70	0.880452	0.880452	0.756936	0.756936	0.027753	0.027753
0.75	0.915596	0.915596	0.647663	0.647663	1.024761	1.024761
0.80	0.945103	0.945103	0.531482	0.531482	1.021209	1.021209
0.85	0.968631	0.968631	0.408506	0.408506	1.017029	1.017029
0.90	0.985842	0.985842	0.278852	0.278852	1.012150	1.012150
0.95	0.996407	0.996407	0.142643	0.142643	1.006499	1.006499
1.00	1.0000000	1.000000	0.000000	0.000000	1.000000	1.000000

Table 9
Comparison between new scheme and HAMfor Numerical values for Nusselt number $\theta'(1)$.

S	β	P_r	E_c	δ	Present results	HAM[7]
-1.5	0.1	0.4	0.2	0.1	-2.748707	-2.735159
-0.5	0.1	0.4	0.2	0.1	-2.700024	-2.698735
0.5	0.1	0.4	0.2	0.1	-2.667891	-2.666799
1.5	0.1	0.4	0.2	0.1	-2.647043	-2.638843
0.5	0.3	0.4	0.2	0.1	-1.037015	-1.050966
0.5	0.5	0.4	0.2	0.1	-0.730584	-0.727899
0.5	0.1	0.1	0.2	0.1	-0.669832	-0.669604
0.5	0.1	0.3	0.2	0.1	-2.003768	-2.002994
0.5	0.1	0.5	0.2	0.1	-3.330131	-3.328686
0.5	0.1	0.4	0.1	0.1	-1.333945	-1.333399
0.5	0.1	0.4	0.2	0.1	-2.667891	-2.666799
0.5	0.1	0.4	0.3	0.1	-4.001836	-4.000199
0.5	0.1	0.4	0.2	0.2	-2.792068	-2.790951
0.5	0.1	0.4	0.2	0.3	-2.999028	-2.997870

7. Conclusion

In this paper, Casson fluid, i.e., heat transfer analysis for the squeezing flow of a non-Newtonian fluid between parallel circular plates is presented. New analytical technique to obtain analytical approximate solution of the physics problem is discussed . The solution obtained by new analytical technique is an infinite power series for appropriate initial approximation. The construction of this technique has a good convergent series and the convergence of the results is explicitly shown. The analysis of the converge confirms that the new analytical technique is an efficient technique as compared with Range- Kutta algorithm with help Shooting algorithm. Graphical results and tables were presented to investigate the influence of physical parameters on the velocity and temperature distributions. Important results illustrating the behavior of the velocity and temperature distributions curves on the physical parameters obtained from the drawings are listed below

i. The velocity distribution increases with the increase of S while the temperature distribution decreases with the increase of S.
ii. The velocity distribution increases with the increase of Casson fluid number β and the temperature distribution decreases with the increase of Casson fluid number β.
iii. The temperature distribution increases with the increase of Prandtl number P_r and Eckert number E_c.
iv. Results obtained by new analytical technique are in excellent agreement with numerical solution obtained.

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