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# Numerical Simulation of Heat Transfer using Finite Element Method

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**ABSTRACT**

The Finite Element Method (FEM) is a method that is widely used in engineering and mathematical modelling. Some of the problems that can be solved using this method are structural analysis, electromagnetic field, and heat transfer. Typically heat transfer problems can be in the form of irregular shapes or complex geometry domains. However, it is very difficult to solve this problem analytically due to its irregular domain or complex geometry. Thus, the domain needs to be partitioned into a smaller size before it can be solved systematically. This research is aimed to solve the heat transfer problem in simple 2D irregular geometry by applying FEM using the approach of Galerkin's method.

## 1. Introduction

A numerical approach is a method for obtaining approximate solutions to various engineering problems. The need for numerical methods emerges from the fact that analytical solutions for most practical engineering problems do not exist [1]. They are also difficult to be solved even though they might have a solution usually from the governing equation and boundary conditions [2]. One of the most used problems in the engineering field is thermal study, which involves a heat transfer equation related to partial differential equations (PDEs) [3,4]. Heat transfer is a study of thermal energy transport within a medium of molecular interaction, fluid motion and electromagnetic waves resulting from a spatial variation in temperature [5,6]. In this study, we were focus on the steady state heat conduction problem, which is independent of time. The heat conduction problem is presented as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+q=0 \qquad (1)$$

where $T$ is the temperature and $q$ represent an equation for heat source [7]. There are numerous applications of heat conduction in modern technology, namely, in geological sciences, mechanical engineering, and many other fields. Some examples are cooling fins or extended surfaces, solidification and melting of metals, welding, metal cutting, nuclear heating, and heating and cooling

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buildings [8]. Regarding all of these applications, the governing equation and boundary condition for each problem were initially solved using empirical methods. However, this method is quite complicated. For better and more practicality in solving this problem, the numerical method can be the alternate solution which is much easier and time-saving.

Most commonly encountered numerical methods including Finite Difference Method (FDM), Finite Volume Method (FVM), and Finite Element Method (FEM). FVM and FEM are suited for unstructured meshing and FEM is highly recognized in solving complex boundary value problems. However, experimental and analytical approaches are needed when it comes to the validation stage of the numerical methods [9]. From all of those methods, solving the heat transfer problem using FEM will be our main interest. Many studies have been found that FEM can be used to solve the heat conduction problem. In the research papers by Papathanasiou *et al.,* [10], Azmi [11], Choi *et al.,* [12] and Ngarisan *et al.,* [13], FEM was used to approximate the differential equation of boundary value problem for heat conduction in two-dimensional space.

Basically, the fundamental idea of the FEM is to discretise the domain into several sub-domains, or finite elements [17]. The elements will be calculated by a small partition of mesh generation and convert PDE problem before reassembling them as nodes into the initial domain [18]. The mesh generation can be formed from various types of shapes namely triangular and rectangular or quad mesh for two-dimensional geometry. By referring to Rao [14], Rosero and Zsaki [15] and Devi *et al.,* [16] different type of mesh and higher order elements produced different results. FEM provides general techniques in discretising the algorithm for approximating any differential equation problem that the computer can automatically read. However, it might need human involvement in developing the programming solution that requires mathematical skills [19]. The calculation for this research then will be conducted using MATLAB. The aim of this research mainly highlights the application of FEM for solving heat transfer in two-dimensional irregular geometry besides comparing with the previous study in terms of results and meshing method.

The work is organized as follows. In Section 2, the shape model of two-dimensional irregular geometry, boundary value problem, and mathematical model in this research were proposed. Besides, the basic step using FEM including pre-processing, post-processing, and processing were also discussed in this section. In Section3, the results verification was done with a previous research paper and some discussion on the comparison of the meshing method. Last but not least, in Section 4, the conclusion and recommendation for future works are discussed.

## 2. Methodology

A heat conduction problem was implemented in simple 2D irregular geometry which is shown in Figure 1. The mathematical model for the heat conduction problem is given as:

$$k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} = -q \tag{2}$$

where $k$ is the material's thermal conductivity, which is constant and set as 1, $q$ is the additional heat source on ($x_q$,$y_q$) and $T$ is the temperature. The range of $x$ and $y$ for the problem is given as $0 \le x \le 3$, $0 \le y \le 2$. Equation of $q$ and boundary conditions are given as

$$
\begin{aligned}
&q = \frac{f}{(x-x_Q)^2-(y-y_Q)^2+0.1} && \text{where } f \text{ is constant} \\
&\frac{\partial T}{\partial x} = 0 && \text{at the insulted part} \\
&T_0 = 0 && \text{at the upper part} \\
&T_1 = 100 && \text{at the lower part}
\end{aligned} \tag{3}
$$

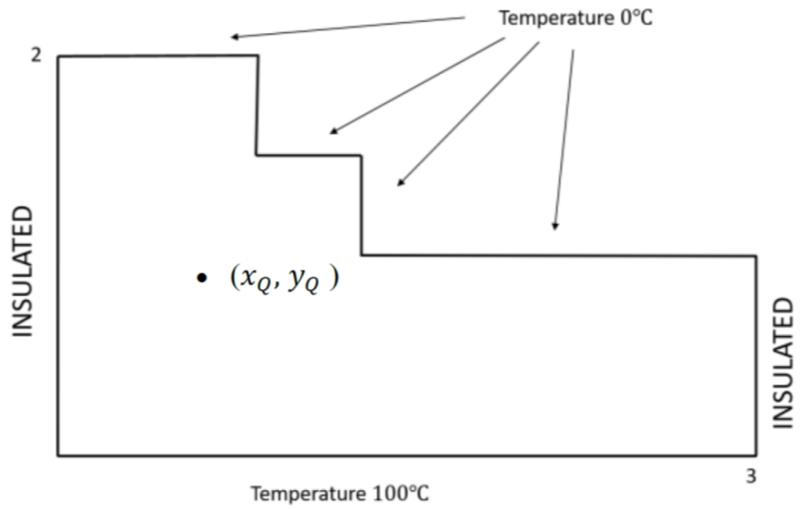


**Fig. 1.** Simple 2D irregular geometry heat conduction problem

### *2.1 Flow Chart of Finite Element Method*

The flow chart for the whole Finite Element Method in this problem is shown in Figure 2. It represents and simplified all important steps beginning with pre-processing, processing and post-processing parts as described by Hutton [20]. This flow chart is modelled based on the approach of Galerkin Finite Element Method that used to solve this problem.

### *2.2 Pre-processing*

Pre-processing part is the initial step in solving a problem using FEM. Following are the steps that have been used to determine the solution to the problem.

(a) Analyse Geometry, Boundary and Initial Condition

The geometry, boundary condition and initial condition of the problem given in the previous part will be studied and later used in the processing part to solve the equation system.

(b) Generating rectangular mesh

The irregular geometry then divided into parts or elements to form grid representation according to the rectangular mesh implemented in this problem. Note that mesh is represented by 3 x *n*size (*x*-axis) and 2 x *n*size (*y*-axis), where *n*size = 2 has been used to solve this problem. The mesh and nodes representation of every element is shown in Figure 3 and Figure 4, respectively. From the mesh, 28 nodes and 17 elements were obtained. Note that every element generated in this geometry are of the same size.

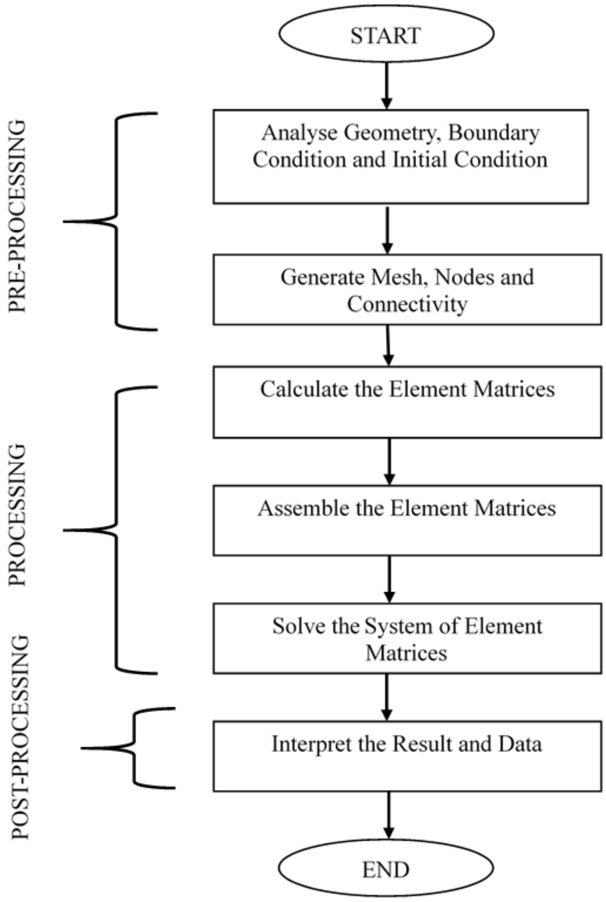


**Fig. 2.** Flow Chart for Finite Element Method

(c) Generate Nodes Connectivity

The connection of all nodes and elements is tabulated as shown in Table 1 for simplification and reference used in the processing part. Since rectangular mesh has been used in this project, every element must consist of 4 nodes from 4 basis local nodes (1, 2, 3 and 4), leading to an abundance of global nodes. For example, the connected nodes are 1, 2, 9 and 8 in the anti-clockwise direction for element 1. For element 2, the connected nodes are 2, 3, 10 and 9 which is also the same in the anti-clockwise rotation.

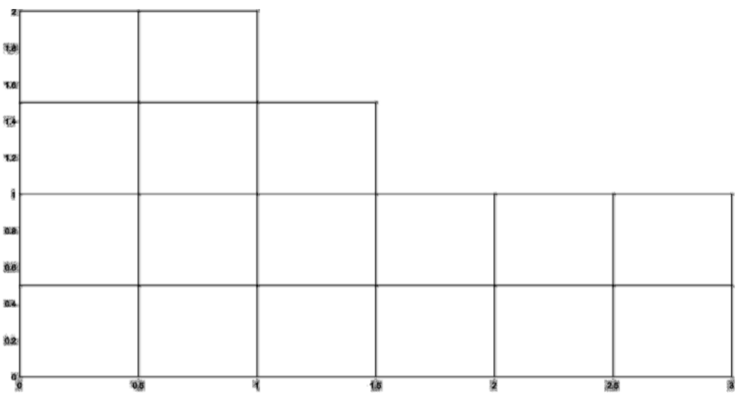

**Fig. 3.** Mesh Generation for irregular geometry

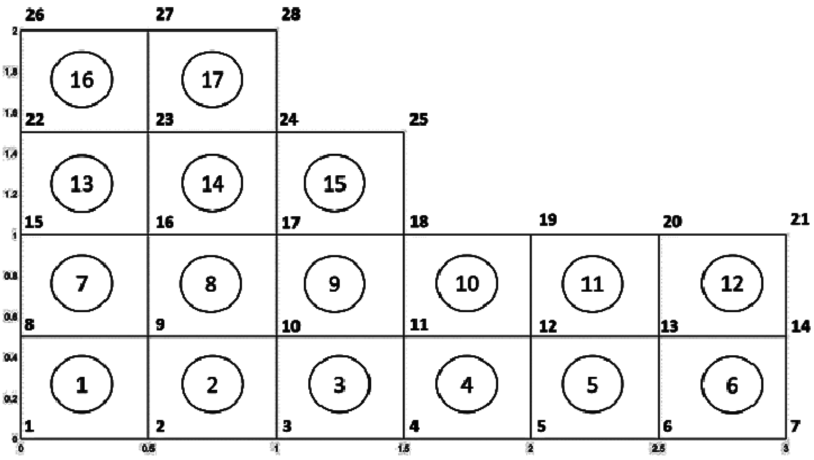


**Fig. 4.** 28 nodes and 17 elements generated

**Table 1**
Elements and Nodes Connectivity

| Elements | 1 | 2 | 3 | 4 | LOCAL NODES |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 9 | 8 | GLOBAL NODES |
| 2 | 2 | 3 | 10 | 9 | |
| 3 | 3 | 4 | 11 | 10 | |
| 4 | 4 | 5 | 12 | 11 | |
| 5 | 5 | 6 | 13 | 12 | |
| 6 | 6 | 7 | 14 | 13 | |
| 7 | 8 | 9 | 16 | 15 | |
| 8 | 9 | 10 | 17 | 16 | |
| 9 | 10 | 11 | 18 | 17 | |
| 10 | 11 | 12 | 19 | 18 | |
| 11 | 12 | 13 | 20 | 19 | |
| 12 | 13 | 14 | 21 | 20 | |
| 13 | 15 | 16 | 23 | 22 | |
| 14 | 16 | 17 | 24 | 23 | |
| 15 | 17 | 18 | 25 | 24 | |
| 16 | 22 | 23 | 27 | 26 | |
| 17 | 23 | 24 | 28 | 27 | |

### *2.3 Processing*

The processing part is the most crucial steps in FEM where all calculations and equations are involved in getting the results. Basically, the calculation of this method will be computed in the matrix system due to the abundance of equations and evaluated via MATLAB Programming software. Following are the steps in this processing part.

(a) Calculate the Element Matrices

Element matrices and calculations in this project were performed based on the equations derived using the Galerkin approach. The scalar element for 4-nodes is given as

$$T = N_1 T_1 + N_2 T_2 + N_3 T_3 + N_4 T_4$$
$$T = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \tag{4}$$
$$T = \{N\}\,\{T\}^T$$

where $N$ is the shape function for rectangular elements and $T$ is the displacement vector or the degree of freedom for this function. The strong form of heat conduction in Eq. (5) is obtained from the mathematical model with applied weighted residual given as follows

$$\frac{\partial^2 \mathrm{T}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{T}}{\partial \mathrm{y}^2} = -q$$
$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} T = -q \tag{5}$$
$$\{\partial\}\,\{\partial\}^T\, T = -q$$
$$\{\partial\}\,\{\partial\}^T \{N\}\,\{T\}^T = -q$$
$$\{\partial\}\,\{\partial\}^T \{N\}\,\{T\}^T + q = 0 = R$$

By using Galerkin residual method given by

$$\int_x \int_y \{N\}^T\, R\, dydx = 0 \tag{6}$$

the governing equation yields a weak form given by

$$\int_x \int_y \{N\}^T\, \{\partial\}\,\{\partial\}^T \{N\}\,\{T\}^T dydx = \int_x \int_y \{N\}^T q\;\, dydx + \int_s \{N\}^T\, q_n \tag{7}$$

where $q_n$ is the natural boundary condition in which $q_n = \frac{\partial T}{\partial x} = 0$, which can be neglected. Hence,

$$\int_x \int_y \{N\}^T\, \{\partial\}\,\{\partial\}^T \{N\}\,\{T\}^T dydx = \int_x \int_y \{N\}^T q\;\, dydx$$
$$k_T = r_T$$
$$k_T = \int_x \int_y \{N\}^T\, \{\partial\}\,\{\partial\}^T \{N\}\, dydx \tag{8}$$
$$r_T = \int_x \int_y \{N\}^T q\;\, dydx$$

where $k_T$ is the stiffness equation for every element and $r_T$ is the force vector.

(b) Assemble the Element Matrices

Every element matrices that have been calculated will be assembled and combined together into the global system using the following function

$$K_e = \textstyle\sum_e\, K_e + k_T$$
$$R_e = \textstyle\sum_e\, R_e + r_T \tag{9}$$

where $K_e$ is the assemble of the global stiffness element and $R_e$ is the assemble of the global force vector.

(c) Solve the System of Element Matrices

The system of element matrices for this problem can be solved using

$$K_e T = R_e \tag{10}$$

### *2.4 Post-Processing*

Data and results obtained will be compared to the existing results, which implemented the same boundary problem and condition with a slightly different on model shape and meshing method.

#### *2.4.1 Problem model*

The shape of the model presented by Ngarisan *et al.,* [13] was slightly different on the B4 part, as shown in Figure 5, where the triangular mesh is used with 28 nodes that can be seen in Figure 6 compared to this research using the rectangular mesh method. However, the problem remains the same with boundary conditions and additional heat source, *q* at the same spot, ($x_q$, $y_q$).

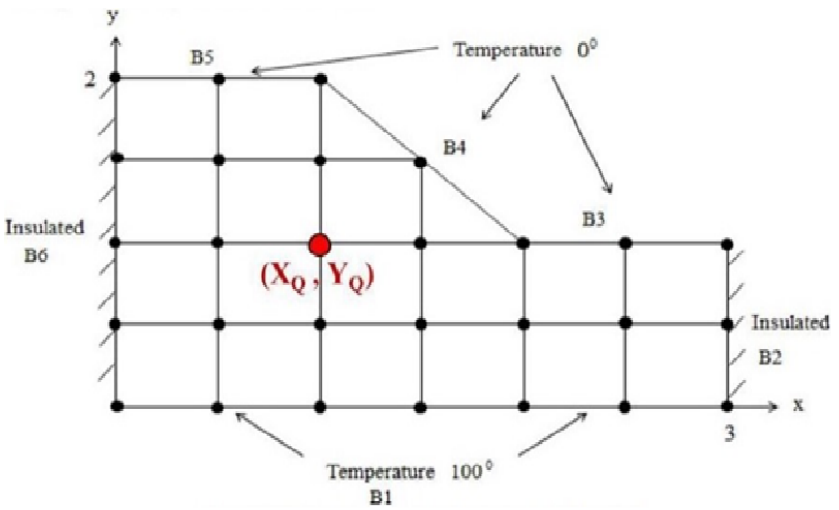


**Fig. 5.** Problem Model by Ngarisan *et al.,* [13]

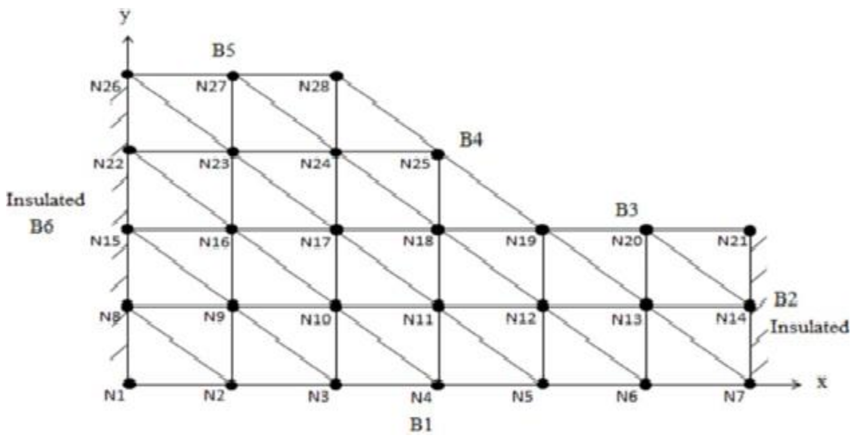


**Fig. 6.** Triangular mesh by Ngarisan *et al.,* [13]

## 3. Results and Discussion

### *3.1 Verification*

To validate our results, the outcome of this study will be compared with a previous research paper by Ngarisan *et al.,* [13] which used the triangular mesh method instead of the rectangular mesh method. The results were obtained by putting the additional heat source, $q$ at $(x_q, y_q)$ that set in a circular shape which is given as

$$q = \frac{f}{\left(x - x_q\right)^2 + \left(y - y_q\right)^2 + 0.1}$$

where $f$ is the manipulated variable set to 10, -10, 100 and -100 to observe the heat distribution and heat flux differences when the value is positive and negative.

(a) Value of $f$ = 10

In Figure 7, the heat distribution pattern for both mesh methods are about the same, where the heat with higher degree temperature is at the lower part and lower degree temperature at the upper part. It can be said that heat moves from the bottom and rise to the top of the shape, respectively. There is a slight difference in heat distribution for both methods. If we look closely at the middle area, $y$ = 1, where an additional heat source, $q$, is projected, the temperature for the rectangular mesh method is about 50 °C. In comparison, the triangular mesh is 46.7 °C and below. The temperature at the bottom of the shape for the rectangular mesh the is 100 °C while the triangular mesh is 93.3 °C which is the maximum temperature for both methods. However, on the top area of the shape recorded the same temperature, which is 0 °C due to boundary conditions that have been given.

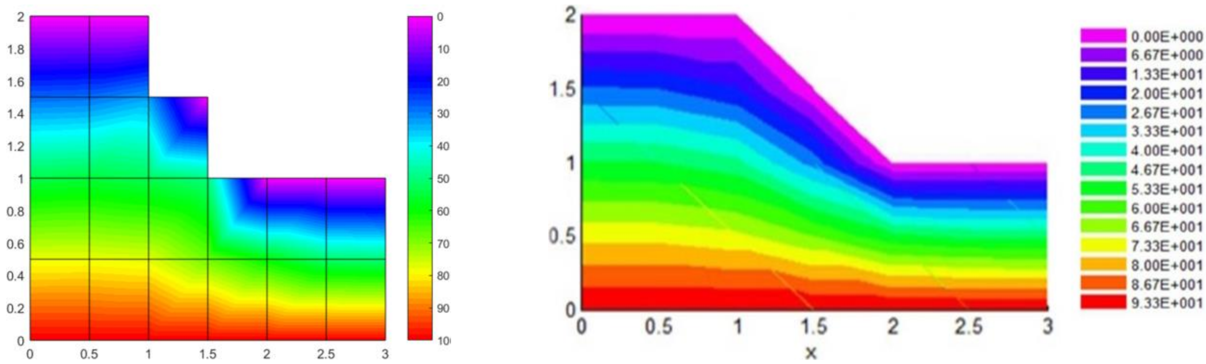


**Fig. 7.** Heat Distribution in Rectangular Mesh FEM (left) and Triangular Mesh FEM for f = 10 (right) by Ngarisan *et al.,* [13]

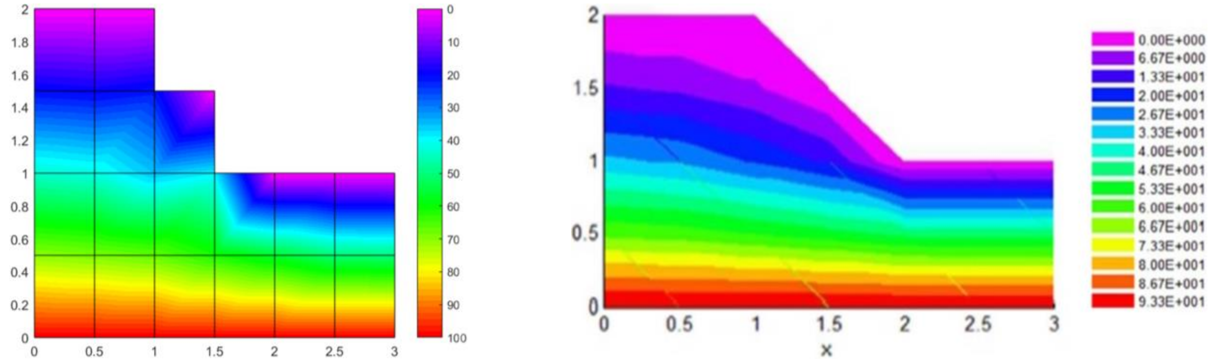


**Fig. 8.** Heat Distribution in Rectangular Mesh FEM (left) and Triangular Mesh FEM for f = -10 (right) by Ngarisan *et al.,* [13]

(b) Value of *f* = -10

The same goes with the value of *f* = -10. The pattern of heat distribution pattern is the same where the heat moves from the bottom to the top of the shape for both methods, as shown Figure 8. The maximum and minimum temperature for the rectangular mesh method is 100 °C and 0 °C, while the triangular mesh method is 93.3 °C and 0 °C, respectively.

(c) Value of *f* = 100

For *f* = 100, the heat distribution pattern for both methods shown in Figure 9 shows that the heat is extremely hot at the middle area and distributed to other parts with lower temperatures. The big difference can be seen in the middle part for the triangular mesh method, where the heat is distributed evenly at the centre, forming a square shape. Meanwhile, in the rectangular mesh method, the maximum temperature is concentrated at one point only (see *x* = 1, *y* = 1). For a better comparison, we take a look at point (*x* = 0, *y* = 1). The temperature at the given point for the rectangular mesh method is about 100 °C. However, the triangular mesh method yields 130 °C, in which there exists a huge gap between the two methods. On the other hand, the maximum temperature for the rectangular mesh method is 134 °C, while the triangular mesh method is 140 °C. The minimum temperature for both methods remain the same which is 0 °C at the upperpart of the shape. However, if we compare roughly the whole heat map, the result is still acceptable where the heat distribution is still delivered at the right place for every element of the shape.

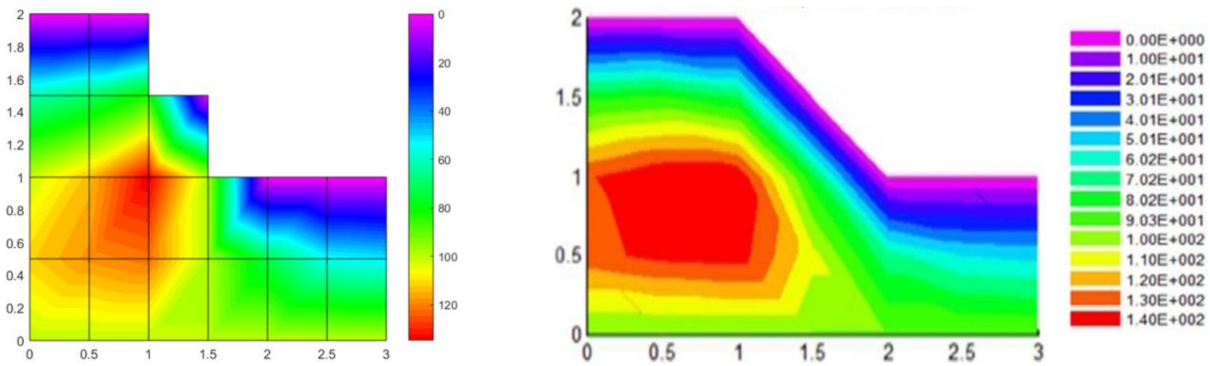


**Fig. 9.** Heat Distribution in Rectangular Mesh FEM (left) and Triangular Mesh FEM for *f* = 100 (right) by Ngarisan *et al.,* [13]

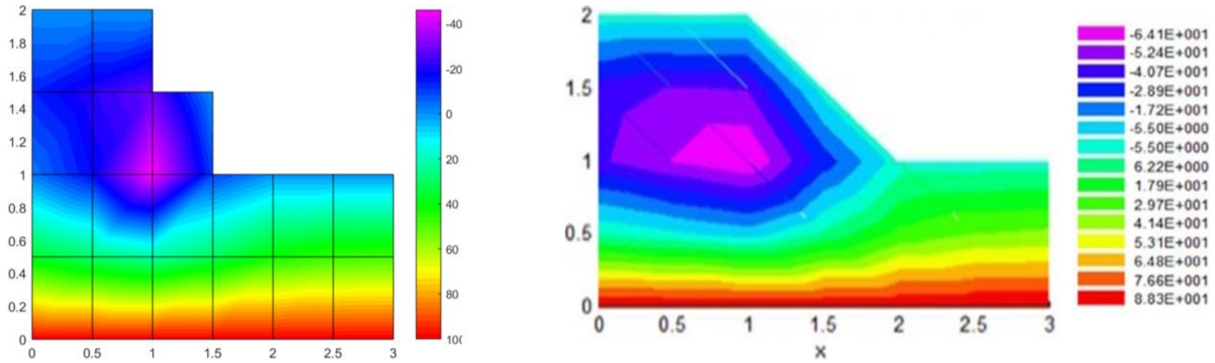


**Fig. 10.** Heat Distribution in Rectangular Mesh FEM (left) and Triangular Mesh FEM for f = -100 (right) by Ngarisan *et al.,* [13]

(d) Value of *f* = -100

For the value of *f* = -100, both methods show the same distribution pattern according to the comparison of the heat map shown in Figure 10. We can see that the heat distribution in the middle area is extremely cold in both methods. Minimum temperature for rectangular mesh method is -46 °C, while for the triangular mesh method is -64.1 °C. As for maximum temperature, each method shows different results which is 100 °C for the rectangular mesh method and 88.3 °C for the triangular mesh method.

*3.2 Triangular vs Rectangular Mesh*

In comparison to element configuration in this study, the rectangular mesh created 17 elements whereas the triangular mesh created 36 elements overall. However, both methods generated 28 nodes respectively, thus make them comparable in a good way. Moreover, the shape of the domain is different at a certain part between both methods due to several factors, mainly caused by the shape selection of the geometry. The shape was a little bit distinct on the slope at B4 part and this can be seen in Figure 5. Since both methods used a simple structure, so the elements that are formed are quite stiffer and rigid, so the slope was removed for the simplification of rectangular mesh. Otherwise, the mesh refinement is needed for the rectangular mesh to be in the exact shape [13]. Therefore, increasing the number of nodes to compare both methods is unsuitable.

For results accuracy, the question is more likely on which method is more accurate in solving this problem. Thus, the data must rely on the agreement with the experimental or analytical solution. Since there is no experiment conducted for this research, the accuracy of the approach remains silent. But referring to Gokhale [21], quad or rectangular element is more accurate than triangular element due to better interpolation function. Apparently, the results comparison shows a good explanation about heat transfer problem even though there might be slightly different temperature on certain part of the nodes.

## 4. Conclusions

FEM are applicable to solve heat transfer problem in two-dimensional irregular geometry. The results obtained show a good and clear explanation about heat simulation involving heat conduction problems using FEM. The heat moves from hot to cold area and distributes evenly all over the space for every value of *f* in the additional heat source, *q* that has been set. There are many ways in determining solutions using FEM regarding element mesh of the shape, for instance, triangular mesh and rectangular mesh. It depends on the structure of the geometry problem so that it is easier to mesh up the shape and get the solution. Rectangular mesh FEM was implemented in this study and compared with a previous research paper by Ngarisan *et al.,* [13], which proposed a triangular mesh approach. As a result, the comparison shows good agreement between them in terms of the heat distribution pattern and temperature. During conducted this research, we encountered some issues especially in method accuracy, the analytical solution might need for a solid justification. In continuous of the further study, it is recommended to explore more on mesh refinement and optimization.

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