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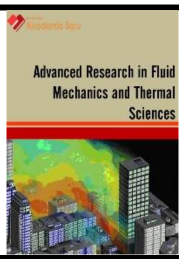

Convergence Order Prediction of CVFEM Solutions Using the Richardson Extrapolation Method on Unstructured Grids

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ABSTRACT

In this work, our proposed idea is based on using the Richardson Extrapolation (RE) method for predicting convergence the *pth* order solution on unstructured grids [1]. Practically, we want to analyze the variation the convergence order compared the solution accuracy numerically of the Control Volume Finite Element Method (CVFEMs) [2, 3]. To this effect, the proposed model a two-dimensional solving Navier-Stokes (N-S) equations coupled with the energy equation, with an irregular domain using six cases different unstructured meshes. All the numerical results are presented and discussed; they have been obtained from our code FORTRAN program.

1. Introduction

The problems of physical phenomena, concerning fluid mechanics, are described by partial differential equations. Generally, we are not able to solve them analytically, or with approximate numerical solutions. In the most common approaches, the simplest form of the equations described is generally used. Thus, to obtain an approximate numerical solution is a function of a discretization of the method, it is necessary to approximate these equations with the initial and boundary conditions, in the form of algebraic approximation equations.

This algebraic approximation is the starting point of any numerical method, and the final point is the mathematical model. The best known, whose numerical methods are: Finite Control Volume method [4,5], Finite Difference method [6-8] and Finite Element method [9-11], etc. All these numerical methods mentioned above have disadvantages and advantages in solving convection-diffusion problems. Among the disadvantages, instability at convergence towards a satisfactory digital solution may be difficult to achieve by hard non-linearities under convection terms. In addition, with the lack of information, when applied in a complex geometry [12-14] , the sensitive variations in fluid properties [12-18], etc., leads to the expensive solution, is sometimes impossible. To avoid or minimize these disadvantages, researchers are researching a new idea based on hybrid

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numerical methods, for example, the approximation of the solution of the diffusion equation by the hybrid called the Finite Control Volume Difference method [19].

In this study, we focused on a method called (CVFEM) which is a hybrid of two methods, the Finite Control Volume method, and the two-dimensional Finite Element method. From the deep point of view, the effect of combining convection and diffusion in solving a fluid flow problem by the action of external forces can be determined by surface forces such as pressure and body forces by gravity [2-22], sometimes this forces represent by combining the electromagnetic and gravitational forces [17-26]. It is well known that the accuracy of numerical solutions depends on the quality of the discretization (the numerical method used). Therefore, each numerical solution contains errors; it is important to know how accurate the errors are, how effective the solution is and whether it is acceptable in the application concerned [27,28].

The Richardson Extrapolation method, first applied by [29]. The basic idea is to combine two discrete solutions f1 and f2, for two different fine and coarse discrete grids with spacing h1 and h2, respectively. In such a way that, eliminate the leading order error, and get exact solution value with great accuracy [30]. From a mesh size perspective, you can apply Richardson Extrapolation in uniform and non-uniform grids. It is generally possible, by a second-order process of the Richardson Extrapolation method, with three successive grids, to calculate the order convergence in higher order terms, and to calculate the exact solution, provided that all three are sufficiently fine [1,30,31]. It is therefore interesting to apply this process in this paper. Although the errors in the solutions are of a different order of magnitude [32]. In addition, the solution error is close to the relative error and the actual fractional error; with the two latter, the convergence index of the fine grid can be determined. The objective is to apply all the above, focusing on the numerical method used in particular on the application of Richardson's extrapolation to evaluate the order of convergence on irregular geometry with unstructured grids.

2. Governing Equations

In this present research, the fluid is assumed to be steady-state, two-dimensional, laminar, Incompressible Newtonian fluid. The equations that govern a dependent variable ϕ will be written into the according to the compact [33].

$$\vec{\nabla}.\vec{J} = S_\phi \tag{1}$$

Where S_ϕ the source term, and $\vec{J}$ is the overall flow consisting of the diffusion and convection flow. So, $\vec{J}$ writing in this form.

$$\vec{J} = \rho\vec{V}\phi - \Gamma\,\vec{\nabla}\phi \tag{2}$$

Where ρ and $\vec{V}$ representing the density and velocity of the fluid, respectively. Γ equals the thermal diffusivity in the energy equation, on the other hand, in momentum equations is the dynamic viscosity. $\vec{J}$ can be expressed as

$$\vec{J} = J_x\vec{i} + J_y\vec{j} \tag{3}$$

Eq. (3) is represented the overall flow in the Cartesian components compared the $\vec{i}$ and $\vec{j}$ the unit vector.

In this context, we present the Control-Volume Finite-Element Method (CVFEM), by the formulation adopted; for that, Eq. (1) permitted the departure.

$$\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}=S_\phi \tag{4}$$

When projecting the velocity using the two components, in the *x*, *y* directions, respectively. So, J_x and J_y is given by

$$J_x=\rho u\phi-\Gamma\frac{\partial \phi}{\partial x}\ , \qquad J_y=\rho v\phi-\Gamma\frac{\partial \phi}{\partial y} \tag{5}$$

If integrated the Eq. (5) to Eq. (4), with taking it as a consideration $\phi=(u,v)$, add on the right side of Eq. (4) the gradient of pressure $-\nabla p=(-\partial P/\partial x,-\partial P/\partial y)$, this gives the Stokes equations for Incompressible flow, All terms in Eq. (4) are written according to the unknowns related to the dependent variables and in the of partial derivatives form; more of that, Eq. (4) is presented in a highly conservative in the two-direction *x*, *y*. But with existed source terms push in the weak direction.

3. Discretization of Numerical Method

As part of discretization Eq. (4) by the method concerned, call the Volume Finite Element Method (CVFEMs). Of point seen numerical, we used the process co-located, equal-order [2-20], that means that all interesting values for the dependent variables are stockpile at nodes interested. For the mesh of the domain are realized by a mesh with three-node triangles non-uniform and with integrating Eq. (4) on over control volume associated at each node [34] allows giving. The final algebraic approximation to the total flux in *aoc* of the element internal *ijk* as follows.

$$\int_a^o \vec{J}.\vec{n_i}ds+\int_o^c \vec{J}.\vec{n_k}ds-\int_{iaoc} S_\phi dV=C_i^{\phi_i}\phi_i+C_j^{\phi_j}\phi_j+C_k^{\phi_k}\phi_k+D^\phi \tag{6a}$$

Likewise, can have the algebraic approximation for the boundary node *l* for element *l l-1K*

$$\int_l^a \vec{J}.\vec{n}_{l-1}ds+\int_a^{o_{l-1}} \vec{J}.\vec{n}_K ds+\int_{o_{l-1}}^c \vec{J}.\vec{n}_{K+1}ds-\int_{lal-1c} S_\phi dV=C_l^{\phi}\phi_l+C_{l-1}^{\phi_{l-1}}\phi_{l-1}+C_k^{\phi_k}\phi_k+D^\phi \tag{6b}$$

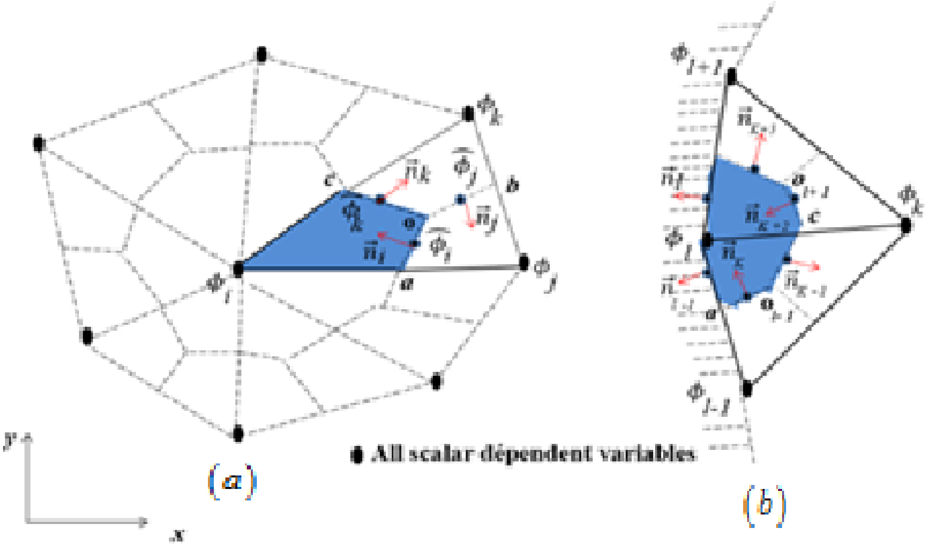

Fig. 1. Schematic representation of notation used: (a) for each node inside; (b) node the boundary conditions

All calculations with reference to Figure 1, and after summing up all cell face fluxes and sources for both boundary nodes and internal nodes in Eq. (6a) and Eq. (6b), the discretized is given by Acharya *et al.,* [35] the following general form.

$$a_i^{\phi}\phi_i = \sum_{nb,i} a_{nb,i}^{\phi}\phi_{nb,i} + D_i^{\phi} \tag{7}$$

The coefficient a_i^{ϕ} are determined by the neighbor coefficients $a_{nb,i}^{\phi}$. So, if we using the process defined by Eq. (7) into obtained the velocity components for the momentum conservation equations. By following, to the estimation and calculation of convective terms with a procedure called "Mass-Weighted Upwind" scheme, the advantage this scheme tailoring of the positive coefficients in the algebraic discretization equation [19-36]. With the same idea, of Eq. (6a) and Eq. (6b) is applicated and used the mass flux vector $\vec{J}$ equal $\rho\vec{V}$, we get the discretized equation of pressure, the detail in the reference [19] see Table 1.

Table 1
Representation of the velocity components and pressure

ϕ_i	$\phi_{nb,i}$	a_i^{ϕ}	$a_{nb,i}^{\phi}$	D_i^{ϕ}
u_i	$u_{nb,i}$	a_i^u	$a_{nb,i}^u$	$b_i^u - \left(\overline{\partial P/\partial x}\right)_{V_i} V_i$
v_i	$v_{nb,i}$	a_i^v	$a_{nb,i}^v$	$b_i^v - \left(\overline{\partial P/\partial y}\right)_{V_i} V_i$
p_i	$p_{nb,i}$	a_i^p	$a_{nb,i}^p$	D_i^p

In Table 1, the terms $\left(\overline{\partial P/\partial x}\right)_{V_i}$ and $\left(\overline{\partial P/\partial y}\right)_{V_i}$ are the pressure gradients associated with the volume-averaged values the total volume V_i. Of more, the terms b_i^u and b_i^v are the volumetric integral of all body forces. For D_i^p in pressure are the function of the pseudo-velocity fields.

4. Richardson Extrapolation Method

We can determine the error estimation discretization on grid size level *k* by the difference enters the exact solution and the discrete solution; the error Er_k, is given by:

$$Er_k = \phi_{exact} - \phi_k \tag{8}$$

where ϕ_{exact} is the exact solution and ϕ_k is a discrete solution value on grid size level *k*. You can use Eq. (8) point-by-point basis locally if your domain has of the global quantities. Other writing, using the Richardson Extrapolation (RE) method [1-38], basic on the Taylor expansion in the discretization space with the order of the truncation error term; as follows.

$$\phi_{exact} - \phi_k = \alpha_p h_k^p + \mathrm{O}\left(h_k^{p+1}\right) \quad k = 1, 2, 3, ...etc \tag{9}$$

The term $\alpha_p h_k^p$ must dominate the discretization error, and $\mathrm{O}\left(h_k^{p+1}\right)$ are the higher-order terms. For *p* is the order of the truncation error and expressed the convergence order of the numerical method used. Can be replacing ϕ_{exact} by the solution extrapolated ϕ_{ext} when h_k is small.

In this case, select three significantly different set of grids size as $h_k > h_{k+1} \quad k = 1, 2$ (coarse > fine), also be verified in the asymptotic grid convergence range. As can defining the difference between a solution variable on mesh *k+1* and mesh *k* levels as

$$\zeta_{k+1,k} = \phi_{k+1} - \phi_k \qquad k = 1, 2 \tag{10}$$

It is desirable that the grid refinement factor are, $r_{f,c} = h_{coarse} / h_{fine}$, and the variable ϕ critical of the simulation numerical being reported. The *pth* order of the convergence the scheme can be expressed as

$$P = \left| \left[\log \left| \frac{\zeta_{21}}{\zeta_{32}} \right| \Big/ \log\left(r_{32}\right) \right] - F\left(p\right) \right| \tag{11a}$$

Where the term $F(p)$ equal

$$F\left(p\right) = \log\left|\left(r_{21}^p - 1\right) / \left(r_{32}^p - 1\right)\right| \Big/ \log\left(r_{32}\right) \tag{11b}$$

If the higher-order terms are neglected, and then solving the coefficient α_p. In the end, the exact solution ϕ_{exact} results in

$$\phi_{exact} \cong \left[\phi_{k+1} + \left(\phi_{k+1} - \phi_k\right) / \left(r_{k+1,k}^p - 1\right)\right] \quad k = 1, 2 \tag{12}$$

The Richardson extrapolation (RE) method based on the results in computing the numerical solutions of our method $\phi_k (1 \le k \le 3)$ which continuous of the initial and boundary values, on h_k different nested uniform or non-uniform mesh or grids of size [1-39]. In that case with we have h_1 and h_3 are the coarsest grid the finest one, respectively. Can be given by

$$h_k = \left[\frac{1}{N}\sum_{i=1}^{N}(\Delta V_i)\right]^{1/2} \quad k = 1,2,3 \tag{13}$$

In the above expression, ΔV_i the total of the sub-control volume surrounding at node *i*, for *N* is the total number of control volume used for the simulation.

To give a quick and easy way of comparing the result values by this method and using the Richardson Extrapolation (RE) method for each quantity presented in this paper, we have added computed the Grid Convergence Index the fine grid.

4.1 Grid Convergence Index the Fine Grid

In general, the measurement of error of the fine grid solution is an approximate relative error for any two grids is defined as

$$Er_a = \left|\left(\phi_f - \phi_c\right)/\phi_f\right| \tag{14}$$

Where ϕ_c and ϕ_f are the solutions computed with the coarse and the fine grids, respectively. As can be defined, the Actual fractional error or Extrapolated relative error A_{ext}^{f} for the fine grid solution.

$$A_{ext}^{f} = \left|\left(\phi_{ext} - \phi_f\right)/\phi_{ext}\right| \tag{15}$$

The computation of the Convergence Index the fine grid, GCI_f is suggested by the first author [1-30], he has determined their expression with by the relative error Er_a is equal to

$$GCI_f^{a} = 3\left|Er_a\right|/\left(r_{f,c}^{p} - 1\right) \tag{16a}$$

If we use Eq. (12) in Eq. (15) and with the principle the Convergence Index the fine grid [1-30], so the Grid Convergence Index extrapolated GCI_f^{ext} is

$$GCI_f^{ext} = 3\left|A_{ext}^{f}\right|/\left[\left(r_{f,c}^{p} - \phi_c/\phi_f\right)/\left(r_{f,c}^{p} - 1\right)\right] \tag{16b}$$

Thus, the results the Richardson's extrapolation (RE) method can be used to compute an approximate relative error obtained on successively refined coarse grids Eq. (14). As a consequence, the Richardson extrapolation (RE) method can determine grid convergence index for the Fine Grid Solution $(GCI)_f$ depending on relative and Actual fractional error, based on the grid refinement factor with *P* order accurate solution of the fine grid Eq. (16a) and Eq. (16b).

5. Convergence Analysis in Two-Dimensional

In this study, we consider the flow state and the fluid-property cited before in the second section for applied in a cavity with a vertical complex wavy wall of height L, with λ the fundamental wavelength associated with the wavy surface, and *a* the amplitude of the complex wavy surface, average width W as shown in Figure 2. In the Table 2 below, the following nondimensionalization was employed:

$$X = \frac{x}{W},\ Y = \frac{y}{W},\ U = \frac{u\,\lambda}{\alpha},\ V = \frac{v\,\lambda}{\alpha},\ P = \frac{P^{*}\,\lambda^{2}}{\rho\alpha^{2}},\ \theta = \frac{T - T_C}{\Delta T},\ Ar = \frac{\lambda}{W},\ \eta = \frac{a}{W},\ A = \frac{L}{W}$$

Where Ar, η and A are surface wavelength waviness, surface amplitude waviness and the geometric quantity aspect ratio, respectively. The physical magnitudes nondimensionalization: $\alpha = K/\rho C_p$, the thermal diffusivity, $\Pr = \upsilon/\alpha$, the Prandtl number, and $Ra = g\beta\Delta T\lambda^{3}/\upsilon\alpha$, the Rayleigh number. The flux and the source term are written in dimensionless form see Table 2.

Table 2
Summarize the system of the equations the fluid flow and heat transfer

	J_x	J_y	S_ϕ
Continuity	U	V	0
X-momentum	$UU - \Pr.Ar\frac{\partial U}{\partial X}$	$VU - \Pr.Ar\frac{\partial U}{\partial Y}$	$-\frac{\partial P}{\partial X}$
Y-momentum	$UV - \Pr.Ar\frac{\partial V}{\partial X}$	$VV - \Pr.Ar\frac{\partial V}{\partial Y}$	$-\frac{\partial P}{\partial Y} + \frac{Ra\Pr}{Ar}\theta$
Energy	$U\theta - Ar\frac{\partial \theta}{\partial X}$	$V\theta - Ar\frac{\partial \theta}{\partial Y}$	0

The associated, the velocity and temperature boundary conditions take the following form:

$$0 \le Y \le A \quad X = \eta\left[1 - \sin 2\pi\left(\frac{Y}{Ar}\right)\right] \tag{17a}$$
$$U = V = 0,\ \theta = 1$$

$$0 \le Y \le A,\ X = (1-\eta) - \eta\left[1 - \sin 2\pi\left(\frac{Y}{Ar}\right)\right] \tag{17b}$$
$$U = V = 0,\ \theta = 0$$

$$Y = 0,\ \eta\left[1 - \sin 2\pi\left(\frac{Y}{Ar}\right)\right] \le X \le (1-\eta) - \eta\left[1 - \sin 2\pi\left(\frac{Y}{Ar}\right)\right] \tag{17c}$$
$$U = V = 0,\ \frac{\partial \theta}{\partial Y} = 0$$

$$Y = A \quad \eta\left[1 - \sin 2\pi\left(\frac{Y}{Ar}\right)\right] \le X \le (1-\eta) - \eta\left[1 - \sin 2\pi\left(\frac{Y}{Ar}\right)\right] \tag{17d}$$
$$U = V = 0,\ \frac{\partial \theta}{\partial Y}$$

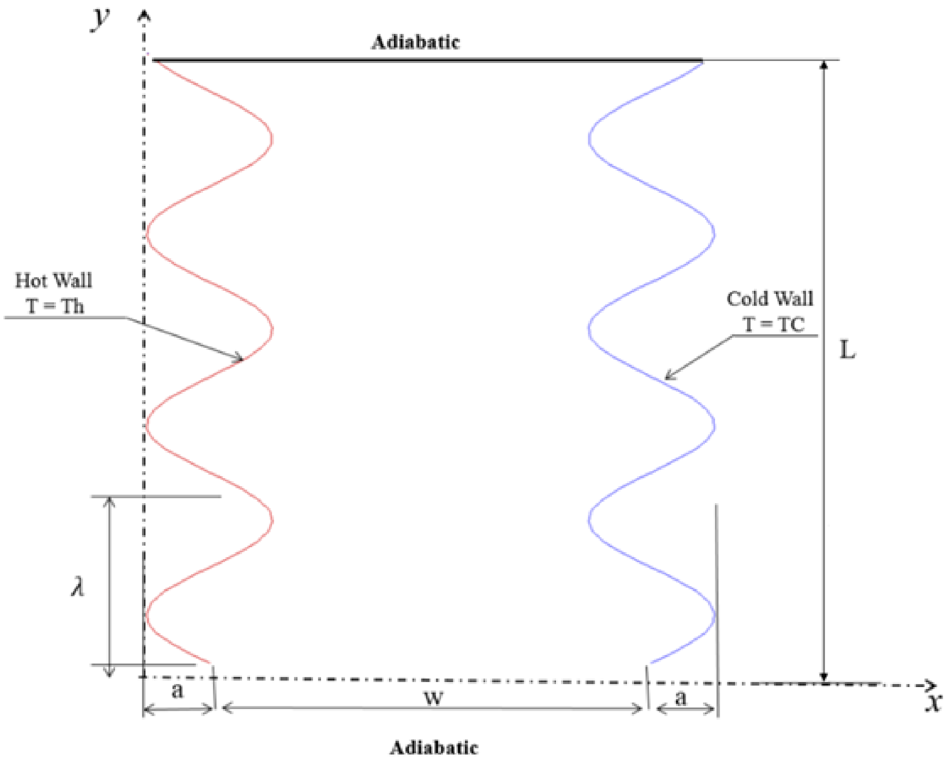

Fig. 2. Physical model of vertical wavy enclosure

The numerical results are presented with the fix of all the parameters geometrical as following η = 0.1, A =1, Ar = 0.3125 of the wavy surfaces. The procedure used in the solution of the nonlinear in coupled the equations see Table 2 the Sequential Iterative Variable Adjustment (SIVA) [20]. In every iteration cycle, the underrelaxation scheme is used in this method, in equations velocity equals 0.6, for pressure in equation continuity no underrelaxation. Results accepted by convergence when the residuals the continuity equation and the momentum equations, were all less than 10^{-5} in all cases, the majority a simulations convergence between 150-1500 iterations and time 10 sec - 2h. The coding is done in FORTRAN Modular programming a maximum field length of 150 (octal) words, it runs on a computer Pentium Dual-Core CPU E5700 and 4,00 Go RAM 3.00 GHz. The control parameter is the average number convergence of Nusselt in the hot wall is calculated by the following expression.

$$\overline{Nu}_{cg} = \frac{1}{A}\int_{0}^{A} Nu_{l,cg}\, dl \tag{18}$$

where $Nu_{l,cg}$ the convergence local Nusselt number at the curve length *l*.

Solutions of the average Nusselt number were obtained for six case different grid size. Table 3 illustrates the calculation procedure the Average Nusselt number Eq. (18) and the meshes used at the Rayleigh number of $Ra = 10^3, 10^4, 10^5$ and 10^6, and $\Pr = 0.71$.

Table 3
Comparison of convergence average Nusselt number on wavy hot wall for η = 0.1, A = 1, Ar = 0.3125 the coarse to fine grids of size

Ra	Case	$\sum_{i=1}^{N}(\Delta V_i)$	h_k	$\overline{Nu}_{cg}$	Case	$\sum_{i=1}^{N}(\Delta V_i)$	h_k	$\overline{Nu}_{cg}$
10^3	1	0.01469	0,0135	1.217	4	0,00768	0,0070	1.295
10^4				2.368				2.532
10^5				4.659				5.039
10^6				9.085				10.312
10^3	2	0.01174	0,0108	1.248	5	0,00624	0,0056	1.312
10^4				2.438				2.565
10^5				4.837				5.11
10^6				9.67				10.566
10^3	3	0.00927	0,0085	1.274	6	0,00508	0,0045	1.327
10^4				2.493				2.591
10^5				4.962				5.152
10^6				10.063				10.649

6. Results and Discussion

Application of the Richardson extrapolation uses various grids (coarse to fine) for giving the convergence order and extrapolated values the integral quantities.

6.1 Influence on the Richardson Extrapolation Convergence Order

In this section, we are interested to see the influence the extrapolated average Nusselt number values using Table 4 compared with the convergence orders. The general conclusion from Table 4, increase convergence orders parallel the values calculated by the Richardson Extrapolation method of the Average Nusselt number. As a consequence, when the convergence orders approximated to 2.060 for cases (1,2,3) at Ra = 10^6 likewise for cases (4,5,6) is 2.431 at Ra = 10^5 corresponded stability of this number. Other by, for the values more than 2 e.g. cases (4,5,6) at Ra = 10^6 we have a 1/100 influence.

Table 4
Richardson extrapolation applied to average Nusselt number for η = 0.1 , A =1, Ar = 0.3125

Ra	Cases used	P	$\overline{Nu}_{ext}$	Cases used	P	$\overline{Nu}_{ext}$
10^3	1,2,3	1.136	1.356	4,5,6	0.562	1.444
10^4		1.407	2.629		1.091	2.689
10^5		1.875	5.181		2.431	5.213
10^6		2 .060	10.676		5.205	10.689
10^3	2,3,4	0.074	2.661	1,3,5	0.614	1.441
10^4		0.709	2.781		0.962	2.709
10^5		1.396	5.271		1.355	5.303
10^6		1.260	11.157		1.232	11.308
10^3	3,4,5	1.252	1.367	2,4,6	0.763	1.409
10^4		1.046	2.695		0.953	2.707
10^5		0.646	5.586		1.221	5.316
10^6		0.176	17.150		1.370	11.070

6.2 Richardson Extrapolation of Uncertainty Estimates

It is important to calculate various uncertainty estimates, who are in function of convergence order of this the numerical method used. The values calculated are of the estimations in the form of relative error and Extrapolated error of the fine grid solution, in order to establish the Convergence Index for Fine Grid Solution (GCI). The figure below represents all these variations for different combined grid size.

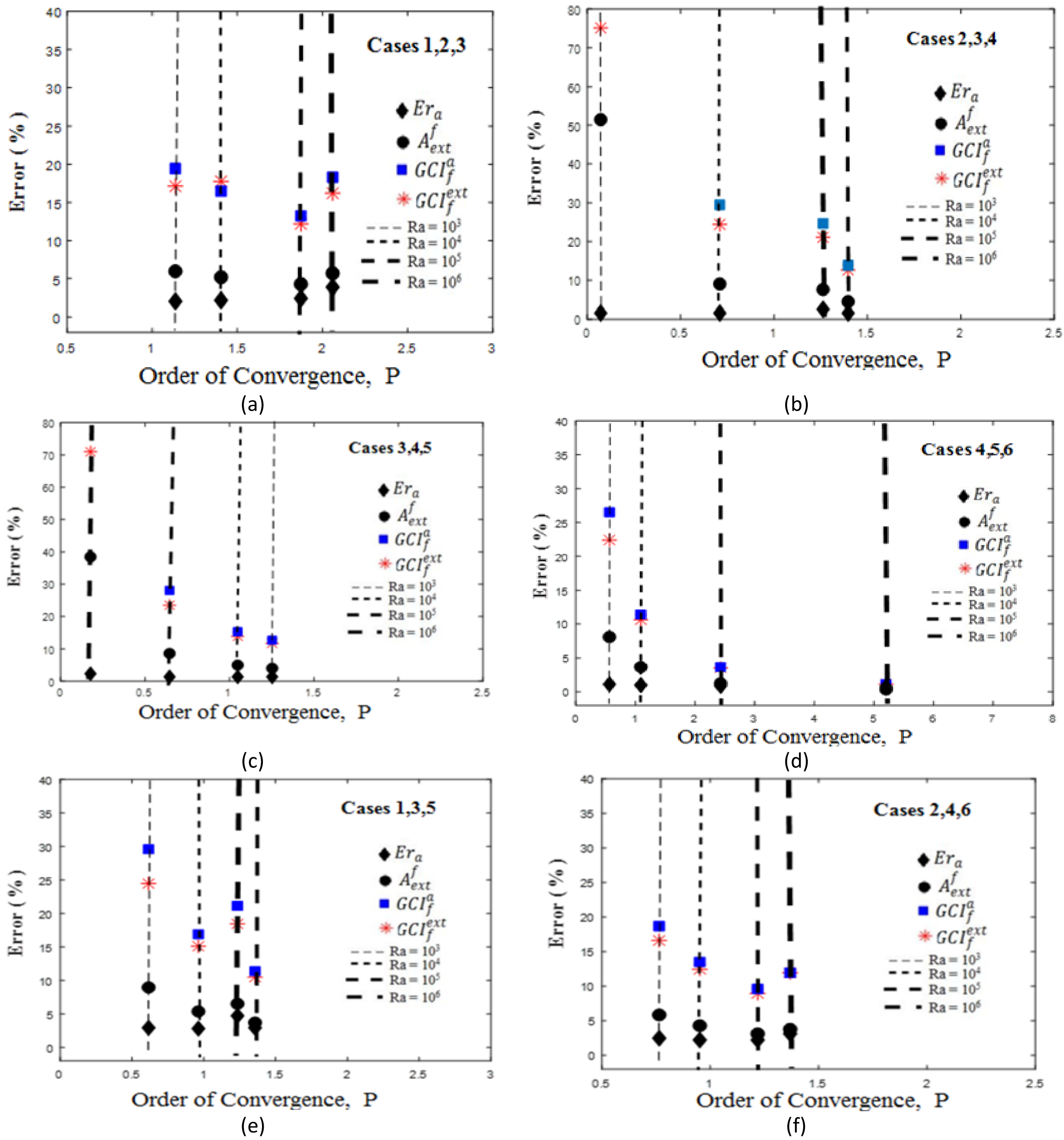

Fig. 3. Variation of various uncertainty estimates with *pth* order of Convergence η = 0.1, A = 1, Ar = 0.3125

It is shown in Figure 3 the Extrapolated errors are always superior to relative errors, therefore, the gap decreases with increasing the order of convergence. The same conclusion for the Grid Convergence Index relative errors compared with the Grid Convergence Index extrapolated. In Figures 3(c) and 3(d) the order of convergence less than 0.5 the Grid Convergence Index are more than 70% relative or extrapolated and the Actual fractional errors are more than 45%.

7. Conclusions

In this paper, we use Richardson's theory of generalized extrapolation, which is based on the calculation, convergence order and refinement error estimator of the grid. In this latter the errors are integrated into the Grid Convergence Index the fine grid of Eq. (19a) or Eq. (19b) is presented in details. The three important findings are summarized below.

For that, it has been proposed in the results the average Nusselt number at $Ra=10^3$-10^6, Pr =0.71 the problem two-dimensional steady-state of fluid flow and heat transfer with the geometrical, the parameters wavy walls is η = 0.1, A=1, Ar= 0.3125.

i. We encountered stability in computation, so these results are simultaneously good.
ii. This method is second-order Convergence (the Mass-Weighted Upwind scheme has not influenced more than 2 an order convergence) in this computation problem.
iii. The Grid Convergence Index will often be less optimistic when the convergence order confined (0.5 - 2).

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