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Fractional Model for the Flow of Brinkman-Type Fluid with Mass Transfer

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ABSTRACT

The flow of fluid with mas transfer is an important phenomenon in the industries and nature. A new scheme for formulating the Caputo time fractional model for the flow of Brinkman-type fluid between the plates was introduced using the generalized Fick's law. Within a channel, the free convection flow of the electrically conducted Brinkman-type fluid was considered. The governing equations were solved by the techniques of Fourier sine and the Laplace transform. In terms of the special function, namely, the Mittag-Leffler function, final solutions were obtained. To explain the conceptual arguments of the embedded parameters, separate plots are represented in figures. It is worth noting that for increasing the values of the Brinkman-type fluid parameter, the velocity profile decreases.

1. Introduction

Because of its flexible and special properties, fractional calculus has evolved tremendously nowadays. The non-integer derivatives of all the orders are solved utilizing fractional calculus techniques. Fractional calculus has a history of around three centuries. Fractional calculus is a versatile and vital method to explain many processes, including memories [1, 2]. In recent years, fractional calculus has been used for many applications in different areas, such as electrochemistry, ground-level water distribution, electromagnetism, elasticity, diffusion, and heat stream conduction [3-5]. Many researchers use the fractional derivatives approach for the flow problems, like Ali, *et al.,* [6] and Ali, *et al.,* [7]. Another approach of fractional derivatives is Caputo-Fabrizio fractional derivatives [8] which are also termed a derivative with a non-singular kernel in the literature. The numerical and analytical solvers use this approach for different phenomena in real life. For different situations and analyses, CF derivatives are widely used, like Ali, *et al.,* [9], Fetecau, *et al.,* [10], and Sheikh, *et al.,* [11]. The modern concept of fractional derivatives using the generalized exponential functions (Mittag Leffler function) was created in 2016 by Atangana and Baleanu [12]. The kernel of integral associated with that derivative is nonlocal and non-singular. Atangana and Koca [13] used the modern concept of a fractional derivative to a simple non-linear system to demonstrate the

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presence and uniqueness of a solution for the problem. Afterward, some authors used a fractional derivative of Atangana-Baleanu in their study, for instance, Sheikh *et al.,* [14, 15].

In the manufacturing and scientific fields, the physical properties of non-Newtonian fluids strongly affect [16-22]. Moreover, in magnetohydrodynamic (MHD) flows, heat transfer effects include a wide variety of applications from geosciences to engineering and the chemical sciences field [23-25]. Non-Newtonian fluids demonstrate a complex process that ultimately needs to be described and represented by mathematical modeling. In a porous medium or clay water contact, this phenomenon of non-Newtonian flow becomes much more complex [18, 19]. Darcy [26] identified the theoretical analysis and the respective mathematical model of the viscous fluid flow through a medium containing pore. In general, Darcy's implemented law can describe the flows passing over a low permeable region. But Darcy's law is not practical and applicable to fluids that move through a medium with high porosity, so Brinkman's model [27] is valid and suitable for such liquids. Extensive research has been done using the Brinkman model on fluid flow through a porous medium. Using the Brinkman-type fluid model, many researchers have presented their studies, for example, Ali, *et al.,* [28], Jan, *et al.,* [29], Aamina, *et al.,* [30], Ali, *et al.,* [31].

This article considers the unsteady MHD flow of Brinkman-type fluid with mass transfer, keeping in mind the above discussion. Using the Caputo time-fractional derivative, the governing equations are translated into fractional PDEs. With the joint applications of the Laplace and Fourier sine transformations technique, the equations are solved.

2. Mathematical Modelling

We considered Brinkman-type fluid motion in a vertical channel. It is assumed that the flow is in the x -axis direction, while the y-axis is taken perpendicular to the plates. With ambient concentration C_1, both the fluid and plates are at rest when $t \le 0$. At $t = 0^+$, the plate at $y = d$ starts to move with velocity $Uh(t)$ in its own plane. At $y = d$, the rate of concentration at the plate rose over time t to $C_1 + (C_2 - C_1)g(t)$.

The free convection flow of fluid of the Brinkman-type, along with the mass transfer, is regulated by the following partial differential equations [30]

$$\rho \frac{\partial u(y,t)}{\partial t} + \beta_r u(y,t) = \mu \frac{\partial^2 u(y,t)}{\partial y^2} \tag{1}$$
$$-\sigma B_0^2 u(y,t) + \rho \beta_C g(C - C_1),$$

$$\frac{\partial C(y,t)}{\partial t} = -\frac{\partial j(y,t)}{\partial y}, \tag{2}$$

$$j(y,t) = -D \frac{\partial C(y,t)}{\partial y}, \tag{3}$$

with the initial and boundary conditions

$$\left.\begin{aligned} u(y,0)=0, \qquad & C(y,0)=C_1, \\ u(0,t)=0, \qquad & C(0,t)=C_1, \\ u(d,t)=Uh(t), \quad & C(d,t)=C_1+\left(C_2-C_1\right)g\left(t\right), \end{aligned}\right\}. \tag{4}$$

Introducing the following dimensionless variables

$$v=\frac{u}{U},\ \xi=\frac{y}{d},\ \tau=\frac{\nu}{d^2}t,\ \Phi=\frac{C-C_1}{C_2-C_1},\ \lambda=\frac{jd}{D\left(C_2-C_1\right)},$$
$$g(\tau)=g\left(\frac{d^2}{\nu}t\right),\ h(\tau)=h\left(\frac{d^2}{\nu}t\right).$$

into Eqs. (2)-(6), we get

$$\frac{\partial v(\xi,\tau)}{\partial\tau}=\frac{\partial^2 v(\xi,\tau)}{\partial\xi^2}-\mathrm{A}v(\xi,\tau)+Gm\Phi(\xi,\tau), \tag{5}$$

$$\frac{\partial\Phi(\xi,\tau)}{\partial\tau}=-\frac{1}{Sc}\frac{\partial\lambda(\xi,\tau)}{\partial\xi}, \tag{6}$$

$$\lambda(\xi,\tau)=-\frac{\partial\Phi(\xi,\tau)}{\partial\xi}, \tag{7}$$

$$\left.\begin{aligned} v(\xi,0)=0, \quad & \Phi(\xi,0)=0, \\ v(0,\tau)=0, \quad & \Phi(0,\tau)=0, \\ v(1,\tau)=h(\tau), \quad & \Phi(1,\tau)=g(\tau), \end{aligned}\right\}, \tag{8}$$

where $Gm=\frac{gd^2\beta_C}{\nu U}\left(C_2-C_1\right)$ is the mass Grashof number, $\gamma_B=\dfrac{\beta_r d^2}{\nu}$ is the Brinkman-type fluid parameter, $M=\frac{\sigma B_0^2 d^2}{\mu}$ is Hartman number, $Sc=\frac{\nu}{D}$ is the Schmidt number and $\mathrm{A}=M+\gamma_B$.

3. Fractional Model

The generalized laws of Fourier and Fick are utilized as follow to establish a fractional model for the convective part of the referred flow problem

$$\lambda(\xi,\tau)=-{}^{C}\wp_\tau^{1-\alpha}\left(\frac{\partial\Phi(\xi,\tau)}{\partial\xi}\right);\ 0<\alpha\leq 1, \tag{9}$$

where ${}^{C}\wp_\tau^{\alpha}(.)$ is the time fractional operator developed by Caputo [32] and is described as [6]

$$ {}^{C}\wp_t^{\alpha} r(y,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \dot{r}(y,s)(t-s)^{-\alpha} ds \\ = \eta_\alpha(t) * \dot{r}(y,t);\ 0 < \alpha \le 1, \tag{10} $$

here $\eta_\alpha(t) = \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}$ is the singular Power law kernel. Furthermore,

$$ L\{\eta_\alpha(t)\} = \frac{1}{s^{1-\alpha}}, \{\eta_{1-\alpha} * \eta_\alpha\}(t) = 1, \\ \eta_0(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1, \eta_1(t) = L^{-1}\{1\} = \zeta(t), \tag{11} $$

here $L\{.\}$ is the Laplace transform, $\zeta(.)$ is the Dirac's delta function and s is the Laplace transform parameter.

Using the above properties and the second form Eq. 10, it is convenient to show that

$$ {}^{C}\wp_t^{0} r(y,t) = r(y,t) - r(y,0), \tag{12} $$

$$ {}^{C}\wp_t^{1} r(y,t) = \frac{\partial r(y,t)}{\partial t}. \tag{13} $$

Utilizing the definition of Caputo time fractional operator form Eq. (10), Using Eqs. (6 and 9) we arrived at

$$ \frac{\partial \Phi(\xi,\tau)}{\partial t} = \frac{1}{Sc} {}^{C}\wp_\tau^{1-\alpha}\left(\frac{\partial^2 \Phi(\xi,\tau)}{\partial^2 \xi}\right), \tag{14} $$

We recalled the time fractional integral operator to get the finest form for the last two equations

$$ \Im_t^{\alpha} r(y,t) = (\eta_{1-\alpha} * r)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t r(y,s)(t-s)^{\alpha-1} ds, \tag{15} $$

This is the inverse operator of the derivative operator ${}^{C}\wp_t^{\alpha}(.)$. Using the properties from Eq. (11) we have

$$ \left(\Im_t^{\alpha} \circ {}^{C}\wp_\tau^{\alpha}\right) r(y,t) = \Im_t^{\alpha}\left({}^{C}\wp_\tau^{\alpha} r(y,t)\right) \\ = \left[\eta_{1-\alpha} * \left(\eta_\alpha * \dot{r}\right)\right](t) = \left[(\eta_{1-\alpha} * \eta_\alpha) * \dot{r}\right](t) \\ = \left[1 * \dot{r}\right](t) = r(y,t) - r(y,0), \tag{16} $$

$$ \Rightarrow \left(\Im_t^{\alpha} \circ {}^{C}\wp_\tau^{\alpha}\right) r(y,t) = r(y,t) \text{ if } r(y,0) = 0. \tag{17} $$

Using the property,

$\Im_t^{1-\alpha}\dot{r}(y,t)=\left(\eta_\alpha*\dot{r}\right)(t)={}^{C}\wp_t^{\alpha}r(y,t)$, Eq. (14) can be written as

$$ {}^{C}\wp_\tau^{\alpha}\Phi(\xi,\tau)=\frac{1}{Sc}\frac{\partial^2\Phi(\xi,\tau)}{\partial^2\xi}, \tag{18}$$

4. Solution of the Problem
4.1 Concentration Field

Using the following transformation

$$\Psi(\xi,\tau)=\Phi(\xi,\tau)-\xi g(\tau), \tag{19}$$

Eq. (18) takes the form

$$ {}^{C}\wp_\tau^{\alpha}\Psi(\xi,\tau)-\xi\,{}^{C}\wp_\tau^{\alpha}g(\tau)=\frac{1}{Sc}\frac{\partial^2\Psi(\xi,\tau)}{\partial^2\xi}, \tag{20}$$

With the corresponding initial and boundary conditions as

$$\Psi(\xi,0)=0,\ \Psi(0,\tau)=0,\ \Psi(1,\tau)=0. \tag{21}$$

Applying the Laplace and Fourier sine transform, we get

$$\bar{\Psi}_F(n,s)=s\bar{g}(s)\frac{(-1)^n}{n\pi}\frac{s^{\alpha-1}}{s^\alpha+\frac{(n\pi)^2}{Sc}}, \tag{22}$$

Inverting the integral transformations of Eq. (22), we have

$$\begin{aligned}\Psi(\xi,\tau)=&\,2\sum_{n=1}^{\infty}\frac{(-1)^n\sin(\xi n\pi)}{n\pi}\\&\times\int_0^\tau\dot{g}(\tau-t)E_{\alpha,\alpha-1}\left(-\frac{(n\pi)^2}{Sc}t^\alpha\right)dt,\end{aligned} \tag{23}$$

The final solution for the concentration equation is

$$\Phi(\xi,\tau)=\Psi(\xi,\tau)+\xi g(\tau), \tag{24}$$

4.2 Velocity Profile

Applying the Laplace and Fourier transform to Eq. (5) using Eq. (8) we arrived at

$$\overline{v}_F(n,s)=\frac{(-1)^{n+1}\overline{h}(s)}{n\pi}+\left(\frac{\Re_5}{s}+\frac{\Re_6}{s+\Re_1}\right)\times\frac{(-1)^n s\overline{h}(s)}{n\pi}+\frac{Gm\overline{\Phi}_F(n,s)}{(s+\Re_1)},\tag{25}$$

where

$$\Re_1=\mathrm{A}+(n\pi)^2,\Re_2=\Re_1(n\pi)^2,\Re_3=1-\frac{(n\pi)^2}{\Re_1},$$
$$\Re_4=\Re_1-(n\pi)^2,\Re_5=\frac{\Re_4}{\Re_1},\Re_6=\frac{(\Re_3\Re_1+\Re_2\Re_1+\Re_4)}{\Re_1},$$

Inverting the Laplace and Fourier sine transformations the final solution is

$$v(\xi,\tau)=h(\tau)\xi+2\sum_{n=1}^{\infty}\frac{\sin(\xi n\pi)(-1)^n}{n\pi}\dot{h}(\tau)*\begin{pmatrix}\Re_5 H(\tau)\\+\Re_6\exp(-\Re_1\tau)\end{pmatrix}$$
$$+2\frac{\sin(\xi n\pi)Gm}{\Re}\sum_{n=1}^{\infty}\left(\begin{array}{l}\frac{(-1)^n}{n\pi}\exp(-\Re_1\tau)\\ *\left(\begin{array}{l}\int_0^\tau\dot{g}(\tau-t)E_{\alpha,\alpha-1}\left(-\frac{(n\pi)^2}{Sc}t^\alpha\right)dt\\+g(\tau)\end{array}\right)\end{array}\right),\tag{26}$$

here $H(\tau)$ is the unit step function and $E_{a,b}(.)$ is the Matage-Leffler function [33].

5. Sherwood Number

The gradient of mass concentration is termed as Sherwood number. In nondimensional form Sherwood number is given by

$$Sh=\left.\frac{\partial\Phi(\xi,\tau)}{\partial\xi}\right|_{\xi=1}.\tag{27}$$

6. Results and Discussion

In this analysis, we address the unsteady flow of Brinkman-type fluid in a vertical channel. Figures demonstrated the effect of different embedded parameters on the velocity and concentration profiles.

The influence of fractional parameter α has been shown in Figure 1. From this figure, the fluid velocity is the increasing function of the fractional parameter. The memory effects are also clearly shown in this figure. All the physical parameters are kept constant and only the fractional parameter is variated. Four different lines are drawn for different values of the fractional parameter for a fixed value of time. Form this figure we can say that while fixing all the physical parameter constant we can obtain different values for velocity to adjust our results with the experimental data, this phenomenon cannot be addressed in classical models due to the absence of memory effect. The variations in velocity profile due to change in Brinkman-type fluid parameter are depicted in Figure 2. From this figure, exciting results are obtained. The velocity decreases with increasing values of the material parameter γ_B . Physically, the increase in this parameter means the increase in the porosity of the media, through which the fluid flow. The influence of fractional α parameter on the concentration profile is shown in Figure 3. The figure shows that the concentration increasing function of the fractional parameter.

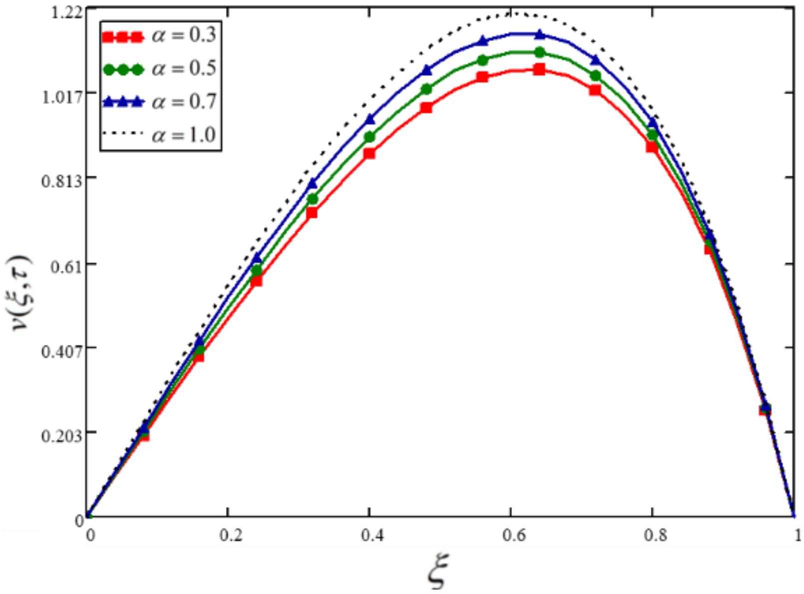

Fig. 1. Variations in velocity profile against ξ if $h(\tau)=0$ for different values of α when $\tau=2$, $\gamma_B=0.5$, $Gm=15$, $Sc=5$ and $M=1.5$

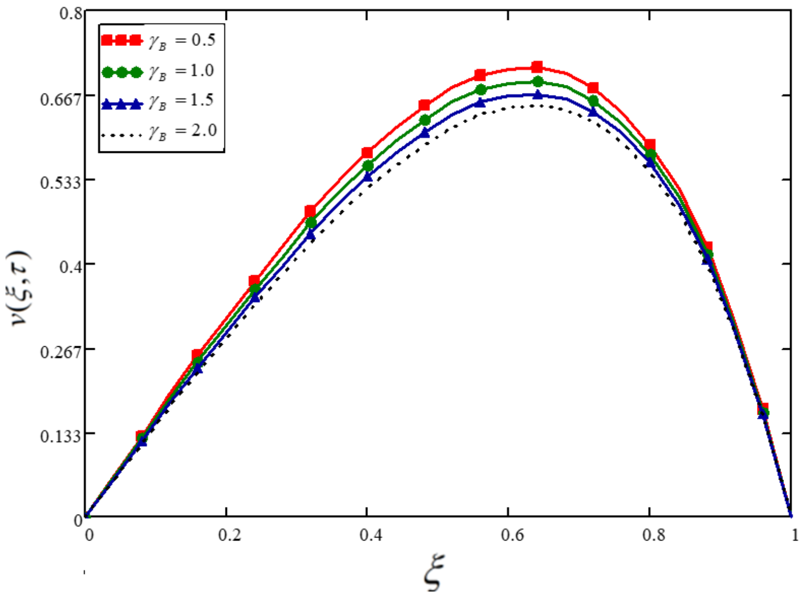

Fig. 2. Variations in velocity profile against ξ if $h(\tau)=0$ for different values of γ_B when $\tau=2$, $\alpha=0.5$, $Gm=15, Sc=5$ and $M=1.5$

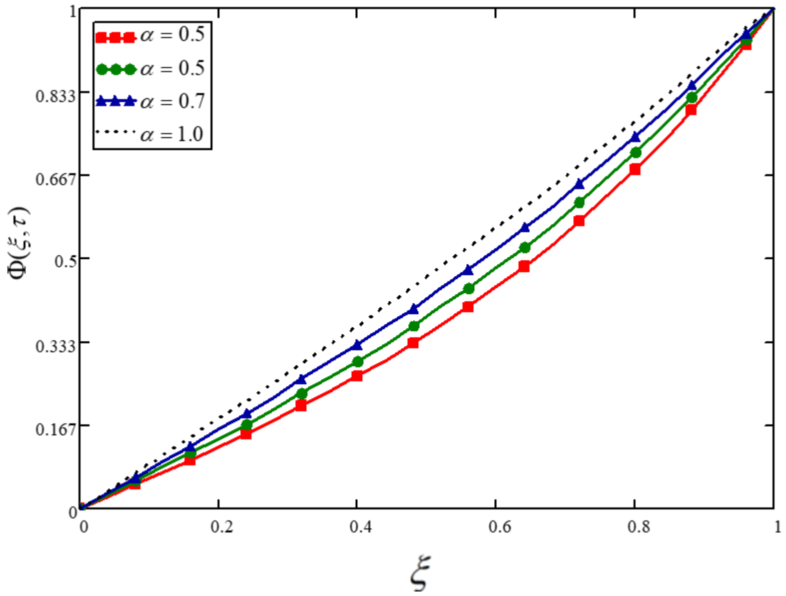

Fig. 3. Variations in Concentration profile against ξ for different values of α when $\tau = 2$ and $Sc = 5$

7. Conclusion and Future Directions

The fractional Brinkman-type fluid model is developed using a new methodology in this research. The Laplace and Fourier transformation methods are used to solving the problem. The results produced are drawn up and displayed in tables. The main findings of this analysis are

i. The new transformation is more reliable for the solution of the fractional model. It is easier to solve the fractional model using this transformation.
ii. This transformation reduces the computational time for finding the exact solutions to such problems.
iii. The velocity reduces with higher values of Brinkman-type fluid parameter.
iv. For various values of α, variations in all three profiles are shown. It is important to note here that we have different lines in the graph for fixed time values. This result demonstrates the fluid's memory effect, which the integer-order model cannot be seen.

This work may be extended for other Newtonian and non-Newtonian flow situations and with other modern fractional approaches. Moreover, Heat and Mass transfer can be considered together in the same flow problem.

References

[1] Atangana, Abdon, and Badr Saad T. Alkahtani. "Analysis of the Keller–Segel model with a fractional derivative without singular kernel." *Entropy* 17, no. 6 (2015): 4439-4453. https://doi.org/10.3390/e17064439

[2] Gómez-Aguilar, José Francisco, Abdon Atangana, and Victor Fabian Morales-Delgado. "Electrical circuits RC, LC, and RL described by Atangana–Baleanu fractional derivatives." *International Journal of Circuit Theory and Applications* 45, no. 11 (2017): 1514-1533. https://doi.org/10.1002/cta.2348

[3] Singh, Harendra. "Analysis for fractional dynamics of Ebola virus model." *Chaos, Solitons & Fractals* 138 (2020): 109992. https://doi.org/10.1016/j.chaos.2020.109992

[4] Abro, Kashif Ali, Ilyas Khan, and Asifa Tassaddiq. "Application of Atangana-Baleanu fractional derivative to convection flow of MHD Maxwell fluid in a porous medium over a vertical plate." *Mathematical Modelling of Natural Phenomena* 13, no. 1 (2018): 1. https://doi.org/10.1051/mmnp/2018007

[5] Khan, Ilyas, Muhammad Saqib, and Aisha M. Alqahtani. "Channel flow of fractionalized H2O-based CNTs nanofluids with Newtonian heating." *Discrete & Continuous Dynamical Systems-S* 13, no. 3 (2020): 769. https://doi.org/10.3934/dcdss.2020043

[6] Ali, Farhad, Nadeem Ahmad Sheikh, Ilyas Khan, and Muhammad Saqib. "Magnetic field effect on blood flow of Casson fluid in axisymmetric cylindrical tube: A fractional model." *Journal of Magnetism and Magnetic Materials* 423 (2017): 327-336. https://doi.org/10.1016/j.jmmm.2016.09.125

[7] Ali, Farhad, Nadeem Ahmad Sheikh, Ilyas Khan, and Muhammad Saqib. "Solutions with Wright function for time fractional free convection flow of Casson fluid." *Arabian Journal for Science and Engineering* 42, no. 6 (2017): 2565-2572. https://doi.org/10.1007/s13369-017-2521-3

[8] Caputo, Michele, and Mauro Fabrizio. "A new definition of fractional derivative without singular kernel." *Progress in Fractional Differentiation & Applications* 1, no. 2 (2015): 73-85.

[9] Ali, Farhad, Madeha Gohar, Ilyas Khan, Nadeem Ahmad Sheikh, Syed Aftab Alam Jan, and Muhammad Saqib. "Magnetite molybdenum disulphide nanofluid of grade two: a generalized model with caputo-fabrizio derivative." *Microfluidics and Nanofluidics* 10 (2018). https://doi.org/10.5772/intechopen.72863

[10] Fetecau, C., A. A. Zafar, D. Vieru, and J. Awrejcewicz. "Hydromagnetic flow over a moving plate of second grade fluids with time fractional derivatives having non-singular kernel." *Chaos, Solitons & Fractals* 130 (2020): 109454. https://doi.org/10.1016/j.chaos.2019.109454

[11] Sheikh, Nadeem Ahmad, Farhad Ali, Ilyas Khan, and Muhammad Saqib. "A modern approach of Caputo–Fabrizio time-fractional derivative to MHD free convection flow of generalized second-grade fluid in a porous medium." *Neural Computing and Applications* 30, no. 6 (2018): 1865-1875. https://doi.org/10.1007/s00521-016-2815-5

[12] Atangana, Abdon, and Dumitru Baleanu. "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model." *Thermal Science* 20, no. 2 (2016): 763-769. https://doi.org/10.2298/TSCI160111018A

[13] Atangana, Abdon, and Ilknur Koca. "Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order." *Chaos, Solitons & Fractals* 89 (2016): 447-454. https://doi.org/10.1016/j.chaos.2016.02.012

[14] Sheikh, Nadeem Ahmad, Farhad Ali, Muhammad Saqib, Ilyas Khan, and Syed Aftab Alam Jan. "A comparative study of Atangana-Baleanu and Caputo-Fabrizio fractional derivatives to the convective flow of a generalized Casson fluid." *The European Physical Journal Plus* 132, no. 1 (2017): 1-14. https://doi.org/10.1140/epjp/i2017-11326-y

[15] Sheikh, Nadeem Ahmad, Farhad Ali, Muhammad Saqib, Ilyas Khan, Syed Aftab Alam Jan, Ali Saleh Alshomrani, and Metib Said Alghamdi. "Comparison and analysis of the Atangana–Baleanu and Caputo–Fabrizio fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction." *Results in physics* 7 (2017): 789-800. https://doi.org/10.1016/j.rinp.2017.01.025

[16] Chhabra, Raj P., and John Francis Richardson. *Non-Newtonian flow and applied rheology: engineering applications*. Butterworth-Heinemann, 2011.

[17] Ali, Farhad, Nadeem Ahmad Sheikh, Muhammad Saqib, and Ilyas Khan. "Unsteady MHD flow of second-grade fluid over an oscillating vertical plate with isothermal temperature in a porous medium with heat and mass transfer by using the Laplace transform technique." *Journal of Porous Media* 20, no. 8 (2017): 671-690. https://doi.org/10.1615/JPorMedia.v20.i8.10

[18] Sochi, Taha. "Flow of non-Newtonian fluids in porous media." *Journal of Polymer Science Part B: Polymer Physics* 48, no. 23 (2010): 2437-2767. https://doi.org/10.1002/polb.22144

[19] Corapcioglu, M. Yavuz. *Advances in porous media*. Elsevier, 1996.

[20] De, Shauvik, J. A. M. Kuipers, E. A. J. F. Peters, and J. T. Padding. "Viscoelastic flow simulations in random porous media." *Journal of Non-Newtonian Fluid Mechanics* 248 (2017): 50-61. https://doi.org/10.1016/j.jnnfm.2017.08.010

[21] Jahan, Sultana, M. Ferdows, M. D. Shamshuddin, and Khairy Zaimi. "Effects of Solar Radiation and Viscous Dissipation on Mixed Convective Non-Isothermal Hybrid Nanofluid over Moving Thin Needle." *Journal of Advanced Research in Micro and Nano Engineering* 3, no. 1 (2021): 1-11.

[22] Idris, Muhammad Syafiq, Irnie Azlin Zakaria, and Wan Azmi Wan Hamzah. "Heat transfer and pressure drop of water based hybrid Al2O3: SiO2 nanofluids in cooling plate of PEMFC." *Journal of Advanced Research in Numerical Heat Transfer* 4, no. 1 (2021): 1-13.

[23] Ali, Farhad, Nadeem Ahmad Sheikh, Muhammad Saqib, and Arshad Khan. "Hidden phenomena of an MHD unsteady flow in porous medium with heat transfer." *Nonlinear Science Letters A* 8, no. 1 (2017): 101-116.

[24] Raftari, Behrouz, Syed Tauseef Mohyud-Din, and Ahmet Yildirim. "Solution to the MHD flow over a non-linear stretching sheet by homotopy perturbation method." *Science China Physics, Mechanics and Astronomy* 54, no. 2 (2011): 342-345. https://doi.org/10.1007/s11433-010-4180-1

[25] Teh, Yuan Ying, and Adnan Ashgar. "Three Dimensional MHD Hybrid Nanofluid Flow with Rotating Stretching/Shrinking Sheet and Joule Heating." *CFD Letters* 13, no. 8 (2021): 1-19. https://doi.org/10.37934/cfdl.13.8.119
[26] Darcy, Henry. "The public fountains of the city of Dijon." *Victor Dalmont, Paris, France* (1856).
[27] Brinkman, Hendrik C. "The viscosity of concentrated suspensions and solutions." *The Journal of chemical physics* 20, no. 4 (1952): 571-571. https://doi.org/10.1063/1.1700493
[28] Ali, Farhad, Ilyas Khan, Nadeem Ahmad Sheikh, Madeha Gohar, and I. Tlili. "Effects of different shaped nanoparticles on the performance of engine-oil and kerosene-oil: A generalized Brinkman-type fluid model with non-singular kernel." *Scientific reports* 8, no. 1 (2018): 1-13. https://doi.org/10.1038/s41598-018-33547-z
[29] Jan, Syed Aftab Alam, Farhad Ali, Nadeem Ahmad Sheikh, Ilyas Khan, Muhammad Saqib, and Madeha Gohar. "Engine oil based generalized brinkman-type nano-liquid with molybdenum disulphide nanoparticles of spherical shape: Atangana-Baleanu fractional model." *Numerical Methods for Partial Differential Equations* 34, no. 5 (2018): 1472-1488. https://doi.org/10.1002/num.22200
[30] Ali, Farhad, Bibi Aamina, Ilyas Khan, Nadeem A. Sheikh, and Muhammad Saqib. "Magnetohydrodynamic flow of brinkman-type engine oil based MoS2-nanofluid in a rotating disk with hall effect." *Int. J. Heat Technol* 4, no. 35 (2017): 893-902. https://doi.org/10.18280/ijht.350426
[31] Ali, Farhad, Madeha Gohar, Ilyas Khan, Nadeem Ahmad Sheikh, and Syed Aftab Alam Jan. "Thermal radiation and magnetic field effects on different channel flows of CNTs Brinkman-type nanofluids with water, kerosene and engine-oil." *City University International Journal Of Computational Analysis* 2, no. 1 (2018): 01-17.
[32] Caputo, Michele, and José M. Carcione. "A memory model of sedimentation in water reservoirs." *Journal of hydrology* 476 (2013): 426-432. https://doi.org/10.1016/j.jhydrol.2012.11.016
[33] Ali, Farhad, Muhammad Saqib, Ilyas Khan, and Nadeem Ahmad Sheikh. "Heat transfer analysis in ethylene glycol based molybdenum disulfide generalized nanofluid via Atangana–Baleanu fractional derivative approach." In *Fractional Derivatives with Mittag-Leffler Kernel*, pp. 217-233. Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-11662-0_13