SEMARAK ILMU
PUBLISHING

Journal of Advanced Research in Fluid Mechanics and Thermal Sciences

Journal homepage:
https://semarakilmu.com.my/journals/index.php/fluid_mechanics_thermal_sciences/index
ISSN: 2289-7879

The Effect on The Volume and Semi Axes of a Conducting Spheroid Due to The Scaling on Its First Order Polarization Tensor

Syafina Ahmad[1], Nurhazirah Mohamad Yunos[2], Taufiq Khairi Ahmad Khairuddin[1,3,*], Mukhiddin Muminov[4], Suzarina Ahmed Sukri[1]

[1] Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia
[2] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Cawangan Johor, Kampus Pasir Gudang, 81750 Masai, Johor, Malaysia
[3] UTM Centre for Industrial and Applied Mathematics (UTM-CIAM), Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia
[4] Mathematical Faculty, Samarkand State University, Samarkand, 140104, Uzbekistan

ARTICLE INFO

Article history:
Received 4 November 2021
Received in revised form 10 February 2022
Accepted 20 February 2022
Available online 1 April 2022

Keywords:
Matrices; Polarization of Pólya-Szegö; Depolarization Factors; Virtual Mass; Integral Equations

ABSTRACT

In order to enhance identification of objects in electrical imaging or metal detection, the polarization tensor is used to characterize the perturbation in electric or electromagnetic field due to the presence of the conducting objects. This is similar as describing the uniform fluid flow that is disturbed after a solid is immersed in the fluid during the study of fluid mechanics. Moreover, in some applications, it is beneficial to determine a spheroid based on the first order polarization tensor in order to understand what is actually represented by the tensor. The spheroid could share similar physical properties with the actual object represented by the polarization tensor. The purpose of this paper is to present how scaling on the matrix for the first order polarization tensor will affect the original spheroid represented by that first order polarization tensor. In the beginning, we revise the mathematical property regarding how scaling the semi axes of a conducting spheroid has an effect to its first order polarization tensor. After that, we give theoretical results with proofs on how scaling the matrix for the first order polarization tensor affects the volume and semi axes of the spheroid. Following that, some numerical examples are provided to further justify the theory. Here, different scalar factors will be used on the given first order polarization tensor before the new volume and semi axes of the spheroid are computed. In addition, we also investigate how the size of the scale on the first order polarization tensor influence the accuracy of computing the related volume and semi axes. In this case, it is found that a large error could occur to the volume and the semi axes when finding them by solving the first order PT with that has being scaled by a very large scaling factor or a too small scaling factor. A suggestion is then given on how to reduce the errors.

* *Corresponding author.*
E-mail address: taufiq@utm.my

https://doi.org/10.37934/arfmts.94.1.118

1. Introduction

Polarization tensor (PT) is widely applied recently, as a consequence of the increasing number of researches in science and engineering. Generally, PT considered in this study is used to characterize the electric or electromagnetic field perturbation of in the existence of conducting objects and thus, PT can be adapted in order to identify the object's position, dimensions, orientation or material properties [1]. It originates from the classical study in fluid mechanics and the theory of electricity, where, in Schiffer and Szegö [2], the terminology called as the virtual mass is used to describe the disturbance on the fluid flow when a solid is placed inside the fluid. On the other hand, if an electrically conducting solid is placed in an electrical field, another terminology called as polarization is used to describe the perturbation on an electrical field [2]. In Schiffer and Szegö [2], both virtual mass and polarization have been studied in a similar fashion, probably influenced at that time by the fluid theory of electricity, where, electricity has also been regarded as fluid.

Polarization in Schiffer and Szegö [2] is also called as the polarization of Pólya-Szegö (see Polya and Szegö [3]) and in the advanced studies of that polarization, the word tensor is added after polarization to show that PT has more than a scalar or a vector and has many other information about the subject. The implementation of PT in enhancing identification of object can be seen for examples in the studies of electrosensing fish, metal detection and medical imaging [4-14]. The method of describing objects specifically using the PT has a lower computational cost as it does not demand on a full image of the object to represent it.

Computational aspects of the PT are also investigated by previous researches. An investigation on the computation of the first order PT specifically for three dimensional objects can be found in the study by Khairuddin *et al.,* [15]. The software BEM++ were later applied by the same authors to ease the computation of the PT for sphere in a study by Khairuddin *et al.,* [16] while the numerical results regarding the PT for some ellipsoids obtained based on the method by Khairuddin *et al.,* [15] and also generated by BEM++ were further compared by Khairuddin *et al.,* [17]. BEM++ is a software in the languange C++, developed by Śmigaj *et al.,* [18], to solve problems in the form of boundary integral equations. For complex geometries, the computation of the related PT using BEM++ can be found in the study by Amad *et al.,* [19]. Besides, Sukri *et al.,* [20] has investigated the different orders of Gaussian quadrature in solving integral equations when computing the PT. In contrast to Khairuddin *et al.,* [15-17], Sukri *et al.,* [20] that used linear element in geometrical modelling for all involved objects, Sukri *et al.,* [21] has proposed to use quadratic element in presenting three dimensional objects before computing the related PT.

On the other hand, the effects on a simpler form of PT, called as the first order PT, that is associated with any objects after the objects are transformed such as rotated, translated or scaled, have also been previously studied. The properties and formula for these transformations are given by Ammari and Kang [22]. By refering to Khairuddin *et al.,* [23], we can found the numerical examples on the rotation of the first order PT for spheroids. They also mentioned that the determinant for the first order PT of the spheroid remains the same before and after the spheroid is rotated. Morever, the study by Sukri *et al.,* [24] had numerically shown that the first order PT of a few objects do not depend on the location of objects that is, the first order PT for an object does not change even though the center of gravity for the object is changed. Similarly, the numerical examples of the first order PT for translated and rotated objects was also conducted by Khairuddin *et al.,* [25]. Besides, by using both linear and quadratic elements in numerical method for computing the PT, Sukri *et al.,* [26] have investigated how the first order PT associated to a few objects including sphere, ellipsoid and cube are effected after each object is scaled.

As previous studies concentrated on the transformation of the first order PT for an object after the object is transformed, this study reversely focus on the effect of the object after the first order PT is transformed. At this stage, we restrict this study in analyzing the scaling on the first order PT caused by the perturbation of electrical field by a conducting spheroid. Specifically, by scaling the first order PT, we analyze the effect on volume, depolarization factors, eccentricity, and semi axes of the spheroid. The findings of this study might increase the understanding on how scaling on the first order PT affect the original object represented by the PT for future applications. Here, we only concentrate on the spheroid since previous research has revealed that the first order PT for some objects may be compared to the first order PT for a spheroid [27].

2. Mathematical Formulation

2.1 The First Order Polarization Tensor for Spheroid

Let B be an ellipsoid, built by a material with conductivity k in the three-dimensional space, R^3. Assume that the conductivity for $R^3 - B$ is equal to 1. When an electrical field passes both R^3 and B, the perturbation in the electrical field due to B can be described by the first order PT, which depends on size and material of B. In the Cartesian coordinate system, where the origin is the center of ellipsoid B, the ellipsoid is described as $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$. The semi principal axes of B are represented as a, b and c. The simpler explicit formula of the first order PT of B when conductivity, k satisfy $0 < k \neq 1 < \infty$ proposed by Ammari and Kang [22] is as follows.

$$M(k,B) = (k-1)|B| \begin{bmatrix} \frac{1}{(1-P)+kP} & 0 & 0 \\ 0 & \frac{1}{(1-Q)+kQ} & 0 \\ 0 & 0 & \frac{1}{(1-R)+kR} \end{bmatrix}, \tag{1}$$

where $|B| = \frac{4}{3}\pi abc$ denotes the volume B while P, Q and R are constants defined by

$$P = \frac{bc}{a^2}\int_1^\infty \frac{1}{t^2\sqrt{t^2-1+\left(\frac{b}{a}\right)^2}\sqrt{t^2-1+\left(\frac{c}{a}\right)^2}}dt, \tag{2}$$

$$Q = \frac{bc}{a^2}\int_1^\infty \frac{1}{\left(t^2-1+\left(\frac{b}{a}\right)^2\right)^{3/2}\sqrt{t^2-1+\left(\frac{c}{a}\right)^2}}dt, \tag{3}$$

$$R = \frac{bc}{a^2}\int_1^\infty \frac{1}{\sqrt{t^2-1+\left(\frac{b}{a}\right)^2}\left(t^2-1+\left(\frac{c}{a}\right)^2\right)^{3/2}}dt. \tag{4}$$

Apart from that, Mohamad Yunos and Khairuddin [28] had proposed an alternative explicit formula of $M(k,B)$ for a spheroid B with conductivity k, described by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{b}\right)^2 = 1$ in the following formula.

$$M(k,B) = (k-1)|B| \begin{bmatrix} \frac{1}{(1-d_1)+kd_1} & 0 & 0 \\ 0 & \frac{2}{(1+d_1)+k(1-d_1)} & 0 \\ 0 & 0 & \frac{2}{(1+d_1)+k(1-d_1)} \end{bmatrix}, \quad (5)$$

where $|B|$ is the volume B and d_1 is the alternative form of Eq. (9) for $b = c$, which is now called as the depolarization factor of B (further details in a study by Yunos *et al.,* [29] about the depolarization factor for ellipsoids).

2.2 The Method in Determining the Semi Axes of a Spheroid based on the First Order PT

The method presented by Khairuddin *et al.,* [30] and Bahuriddin *et al.,* [31] for determining a spheroid B subject to a specified first order PT is revised in this subsection. At the moment, only spheroid that can be represented as $\left(\frac{x}{a}\right)^2 + \left(\frac{y^2+z^2}{b^2}\right) = 1$ in the Cartesian coordinate will be considered. Thus, in this paper, we will only concentrate on the specified first order PT in the following form

$$M(k_0,B) = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_2 \end{bmatrix}, \quad (6)$$

where the conductivity k_0 is fixed and satisfies $0 < k_0 \neq 1 < \infty$.

Given the first order $M(k_0,B)$ in the form of Eq. (6), two cases will be considered, which are either $M_1 > M_2$ in identifying the semi axes of a prolate spheroid with $a > b = c$ or $M_1 < M_2$ when computing the semi axes $a < b = c$ for an oblate spheroid. However, as presented in Khairuddin *et al.,* [30] and Bahuriddin *et al.,* [31], several other parameters including the volume of the spheroid must be firstly obtained before all semi axes of the spheroid can be determined. Generally, the computation is done by solving the given matrix on the right hand side of Eq. (6) equal to right hand side of Eq. (5).

In order to find the volume and all other parameters of the spheroid, the value of k_0 is firstly specified such that $k_0 > 1$ once $M(k_0,B)$ is a positive definite matrix whereas $k_0 < 1$ if $M(k_0,B)$ is a negative definite matrix. These properties of the conductivity for spheroid that depend on its first order PT have been proven by Khairuddin *et al.,* [32]. Any first order PT must be either a positive definite or a negative definite matrix, where, $M(k_0,B)$ is a positive definite matrix if both M_1 and M_2 are positive whereas $M(k_0,B)$ is a negative definite matrix if both M_1 and M_2 are negative.

After that, from Cramer's rule, the volume of B expressed as $|B|$ and its depolarization factor, d_1 are calculated as

$$|B| = \frac{M_1M_2(k_0+2)}{(k_0-1)(2M_1+M_2)} \quad (7)$$

and

$$d_1 = \frac{M_2-2M_1+M_2k_0}{(k_0-1)(2M_1+M_2)} \quad (8)$$

where d_1 must satisfy $0 < d_1 < 1$. In the studies by Stoner [33] and Milton [34], an explicit formula of d_1 in terms of the eccentricity for the prolate spheroid has been given. Thus, by utilising the calculated value of d_1 in Eq. (8), the eccentricity for the prolate spheroid, ψ is solved from the following equation

$$d_1 = \frac{1-\psi^2}{\psi^2}\left\{\frac{1}{2\psi}\ln\left(\frac{1+\psi}{1-\psi}\right) - 1\right\} \tag{9}$$

specifically for $M_1 > M_2$ by using Newton's method. Meanwhile, from the obtained value of d_1 in Eq. (8), the eccentricity of the oblate spheroid, φ is solved from the equation

$$d_1 = \frac{1}{\varphi^2}\left\{1 - \frac{\sqrt{1-\varphi^2}}{\varphi}\sin^{-1}\varphi\right\}. \tag{10}$$

Lastly, the semi axes a and b of the prolate spheroid are computed such that

$$a = \sqrt[3]{\frac{3|B|}{4\pi(1-\psi^2)}}, \tag{11}$$

$$b = \sqrt{1-\psi^2}\sqrt[3]{\frac{3|B|}{4\pi(1-\psi^2)}}. \tag{12}$$

when $M_1 < M_2$ by using Newton's method while the semi axes a and b of the oblate spheroid are computed where

$$a = \sqrt{1-\varphi^2}\sqrt[3]{\frac{3|B|}{4\pi\sqrt{1-\varphi^2}}}, \tag{13}$$

$$b = \sqrt[3]{\frac{3|B|}{4\pi\sqrt{1-\varphi^2}}}. \tag{14}$$

After the semi axes of spheroid, a and b are obtained with the specified k_0, the first order PT for the spheroid, denoted by $\widehat{M}$, is calculated back using Eq. (5). Then, the following difference is calculated as

$$M - \widehat{M} = \begin{bmatrix} \widehat{m}_{11} & 0 & 0 \\ 0 & \widehat{m}_{22} & 0 \\ 0 & 0 & \widehat{m}_{33} \end{bmatrix}. \tag{15}$$

The diagonals of Eq. (15) will be used to determine the matrix norm, $e = \sqrt{\widehat{m}_{11}^{\ 2} + \widehat{m}_{22}^{\ 2} + \widehat{m}_{33}^{\ 2}}$ to calculate the relative error between the specified first order PT, M and the computed first order PT, $\widehat{M}$.

This method in determining the semi axes of a spheroid will be used throughout this study depend on the specified first order PT along with the implementation of the scaling on the first order PT.

3. The Effect due to the Scaling on the Semi Axes of an Ellipsoid to its First Order PT

In this section, we will present the mathematical property regarding the scaling effect of the semi axes of a conducting ellipsoid on its first order PT based on the previous properties given by Ammari and Kang [22] and Kang [35]. In Ammari and Kang [22]and Kang [35], the effect of scaling the size of any object on its first order PT was investigated. We now prove the next proposition that explains how scaling on the semi axes of an ellipsoid will affect its first order PT where, our result here is a specific case of the result from Ammari and Kang [22] and Kang [35].

Proposition 1

Let B be an ellipsoid given by $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$ in the Cartesian coordinate system and $M(k,B)$ is the first order PT for B at any conductivity, k where $0<k\neq 1<+\infty$. Suppose that $p>0$ is a scalar factor for B such that B becomes $\left(\frac{x}{pa}\right)^2+\left(\frac{y}{pb}\right)^2+\left(\frac{z}{pc}\right)^2=1$, for any $p\in\mathbb{R}$. If $M(k,B_p)$ is the first order PT for B after it is scaled by p then $M(k,B_p)=p^3M(k,B)$.

Proof

By substituting $a=pa$, $b=pb$ and $c=pc$ in the formula of the first order PT for B in Eq. (1), we have

$$M(k,B_p)=(k-1)\cdot\frac{4}{3}\pi(pa)(pb)(pc)\cdot\begin{bmatrix}\frac{1}{(1-P)+kP} & 0 & 0\\ 0 & \frac{1}{(1-Q)+kQ} & 0\\ 0 & 0 & \frac{1}{(1-R)+kR}\end{bmatrix}. \tag{16}$$

Meanwhile, Eq. (2) now becomes

$$P=\frac{pb.pc}{(pa)^2}\int_1^\infty\frac{1}{t^2\sqrt{t^2-1+\left(\frac{pb}{pa}\right)^2}\sqrt{t^2-1+\left(\frac{pc}{pa}\right)^2}}dt,$$

$$=\frac{bc}{a^2}\int_1^\infty\frac{1}{t^2\sqrt{t^2-1+\left(\frac{b}{a}\right)^2}\sqrt{t^2-1+\left(\frac{c}{a}\right)^2}}dt,$$

such that, *P* does not change after the semi axes is scaled. The same happens to each *Q* and *R*.

Next, we have

$$M(k,B_p)=p^3.(k-1)|B|\begin{bmatrix}\frac{1}{(1-P)+kP} & 0 & 0\\ 0 & \frac{1}{(1-Q)+kQ} & 0\\ 0 & 0 & \frac{1}{(1-R)+kR}\end{bmatrix}. \tag{17}$$

Therefore, multiplying Eq. (1) with p^3, we can show that

$$M(k,B_p)=p^3M(k,B). \tag{18}$$

□

4. The Effect due to the Scaling on the first order PT of a Spheroid to its Volume and Semi Axes

In the previous section, after B is scaled by p, we justify that the first order PT after the scaling, $M(k, B_p)$ can be related to the original first order PT, $M(k, B)$. However, in this study, we will reversely investigate the scaling effect on the first order PT for spheroid to the volume, depolarization factors, eccentricity and semi axes of the spheroid. Our main result regarding this is presented here in the next theorem.

Theorem 1

Let k_0 be fixed so that $k_0 > 1$ if $M(k_0, B)$ is a positive definite matrix or $0 < k_0 < 1$ if $M(k_0, B)$ is a negative definite matrix, where $M(k_0, B)$ is given by

$$M(k_0, B) = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_2 \end{bmatrix},$$

for some values M_1 and M_2. Suppose that $M(k_0, B)$ is the first order PT for a spheroid B such that B has volume $|B|$, depolarization factor d_1, eccentricity ψ when $M_1 > M_2$ (or eccentricity φ when $M_1 < M_2$) as well as semi axes *a* and *b*. For a fixed constant p satisfying $p > 0$, if $M(k_0, B_p) = p^3 M(k_0, B)$ such that $M(k_0, B_p)$ is the first order PT for another spheroid B_p that has volume $|B_p|$, depolarization factor d_{1p}, eccentricity ψ_p when $p^3 M_1 > p^3 M_2$ (or eccentricity φ_p when $p^3 M_1 < p^3 M_2$) as well as semi axes a_p and b_p, then, each of the following holds.

(a) $|B_p| = p^3 |B|$ and $d_{1p} = d_1$.
(b) $\psi_p = \psi$ or $\varphi_p = \varphi$.
(c) $a_p = pa$ and $b_p = pb$.

Proof

In order to prove part (a), first of all, note that $|B|$ and d_1 are respectively given by Eq. (7) and Eq. (8). For $M(k_0, B_p) = p^3 M(k_0, B)$ such that

$$M(k_0, B_p) = \begin{bmatrix} p^3 M_1 & 0 & 0 \\ 0 & p^3 M_2 & 0 \\ 0 & 0 & p^3 M_2 \end{bmatrix}, \tag{19}$$

the associated $|B_p|$ and d_{1p} can be obtained by replacing M_1 and M_2 respectively with $p^3 M_1$ and $p^3 M_2$ in Eq. (7) and Eq. (8). This gives

$$|B_p| = \frac{p^3 M_1 \cdot p^3 M_2 (k_0 + 2)}{(k_0 - 1)(2p^3 M_1 + p^3 M_2)}, \tag{20}$$

$$= \frac{p^6 M_1 M_2 (k_0 + 2)}{p^3 (k_0 - 1)(2M_1 + M_2)}, \tag{21}$$

$$= p^3 |B|. \tag{22}$$

and

$$d_{1p} = \frac{p^3(M_2-2M_1+M_2k_0)}{(k_0-1)(2p^3M_1+p^3M_2)}, \tag{23}$$

$$= d_1. \tag{24}$$

For part (b), ψ_p will satisfy the same Eq. (9) for d_{1p} in terms of the eccentricity ψ_p. Meanwhile, φ_p will satisfy the same Eq. (10) of d_{1p} in terms of the eccentricity φ_p. This implies $\psi_p = \psi$ while φ_p= φ when $d_{1p} = d_1$ from part (a).

In order to prove part (c), a_p and b_p are firstly determined when $p^3M_1 > p^3M_2$ by replacing $|B|$ and ψ respectively with $|B_p|$ and ψ_p in Eq. (11) and Eq. (12). This gives

$$a_p = \sqrt[3]{\frac{3|B_p|}{4\pi(1-\psi_p{}^2)}}, \tag{25}$$

$$b_p = \sqrt{1-\psi^2}\sqrt[3]{\frac{3|B_p|}{4\pi(1-\psi_p{}^2)}}. \tag{26}$$

Moreover, since $|B_p| = p^3|B|$ and $\psi_p = \psi$, we then have

$$a_p = \sqrt[3]{\frac{3p^3|B|}{4\pi(1-\psi^2)}}, \tag{27}$$

$$b_p = \sqrt{1-\psi^2}\sqrt[3]{\frac{3p^3|B|}{4\pi(1-\psi^2)}}. \tag{28}$$

On the other hand, when $p^3M_1 < p^3M_2$, the semi axes a_p and b_p are obtained by replacing $|B|$ and φ in (13) and (14) with $|B_p|$ and φ_p. This implies

$$a_p = \sqrt{1-\varphi_p{}^2}\sqrt[3]{\frac{3|B_p|}{4\pi\sqrt{1-\varphi_p{}^2}}}, \tag{29}$$

$$b_p = \sqrt[3]{\frac{3|B_p|}{4\pi\sqrt{1-\varphi_p{}^2}}}. \tag{30}$$

Similarly, since $|B_p| = p^3|B|$ and φ_p= φ, we then obtain

$$a_p = \sqrt{1-\varphi^2}\sqrt[3]{\frac{3p^3|B|}{4\pi\sqrt{1-\varphi^2}}}, \tag{31}$$

$$b_p = \sqrt[3]{\frac{3p^3|B|}{4\pi\sqrt{1-\varphi^2}}}. \tag{32}$$

Both Eq. (27) and Eq. (31) can be simplified to give $a_p = pa$ while Eq. (28) and Eq. (32) conclude that $b_p = pb$. □

The presented Theorem 1 suggests that the volume of the spheroid B denoted by $|B|$ is affected and becomes $p^3|B|$ after the related first order PT, $M(k_0, B)$ is scaled to $p^3M(k_0, B)$, where p is a positive constant. On the other hand, the depolarization factor, d_1 before and after the same scaling do not produce in any significant change, which lead to the eccentricity, ψ or φ also has the same value even after the scaling. In addition, part (c) of Theorem 1 and its proof show that every semi axes of the spheroid B which are a and b, has an effect due to the scaling, that is, after the first order PT is scaled by p^3, the semi axes become pa and pb.

In the next section 5.1, numerical explain are provided to further explain and justify Theorem 1.

5. Numerical Results and Discussion

5.1 General Examples

For this section, we will discuss on the numerical results of the semi axes of a spheroid when the given first order PT, $M(k_0, B)$ is scaled at p^3, which is by multiplying p^3 with Eq. (6). By utilizing the previous method discussed in subsection 2.2 with the existing examples from Khairuddin *et al.,* [30], the numerical results of the volume, depolarization factors, eccentricity and semi axes for prolate spheroid and oblate spheroid calculated based on the positive and negative definite matrices of the first order PT are presented in this section. After choosing a few examples from the study by Khairuddin *et al.,* [30], the volume, depolarization factors, eccentricity and semi axes of a few spheroids are determined after the chosen first order PT are scaled. Then, we make comparisons in a ratio form between the previous results (before scaling) and the current results (after scaling) for the volume, depolarization factors, eccentricity and semi axes of the spheroids. For each choosen the first order PT, we apply three randomly choosen different scalar factor, p in order to investigate how scaling the first order PT affect the original spheroid represented by the first order PT.

First and foremost, we will discuss on the prolate spheroid. The following first order PT indicates the positive definite matrix of a prolate spheroid by setting Eq. (6) equal to

$$\begin{bmatrix} 20 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}. \tag{33}$$

When the conductivity k_0 is fixed at $k_0 = 2$, from a study by Khairuddin *et al.,* [30], the values for the volume, $|B|$, depolarization factor, d_1, eccentricity, ψ and semi axes a and b for the given $M(k_0, B)$ in Eq. (33) are $|B| = 21.8182$, $d_1 = 0.0909$, $\psi = 0.9574$, $a = 3.9664$, $b = 1.1457$ with relative error, $e = 0.0002$. Meanwhile, the values for $|B|$, d_1, ψ, a, b and e when $M(k_0, B)$ is scaled by p^3, where $p = 0.5, 2, ^7/_3$, computed based on the method discussed in subsection 2.2, are shown in Table 1. After that, we compute the ratios for $|B|$, d_1, ψ, a and b by dividing the current values (given in Table 1) with the previous values from Khairuddin *et al.,* [30] and present the ratios in Table 2.

Table 1
The values of $|B|$, d_1, ψ, a, b and e computed after $M(k_0, B)$ for positive definite matrix of a prolate spheroid, given by Eq. (33), is scaled by p^3 where $p = 0.5, 2, ^7/_3$, at fixed $k_0 = 2$

p	$\|B\|$	d_1	ψ	a	b	e
0.5	2.7273	0.0909	0.9574	1.9832	0.5729	0.0000
2	174.5455	0.0909	0.9574	7.9328	2.2915	0.0005
$^7/_3$	277.1717	0.0909	0.9574	9.2549	2.6734	0.0010

Table 2
The ratio values of $|B|$, d_1, ψ, a, and b for $p = 0.5, 2, {}^{7}/_{3}$ when the positive definite first order PT of a prolate spheroid, given by Eq. (33), is scaled by p^3

p	Ratio $\lvert B \rvert$	Ratio d_1	Ratio ψ	Ratio a	Ratio b
0.5	0.1250	1.0000	1.0000	0.5000	0.5000
2	8.0000	1.0000	1.0000	2.0000	2.0001
${}^{7}/_{3}$	12.7037	1.0000	1.0000	2.3333	2.3334

Next, the following first order PT for the negative definite matrix of a prolate spheroid is given by

$$\begin{bmatrix} -15 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{bmatrix}. \tag{34}$$

Firstly, the conductivity k_0 is set to $k_0 = 0.2$. The values for the volume, $|B|$, depolarization factor, d_1, eccentricity, ψ and semi axes a and b of the spheroid, based on the given $M(k_0, B)$, before it is scaled, as presented by Khairuddin *et al.,* [30], are $|B| = 16.5000$, $d_1 = 0.1500$, $\psi = 0.8989$, $a = 2.7375$, $b = 1.1993$ with relative error, $e = 0.0002$. Moreover, the values for $|B|$, d_1, ψ, a, b and e computed after $M(k_0, B)$ is scaled by p^3, where $p = 2.2, 4, {}^{13}/_{3}$, are also computed and then shown in Table 3. Hence, we divide the current results by the previous results to get the ratios for $|B|$, d_1, ψ, a and b, as presented in Table 4.

Table 3
The values of $|B|$, d_1, ψ, a, b and e computed after $M(k_0, B)$ for negative definite matrix of a prolate spheroid, given by Eq. (34), is scaled by p^3 where $p = 2.2, 4, {}^{13}/_{3}$, at fixed $k_0 = 0.2$

p	$\lvert B \rvert$	d_1	ψ	a	b	e
2.2	175.6920	0.1500	0.8989	6.0224	2.6385	0.0002
4	1056.0000	0.1500	0.8989	10.9498	4.7973	0.0038
${}^{13}/_{3}$	1342.6111	0.1500	0.8989	11.8623	5.1971	0.0015

Table 4
The ratio values of $|B|$, d_1, ψ, a, and b for $p = 2.2, 4, {}^{13}/_{3}$ when the negative definite first order PT of a prolate spheroid, given by Eq. (34), is scaled by p^3

p	Ratio $\lvert B \rvert$	Ratio d_1	Ratio ψ	Ratio a	Ratio b
2.2	10.6480	1.0000	1.0000	2.2000	2.2000
4	64.0000	1.0000	1.0000	3.9999	4.0001
${}^{13}/_{3}$	81.3704	1.0000	1.0000	4.3333	4.3334

In different circumstances, we will then discuss on the results for oblate spheroid. The given first order PT in Eq. (6) for the positive definite matrix of an oblate spheroid is set as follows

$$\begin{bmatrix} 15 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} \tag{35}$$

and the conductivity k_0 used in this case is $k_0 = 10$. The values of the volume, $|B|$, depolarization factor, d_1, eccentricity, φ and semi axes a and b for the given $M(k_0, B)$ before it is scaled are $|B| = 8.0000$, $d_1 = 0.4222$, $\varphi = 0.6890$, $a = 1.0009$, $b = 1.3811$ with relative error, $e = 0.0001$, as stated in the study by Khairuddin *et al.,* [30]. In addition, Table 5 shows the values of $|B|$, d_1, φ, a, b and e for the oblate spheroid obtained based on the method discussed in subsection 2.2, when $M(k_0, B)$ is scaled by p^3, where $p = {}^{15}/_{7}, 4.1, 6$. Then, we compute the ratios for $|B|$, d_1, φ, a and b, before showing them in Table 6.

Table 5
The values of $|B|$, d_1, φ, a, b and e computed after $M(k_0, B)$ for positive definite matrix of an oblate spheroid, given by Eq. (35), is scaled by p^3 where $p = {}^{15}/_{7}, 4.1, 6$, at fixed $k_0 = 10$

p	$\|B\|$	d_1	φ	a	b	e
${}^{15}/_{7}$	78.7172	0.4222	0.6890	2.1449	2.9594	0.0008
4.1	551.3680	0.4222	0.6890	4.1039	5.6623	0.0028
6	1728.0000	0.4222	0.6890	6.0057	8.2863	0.0105

Table 6
The ratio values of $|B|$, d_1, φ, a, and b at $p = {}^{15}/_{7}, 4.1, 6$ when the positive definite first order PT of an oblate spheroid, given by Eq. (35), is scaled by p^3

p	Ratio $\|B\|$	Ratio d_1	Ratio φ	Ratio a	Ratio b
${}^{15}/_{7}$	9.8397	1.0000	1.0000	2.1430	2.1428
4.1	68.9210	1.0000	1.0000	4.1002	4.0998
6	216.0000	1.0000	1.0000	6.0003	5.9998

Besides, the given first order PT for the negative definite matrix of an oblate spheroid is as follows

$$\begin{bmatrix} -20 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix}, \tag{36}$$

for $k_0 = 0.3$. Before $M(k_0, B)$ being scaled, the values of the volume, $|B|$, depolarization factor, d_1, eccentricity, φ and semi axes a and b of the spheroid, as mentioned in the study by Khairuddin *et al.,* [30], are indicated by $|B| = 17.9221$, $d_1 = 0.5325$, $\varphi = 0.8714$, $a = 1.0098$, $b = 2.0581$ with relative error, $e = 0.0001$. The obtained values for $|B|$, d_1, φ, a, b and e after $M(k_0, B)$ is scaled by p^3, where $p = {}^{24}/_{7}, 5.9, 8$ are shown in Table 7. Consequently, we obtain the ratios for $|B|$, d_1, φ, a and b and present them in Table 8.

Table 7
The values of $|B|$, d_1, φ, a, b and e computed after $M(k_0, B)$ for negative definite matrix of an oblate spheroid, given by Eq. (36), is scaled by p^3 where $p = {}^{24}/_{7}, 5.9, 8$, at fixed $k_0 = 0.3$

p	$\|B\|$	d_1	φ	a	b	e
${}^{24}/_{7}$	722.3172	0.5325	0.8714	3.4620	7.0562	0.0018
5.9	3680.8184	0.5325	0.8714	5.9575	12.1425	0.0014
8	9176.1039	0.5325	0.8714	8.0780	16.4644	0.0061

Table 8
The ratio values of $|B|$, d_1, φ, a, and b at $p = {}^{24}/_{7}, 5.9, 8$ when the positive definite first order PT of an oblate spheroid, given by Eq. (36), is scaled by p^3

p	Ratio $\lvert B\rvert$	Ratio d_1	Ratio φ	Ratio a	Ratio b
${}^{24}/_{7}$	40.3032	1.0000	1.0000	3.4284	3.4285
5.9	205.3787	1.0000	1.0000	5.8997	5.8999
8	511.9994	1.0000	1.0000	7.9996	7.9998

In the above examples, we choose the first order PT for spheroid from the study by Khairuddin *et al.,* [30] to be equal to Eq. (33), Eq. (34), Eq. (35), and Eq. (36) because all of them will give small error, e for computing the related $|B|$, d_1, ψ or φ and the semi axes a and b. In each case, the value of e is zero at four decimal places. This ensure us that accurate first order PT are used as the reference when doing the scaling.

Based on Table 2, Table 4, Table 6 and Table 8, we can justify that the ratios for $|B|$ are approximately equal to p^3 as suggested by Theorem 1 part (a) that $|B_p|/|B| = p^3$.

This means that the original $|B|$ for all four cases, are affected by the scaling. Meanwhile, the ratios d_1, ψ and φ are exactly equal to 1 as they remain the same values before the scaling as shown in Theorem 1 part (b). In addition, the ratio of semi axes a and b are approximately equal to p, indicating that, scaling the first order PT for a spheroid by p^3 will have an effect on the semi axes of the original spheroid. This also agrees with Theorem 1 part (c).

5.2 The Size of the Scaling Factor of the Scaled First Order PT for Spheroid

According to Ammari and Kang [22], the first order PT is mathematically formulated based on a small object. Generally, the first order PT for a spheroid, that is a positive definite matrix in the form of Eq. (6), represents a large object if all coefficients of the first order PT are very large. On the other hand, it also represents a large object if all of its coefficients are very small when it is a negative definite matrix. In this case, large error could occur when solving these first order PT to identify the related volume, depolarization factors, eccentricity and also all semi axes of the spheroid. Therefore, this section will investigate how the size of the scaling factor on the first order PT for spheroid will affect the computation of the volume and all semi axes of the spheroid.

First of all, the positive definite matrix given in Eq. (33) that represents the first order PT of a prolate spheroid is again considered. Solving Eq. (33) for $k_0 = 150$ with the method presented in Section 2.2 gives $|B| = 5.5644$, $a = 1.2911$ and $b = 1.0141$ with relative error, $e = 0.0002$. Next, after Eq. (33) is scaled with $\alpha^3 = 10^{-6}$, 10^{-4}, 10^{-2}, 10^{2}, 10^{4} and 10^{6}, the same method is used to find the volume and all semi axes for every spheroid at each scaling factor α. For each α, these values are respectively denoted by $|B|_{Np}, a_{Np}$ and b_{Np}. In this case, we can regard $|B|_{Np}, a_{Np}$ and b_{Np} as numerical values based on the numerical method. At the same time, using the results in Theorem 1, the values $|B| = 5.5644$, $a = 1.2911$ and $b = 1.0141$ of Eq. (33) are scaled accordingly for each scaling factor α where, the values after the scaling are respectively denoted by $|B|_{Tp}, a_{Tp}$ and b_{Tp}. These values can be regarded as the actual values for $|B|$, a and b after Eq. (33) is scaled by α^3, that is based on theoretical result of the first order PT for spheroid. Next, the absolute difference $|B|_{Np} - |B|_{Tp}$, $a_{Np} - a_{Tp}$ and $b_{Np} - b_{Tp}$ are calculated for each $p = \alpha^3$. After that, all values for the difference $|B|_{Np} - |B|_{Tp}$ are then plotted in the same graphs against the scaling factor p in Figure 1 while all differences for $a_{Np} - a_{Tp}$ and $b_{Np} - b_{Tp}$ each is also plotted in the same graphs against the scaling factor p in Figure 2.

Next, the negative definite matrix that is the first order PT of an oblate spheroid given by Eq. (36) is again solved with the method presented in Section 2.2 but now for $k_0 = 0.1$, which gives $|B| = 12.7273$, $a = 1.0323$ and $b = 1.7148$ with relative error, $e = 0.0001$. Similarly, after Eq. (36) is scaled by $\alpha^3 = 10^{-6}$, 10^{-4}, 10^{-2}, 10^{2}, 10^{4} and 10^{6}, $|B|_{Np}, a_{Np}$ and b_{Np} are determined based on the method presented in section 2.2 while $|B|_{Tp}, a_{Tp}$ and b_{Tp} are obtained based on the properties in Theorem 1. For each $p = \alpha^3$, all values for each of the absolute difference $|B|_{Np} - |B|_{Tp}$, $a_{Np} - a_{Tp}$ and $b_{Np} - b_{Tp}$ are then plotted in the same graphs against the scaling factor p in Figure 3 and Figure 4.

From Figure 1 and Figure 2, as the size of the scaling, p increased, the difference for each $|B|_{Np} - |B|_{Tp}$, $a_{Np} - a_{Tp}$ and $b_{Np} - b_{Tp}$ also increased. Similarly, each $|B|_{Np} - |B|_{Tp}$, $a_{Np} - a_{Tp}$ and $b_{Np} - b_{Tp}$ also increased when p increased in Figure 3 and Figure 4. This shows that, at a fixed conductivity, the size of the first order PT for the spheroid has an effect when solving the first order PT to obtain the volume and all semi axes of the spheroid. In this case, a large error could occur to the volume and the semi axes when finding them by solving the positive definite first order PT with very large coefficients or solving the negative definite first order PT with very small coefficients. In these examples, the coefficients of the first order PT become very large or very small due to the scaling on the original first order PT.

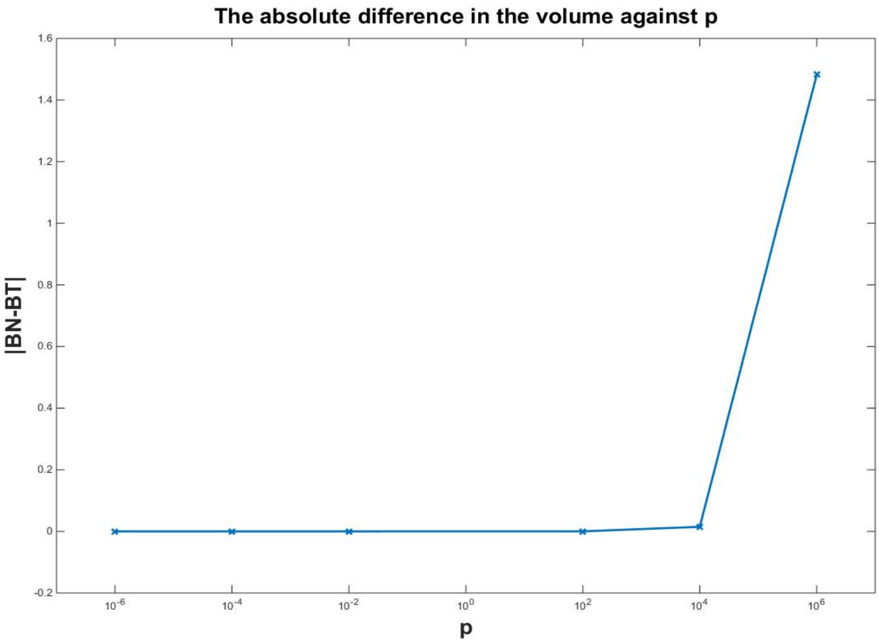
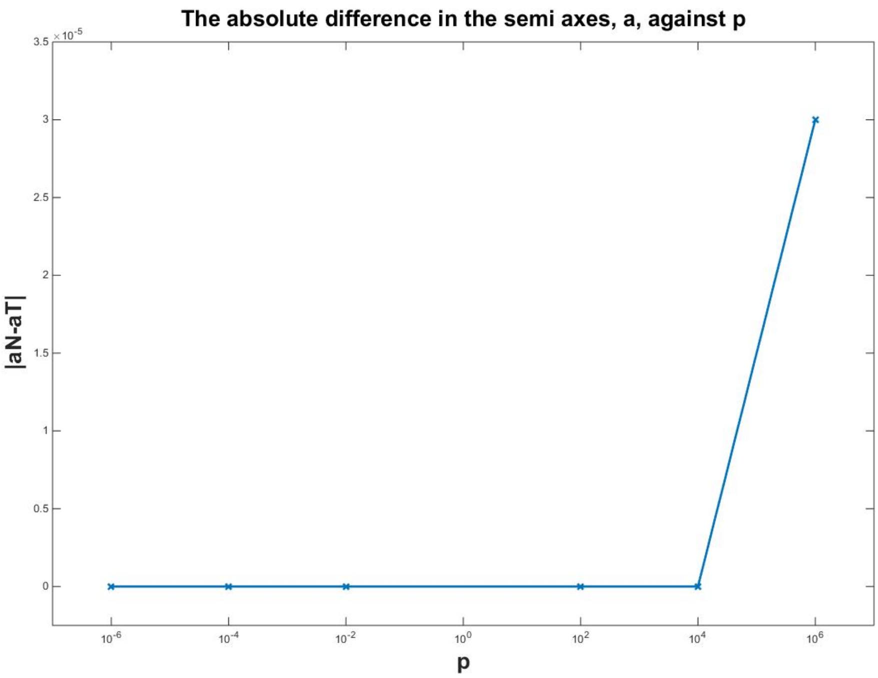
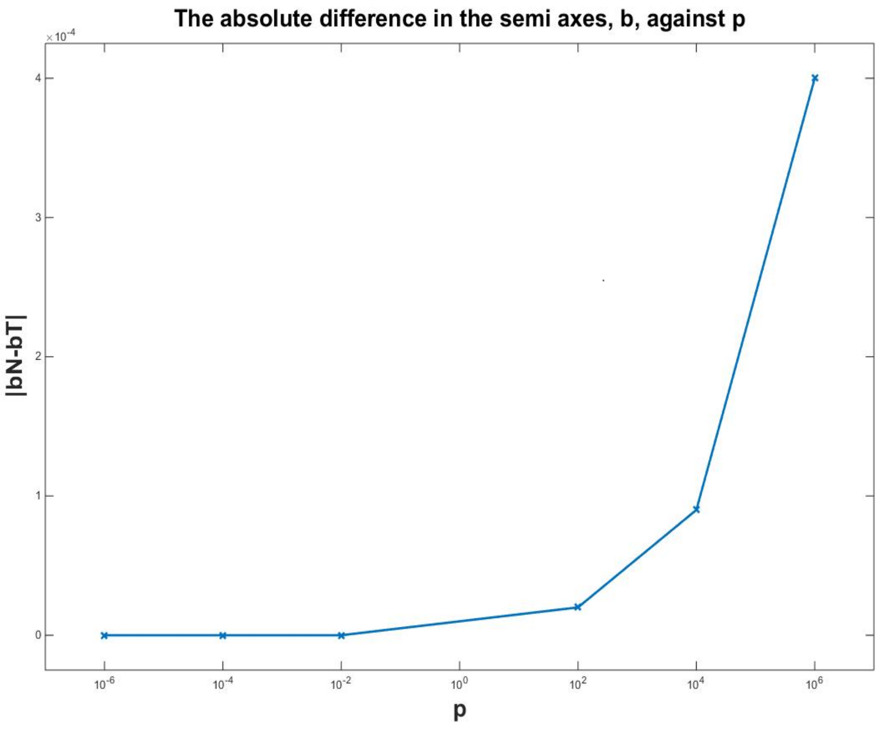


Fig. 1. The absolute difference in the volume, $|B|_{Np} - |B|_{Tp}$ against the scaling factor, p. Here, each $|B|_{Np}$ is obtained by solving Eq. (33) for $k_0 = 150$ after Eq. (33) is scaled by $\alpha^3 = 10^{-6}$, 10^{-4}, 10^{-2}, 10^{2}, 10^{4} and 10^{6}. On the other hand, all $|B|_{Tp}$ are calculated by scaling $|B| = 5.5644$ with α as in Theorem 1, after $|B| = 5.5644$ is obtained by solving Eq. (33) with the method presented in section 2.2



(a)



(b)

Fig. 2. The absolute difference in the semi axes (a) $a_{Np} - a_{Tp}$ and (b) $b_{Np} - b_{Tp}$ against the scaling factor, p. Here, a_{Np} and b_{Np} are obtained by solving Eq. (33) for $k_0 = 150$ after Eq. (33) is scaled by $\alpha^3 = 10^{-6}$, 10^{-4}, 10^{-2}, 10^2, 10^4 and 10^6. On the other hand, a_{Tp} and b_{Tp} are calculated by scaling $a = 1.2911$ and $b = 1.0141$ with α as in Theorem 1, after each $a = 1.2911$ and $b = 1.0141$ is obtained by solving Eq. (33) with the method presented in section 2.2

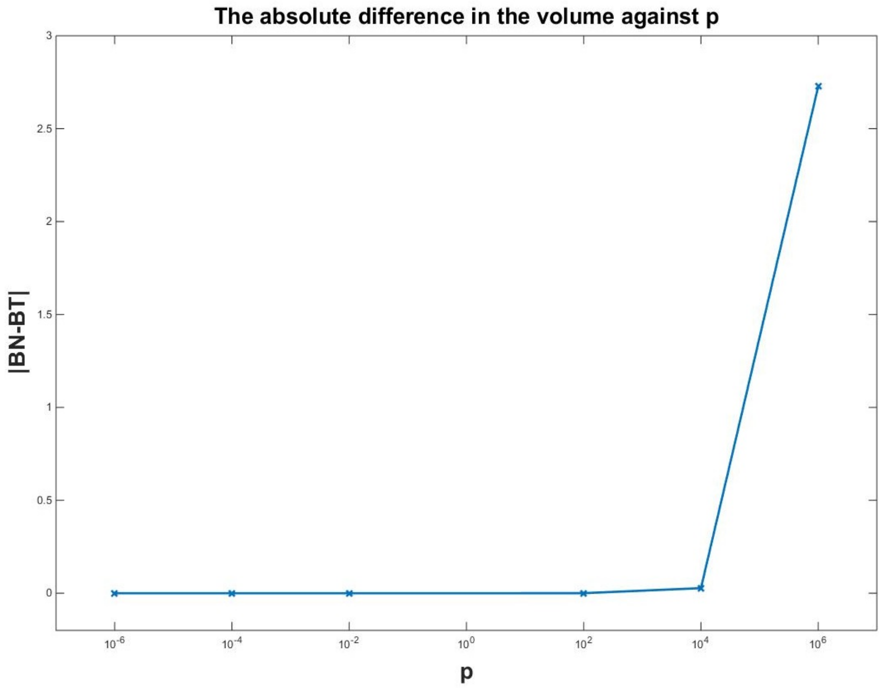
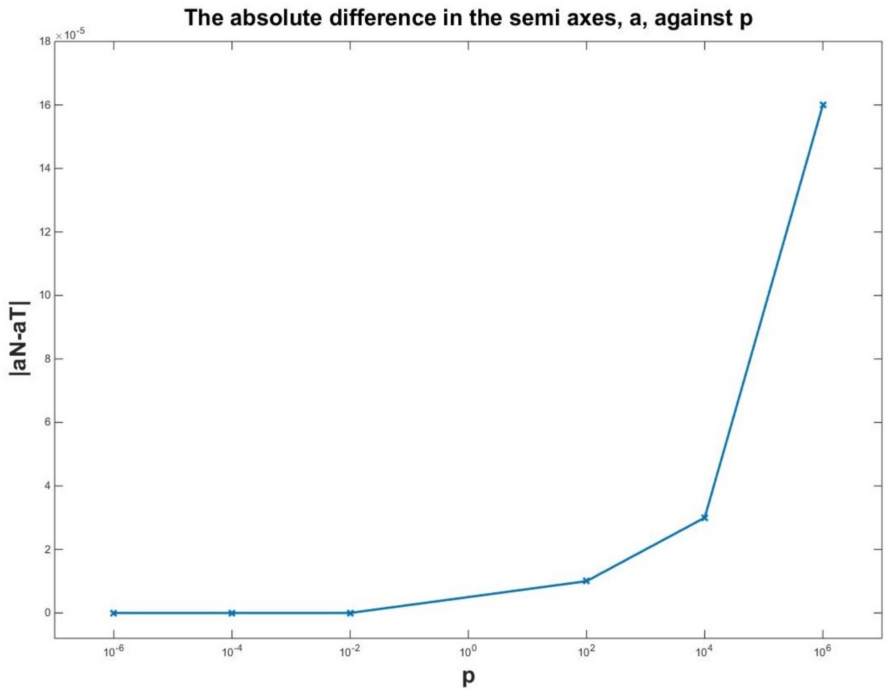


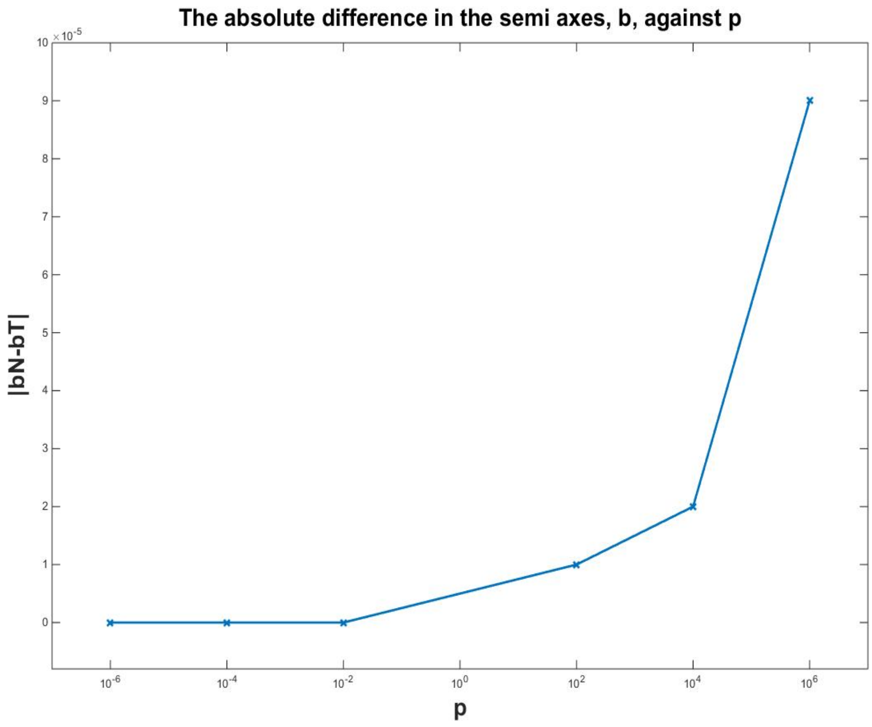
Fig. 3. The absolute difference in the volume, $|B|_{Np} - |B|_{Tp}$ against the scaling factor, p. Here, every $|B|_{Np}$ is obtained by solving Eq. (36) for $k_0 = 0.1$ after Eq. (36) is scaled by $\alpha^3 = 10^{-6}$, 10^{-4}, 10^{-2}, 10^2, 10^4 and 10^6. On the other hand, all $|B|_{Tp}$ are calculated by scaling $|B| = 12.7273$ with α as in Theorem 1, after $|B| = 12.7273$ is obtained by solving Eq. (36) with the method presented in section 2.2

Moreover, only the differences in the volume and all semi axes of the spheroid are investigated here due to the size of the scaling. The values of the depolarization factor and the eccentricity of the spheroid are not considered because they do not change although the first order PT for the spheroid is scaled.

Based on our finding in this section, we can now propose a method to improve the computation of the volume and all semi axes of the spheroid when the positive definite first order PT has too large coefficients or when the negative definite first order PT has too small coefficients at a fixed conductivity. Note that the results in Figure 1 until Figure 4 have suggested that there is no difference or only small difference in $|B|_{Np} - |B|_{Tp}$, $a_{Np} - a_{Tp}$ and $b_{Np} - b_{Tp}$ when p are small. Thus, the positive definite first order PT with too large coefficients can be firstly scaled by a small p so that all coefficients become smaller before being solved to obtain the volume and the semi axes of the spheroid. After that, the obtained volume and semi axes can then be used with Theorem 1 to reobtain the volume and the semi axes that are related to the original first order PT with the large coefficients. On the other hand, when the negative definite first order PT with too small coefficients is given, this first order PT can also be scaled by a small p so that all coefficients become larger before being solved to obtain the volume and the semi axes. Similarly, these volume and semi axes together with Theorem 1 can then be used to regenerate the volume and the semi axes that are related to the original first order PT with the small coefficients. In each case, more accurate results can be obtained by ensuring small value for e when solving for the volume and semi axes after the first order PT is scaled by the appropriate small p, before the volume and semi axes being reused with Theorem 1.



(a)



(b)

Fig. 4. The absolute difference in the semi axes (a) $a_{Np} - a_{Tp}$ and (b) $b_{Np} - b_{Tp}$ against the scaling factor, p. Here, a_{Np} and b_{Np} are obtained by solving Eq. (36) for $k_0 = 0.1$ after Eq. (36) is scaled by $\alpha^3 = 10^{-6}$, 10^{-4}, 10^{-2}, 10^2, 10^4 and 10^6. On the other hand, a_{Tp} and b_{Tp} are calculated by scaling $a = 1.0323$, $b = 1.7148$ with α as in Theorem 1, after each $|B| = 12.7273$, $a = 1.0323$, $b = 1.7148$ is obtained by solving Eq. (36) with the method presented in section 2.2

6. Conclusions

In this paper, we have investigated how the semi axes and other properties of the prolate and oblate spheroid are generated after the scaling on the matrix for the first order PT. In conclusion, some properties of the spheroid such as the volume and the semi axes do have an effect when its first order PT is scaled. Specifically, we found that the scaling factor on the semi axes of the spheroid is proportional to the scaling factor on the related first order PT.

Acknowledgments
The authors would like to thank Ministry of Education, Malaysia and Research Management Centre (RMC) UTM for providing financial support under vote FRGS/1/2019/STG06/UTM/02/4 and FRGS/1/2019/STG06/UTM/02/11 to conduct the study.

References
[1] Ammari, Habib, Hyeonbae Kang, and Mikyoung Lim. "Polarization tensors and their applications." In *Journal of Physics: Conference Series*, vol. 12, no. 1, p. 002. IOP Publishing, 2005. https://doi.org/10.1088/1742-6596/12/1/002
[2] Schiffer, M., and G. Szegö. "Virtual Mass and Polarization." *Transactions of the American Mathematical Society* 67, no. 1 (1949): 130-205. https://doi.org/10.2307/1990422
[3] Polya, G., and G. Szegő. *Isoperimetric Inequalities in Mathematical Physics*. No. 27. Princeton University Press, 1951. https://doi.org/10.1515/9781400882663
[4] Ammari, Habib, Thomas Boulier, Josselin Garnier, and Han Wang. "Shape recognition and classification in electro-sensing." *Proceedings of the National Academy of Sciences* 111, no. 32 (2014): 11652-11657. https://doi.org/10.1073/pnas.1406513111
[5] Khairuddin, Taufiq KA, and W. R. B. Lionheart. "Does electro-sensing fish use the first order polarization tensor for object characterization? Object discrimination test." *Sains Malaysiana* 43, no. 11 (2014): 1775-1779.
[6] Khairuddin, Taufiq Khairi Ahmad, and William R. B. Lionheart. "Polarization tensor: Between Biology and engineering." *Malaysian Journal of Mathematical Sciences* 10 (2016): 179-191.
[7] Khairuddin, Taufiq K. Ahmad, and William R. B. Lionheart. "Characterization of objects by electrosensing fish based on the first order polarization tensor." *Bioinspiration & Biomimetics* 11, no. 5 (2016): 055004. https://doi.org/10.1088/1748-3190/11/5/055004
[8] Peyton, Anthony J., David W. Armitage, Liam A. Marsh, Christos Ktistis, William Robert Breckon Lionheart, and Ari Järvi. "Walk through metal detection system." *U.S. Patent 9,562,986*, 2017.
[9] Marsh, Liam A., Christos Ktistis, Ari Järvi, David W. Armitage, and Anthony J. Peyton. "Determination of the magnetic polarizability tensor and three dimensional object location for multiple objects using a walk-through metal detector." *Measurement Science and Technology* 25, no. 5 (2014): 055107. https://doi.org/10.1088/0957-0233/25/5/055107
[10] Watson, Francis Maurice. "Better imaging for landmine detection: an exploration of 3D full-wave inversion for ground-penetrating radar." *PhD diss., University of Manchester*, 2016.
[11] Dekdouk, Bachir, Liam A. Marsh, David W. Armitage, and Anthony J. Peyton. "Estimating magnetic polarizability tensor of buried metallic targets for land mine clearance." In *Ultra-Wideband, Short-Pulse Electromagnetics* 10, pp. 425-432. Springer, New York, NY, 2014. https://doi.org/10.1007/978-1-4614-9500-0_38
[12] Marsh, Liam A., Christos Ktistis, Ari Järvi, David W. Armitage, and Anthony J. Peyton. "Three-dimensional object location and inversion of the magnetic polarizability tensor at a single frequency using a walk-through metal detector." *Measurement Science and Technology* 24, no. 4 (2013): 045102. https://doi.org/10.1088/0957-0233/24/4/045102
[13] Ammari, Habib. *An introduction to mathematics of emerging biomedical imaging*. Vol. 62. Berlin: Springer, 2008.
[14] Adler, Andy, and David Holder, eds. *Electrical impedance tomography: methods, history and applications*. CRC Press, 2004.
[15] Khairuddin, Taufiq Khairi Ahmad, and William R. B. Lionheart. "Some properties of the first order polarization tensor for 3-D domains." *Matematika* 29, no. 1 (2013): 1-18.
[16] Khairuddin, Taufiq Khairi Ahmad, and William R. B. Lionheart. "Computing the first order polarization tensor: welcome BEM++!." *Menemui Matematik (Discovering Mathematics)* 35, no. 2 (2013): 15-20.

[17] Khairuddin, Taufiq K. A., and William R. B. Lionheart. "Numerical comparisons for the approximated first order polarization tensor for ellipsoids." *Applied Mathematics and Computational Intelligence* 4, no. 1 (2015): 341-354.
[18] Śmigaj, Wojciech, Timo Betcke, Simon Arridge, Joel Phillips, and Martin Schweiger. "Solving boundary integral problems with BEM++." *ACM Transactions on Mathematical Software (TOMS)* 41, no. 2 (2015): 1-40. https://doi.org/10.1145/2590830
[19] Amad, A. A. S., P. D. Ledger, T. Betcke, and D. Praetorius. "Accurate Benchmark Computations of the Polarizability Tensor for Characterising Small Conducting Inclusions." *arXiv preprint arXiv:2106.15157* (2021).
[20] Sukri, Suzarina Ahmed, Yeak Su Hoe, and Taufiq Khairi Ahmad Khairuddin. "Study of different order of Gaussian Quadrature using Linear Element Interpolation in First Order Polarization Tensor." *Malaysian Journal of Fundamental and Applied Sciences* 17, no. 4 (2021): 343-353. https://doi.org/10.11113/mjfas.v17n4.2051
[21] Sukri, Suzarina Ahmed, Yeak Su Hoe, and Taufiq Khairi Ahmad Khairuddin. "Quadratic Element Integration of Approximated First Order Polarization Tensor for Sphere." *Malaysian Journal of Fundamental and Applied Sciences* 16, no. 5 (2020): 560-565.
[22] Ammari, Habib, and Hyeonbae Kang. *Polarization and moment tensors: with applications to inverse problems and effective medium theory*. Vol. 162. Springer Science & Business Media, 2007.
[23] Khairuddin, Taufiq Khairi Ahmad, and Nurhazirah Mohd Yunos. "Describing rotations of the spheroids based on the first order polarization tensor." *eProceedings Chemistry* 2, no. 2 (2017).
[24] Sukri, Suzarina Ahmed, Taufiq Khairi Ahmad Khairuddin, and Yeak Su Hoe. "The Effect of Translation on the Approximated First Order Polarization Tensor of Sphere and Cube." *Open Journal of Science and Technology* 3, no. 3 (2020): 274-282.
[25] Khairuddin, Taufiq KA, Paul D. Ledger, and William RB Lionheart. "Investigating the polarization tensor to describe and identify metallic objects." In *Proceedings of the World Congress on Engineering*, vol. 1, pp. 122-127. 2015.
[26] Sukri, Suzarina Ahmed, Taufiq Khairi Ahmad Khairuddin, Mukhiddin Muminov, Yeak Su Hoe, and Syafina Ahmad. "The Effect of Different Scale on Object to the Approximation of the First Order Polarization Tensor of Sphere, Ellipsoid, and Cube." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 87, no. 1 (2021): 108-117. https://doi.org/10.37934/arfmts.87.1.108117
[27] Khairuddin, T. K. Ahmad, and William R. B. Lionheart. "Fitting ellipsoids to objects by the first order polarization tensor." *Malaya Journal of Matematik* 4, no. 1 (2013): 44-53.
[28] Mohamad Yunos, N., and T. K. Ahmad Khairuddin. "Adapting depolarization factors in the first order polarization tensor for spheroid." *Final Year Project Proceeding (Department of Mathematical Sciences, UTM JB)* 2 (2017): 383-390.
[29] Yunos, Nurhazirah Mohamad, Taufiq Khairi Ahmad Khairuddin, Sharidan Shafie, Tahir Ahmad, and William Lionheart. "The depolarization factors for ellipsoids and some of their properties." *Malaysian Journal of Fundamental and Applied Sciences* 15, no. 6 (2019): 784-789.
[30] Khairuddin, Taufiq Khairi Ahmad, Nurhazirah Mohamad Yunos, and Sharidan Shafie. "Fitting the first order PT by spheroid: A semi analytical approach." In *AIP Conference Proceedings*, vol. 2184, no. 1, p. 060059. AIP Publishing LLC, 2019. https://doi.org/10.1063/1.5136491
[31] Bahuriddin, Nur Safirah, Mukhiddin Muminov, Taufiq Khairi Ahmad Khairuddin, Syafina Ahmad, and Wan Rohaizad Wan Ibrahim. "An Extended Method for Fitting the First Order Polarization Tensor to a Spheroid." *Journal of Advanced Research in Applied Sciences and Engineering Technology* 23, no. 1 (2021): 8-17. https://doi.org/10.37934/araset.23.1.817
[32] Khairuddin, TK Ahmad, N. Mohamad Yunos, Z. A. Aziz, T. Ahmad, and W. R. B. Lionheart. "Classification of materials for conducting spheroids based on the first order polarization tensor." In *Journal of Physics: Conference Series*, vol. 890, no. 1, p. 012035. IOP Publishing, 2017. https://doi.org/10.1088/1742-6596/890/1/012035
[33] Stoner, Edmund C. "XCVII. The demagnetizing factors for ellipsoids." *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 36, no. 263 (1945): 803-821. https://doi.org/10.1080/14786444508521510
[34] Milton, Graeme W. *The Theory of Composites*. Cambridge University Press, 2002. https://doi.org/10.1017/CBO9780511613357
[35] Kang, Hyeonbae. "Layer potential approaches to interface problems." *Inverse problems and imaging, Panoramas et Syntheses, Societe Mathematique de France, to appear* (2015).