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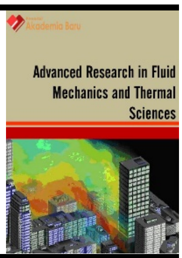

Weak Galerkin and Weak Group Finite Element Methods for 2-D Burgers' Problem

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Najat Jalil Noon[1,*]

[1] Department of Mathematics, College of Education for Pure Science, University of Basrah, Basra, Iraq

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ABSTRACT

In this paper, we apply weak Galerkin finite element method (WGFEM) and weak group finite element method (WGrFEM) for 2-D Burgers' problem by using the weak functions and their corresponding discrete weak derivatives in standard weak form. We consider the fully formulation using the backward-difference form for the time variable and we prove the stability and an error estimate for these methods. The numerical examples of our methods have been carried out through implementation in MATLAB programs and compared with the exact solutions and other available literature to illustrate the efficiency of the proposed methods, the WGFEM and the WGrFEM provide convergent approximations and handles the equation well in different cases, the results obtained of our methods are very acceptable and competent more than the results available in the other literature. Moreover, a WGrFEM is displayed to be better than the GFEM.

1. Introduction

The nonlinear Burgers' equation has been widely used to model several physical flow phenomena in fluid dynamic teaching and in engineering. In this paper, we consider the following time-dependent Burgers' problem in two dimensions

$$u_t - \epsilon\nabla.\nabla \mathrm{u} + \mathrm{u}u_x + uu_y = f \;\; \Omega \times [0,T]\,, \tag{1}$$

with the initial-boundary conditions

$$u(x,y,t) = 0 \text{ on } \partial\Omega \times [0,T],$$

$$u(x,y,0) = u^0(x,y) \text{ on } \overline{\Omega} \times \{0\},$$

* *Corresponding author.*
E-mail address: fini.burg@gmail.com (Najat Jalil Noon)

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where $u = u(x, y, t)$, $\epsilon > 0$ is a viscosity constant, $\Omega \subset R^2$ with boundary $\partial\Omega$, $\bar{\Omega} = \Omega \cup \partial\Omega$, and $f = f(x, y, t) \in L^2(\Omega)$. The grouped form of Burgers' equation is represented as follows, here the uu_x and uu_y terms are replaced by $\frac{1}{2}(u^2)_x$ and $\frac{1}{2}(u^2)_y$, respectively. Burgers' problem is then written as:

$$u_t - \epsilon\nabla.\nabla u + \tfrac{1}{2}(u^2)_x + \tfrac{1}{2}(u^2)_y = f \; on \; \Omega \times [0, T], \tag{2}$$

$u(x, y, t) = 0$, on $\partial\Omega \times [0, T], u(x, y, 0) = u^0(x, y)$, on $\bar{\Omega} \times \{0\}$

The weak variational forms to Eq. (1) and Eq. (2) respectively with fully formulation utilizing backward difference quotient for the time discretization such as the following:

Let $0 = t^0 < t^1 <\ldots< t^M = T$ be a partition for time interval $[0, T]$, where $t^n = n\,\Delta\,t$, Δt is the time step and M is a positive integer, find $u^n \in H^1(0, T; H_0^1(\Omega))$, $(n = 1,2, \ldots, M)$, such that,

$$\left(\bar{\partial}_t u^n, v\right) + \epsilon(\nabla u^n, \nabla v) + (u^n.\nabla u^n, v) = (f^n, v), \;\; \Omega \times [0, T], \forall\, v \in H_0^1(\Omega) \tag{3}$$

$u^n(x, y, t) = 0$, on $\partial\Omega \times [0, T], u(x, y, 0) = u^0(x, y)$, on $\bar{\Omega} \times \{0\}$

$$\left(\bar{\partial}_t u^n, v\right) + \epsilon(\nabla u^n, \nabla v) - \tfrac{1}{2}((u^n)^2, \nabla v) = (f^n, v), \;\; \Omega \times [0, T], \forall\, v \in H_0^1(\Omega) \tag{4}$$

$u^n(x, y, t) = 0$, on $\partial\Omega \times [0, T]$, $u(x, y, 0) = u^0(x, y)$, on $\bar{\Omega} \times \{0\}$

where, $\bar{\partial}_t u^n = \frac{u^n - u^{n-1}}{\Delta t}$, $u^n.\nabla u^n = (u^n, u^n)^t.\nabla u^n$.

The GrFEM or otherwise called product approximation is a finite element (FE) technique for definite sorts of nonlinear PDEs. Experiments with the GrFEM have shown an increase in economy and in the nodal accuracy compared to FE solutions of the Burgers' equations [3,5,6] and for other problems [8,4,9,27,32]. Recently, the weak Galerkin finite element method has catched much consideration in the field of numericaln PDEs , The idea of the weak Galerkin method was first introduced by Wang and Ye [14], the method was applied to the second order elliptic equations [1,15,16,19,20,22,24,28], the Stokes equations[17,21], Parabolic equations [10,11,25], biharmonic equations [23,26], Navier-Stokes equations [7,18,30] and 1-D Burgers' equation [31], etc. In this paper we present (WGFEM) and (WGrFEM) for 2-D Burgers' problem with a fully-discrete approximation for the time variable, the backward-difference formula for the time variable is examined and the stability and error estimate are proved for these methods. The numerical solutions of WGFEM and WGrFEM are compared with the exact solutions and with other available solutions to measure the numerical errors and the efficiency of our methods, the results obtained of WGFEM and WGrFEM are quite satisfactory and competent more than the results available in the other literature, moreover, (WGrFEM) is demonstrated to be better than (WGFEM).

2. Preliminaries and Notations

We use standard definitions for the Sobolev spaces $H^m(\Omega)$, and their associated inner products $(.,.)_{m,\Omega}$, norms $\|.\|_{m,\Omega}$ and semi-norms $|.|_{m,\Omega}$ for $m \geq 0$ any integers $s \geq 0$ the semi-norm $|.|_{s,\Omega}$ is given by,

$$|v|_{s,\Omega} = \left(\sum_{|\sigma|=s} \int_{\Omega} |\partial^{\sigma} v|^2 d\Omega \right)^{\frac{1}{2}},$$

where, $\sigma = (\sigma_1, \sigma_2)$, $|\sigma| = \sigma_1 + \sigma_2$, $\partial^{\sigma} = \partial_x^{\sigma_1} \partial_y^{\sigma_2}$.

The Sobolev norm $\|.\|_{m,\Omega}$ is given by,

$$\|v\|_{m,\Omega} = \left(\sum_{j=0}^{m} |v|_{j,\Omega}^2 \right)^{\frac{1}{2}},$$

The space $H(div, \Omega)$ is defined as the set of vector-valued functions on Ω which together with their divergence are square integrable,

$$H(div, \Omega) = \{v: v \in [L^2(\Omega)]^2 ,\ \nabla. v \in L^2(\Omega)\},$$

the norm in $H(div, \Omega)$ is defined by,

$$\|v\|_{H(div,\Omega)} = (\|v\|^2 + \|\nabla. v\|^2)^{\frac{1}{2}} .$$

3. WGFEM And WGRFEM for 2-D Burgers' Problem

Let $\mathcal{T}_h$ be a shape-regular mesh of the domain $\overline{\Omega}$, with mesh size h, for each element $\tau \in \mathcal{T}_h$, we will indicate by $\tau^0$and τ^{∂} for the interior and the boundary of τ respectively. A weak function on τ is a pair of scalar-valued functions $v = \{v_0, v_b\}$such that $v_0 \in L^2(\tau^0)$ and $v_b \in H^{\frac{1}{2}}(\tau^{\partial})$. Let $P^l(\tau^0)$ be the space of polynomials τ^0 with degree at most $l \geq 0$ and $P^m(\tau^{\partial})$ be the space of polynomials on τ^{∂} with degree at most $m \geq 0$. A discrete weak function is a weak function $v = \{v_0, v_b\}$ such that $v_0 \in P^l(\tau^0)$ and $v_b \in P^m(\tau^{\partial})$. A discrete weak function space on τ is defined as,

$$W(\tau, l, m) = \{v = \{v_0, v_b\}: v_0 \in P^l(\tau^0) , v_b \in P^m(\tau^{\partial})\}.$$

Let $P^r(\tau) \subset H(div, \tau)$ the space of polynomials with degree $\leq r$, where $r \geq 0$. Let $V(\tau, r) \subset [P^r(\tau)]^2$, for any $v \in W(\tau, l, m)$ its discrete weak gradient $\nabla_{W,r} v \in V(\tau, r)$ is defined as,

$$\int_{\tau} \nabla_{W,r} v. w\, d\tau = \int_{\tau^{\partial}} v_b\, w. n\, ds - \int_{\tau} v_0 \nabla. w\, d\tau \ , \forall\, w \in V(\tau, r).$$

By taking $W(\tau, l, m)$, $\forall \tau \in \mathcal{T}_h$, we obtain weak finite element spaces,

$$S_h(l, m) = \{v = \{v_0, v_b\}: v|_{\tau} \in W(\tau, l, m) ,\ \forall \tau \in \mathcal{T}_h\},$$

$$S_h^0(l, m) = \{v = \{v_0, v_b\} \in S_h(l, m): v_b|_{\tau^{\partial} \cap \partial\Omega} = 0,\ \forall \tau \in \mathcal{T}_h\}.$$

To design a WGFEM and WGrFEM, we must follow two basic principles.

i. Replace $H_0^1(\Omega)$ by a space of discrete weak functions defined on the finite element partition $\mathcal{T}_h$ and the boundary of triangular elements.
ii. replace the classical gradient operator by a discrete weak gradient operator $\nabla_{W,r}$ for weak functions on each element τ .

Now, we describe a WGFEM for 2-D Burgers' problem with the fully discrete formulation utilizing backward difference form for the time variable for Eq. (1), seek $u_h = \{u_{h,0}, u_{h,b}\} \in S_h^0(l,m)$ such that satisfying

$$\left(\bar{\partial}_t u_h^n, v_0\right) + A_1(u_h^n, v) = (f^n, v_0) \text{ on } \Omega \times [0,T]\,, \forall\, v = \{v_0, v_b\} \in S_h^0(l,m), \tag{5}$$

$$u_h^n(x,y,t) = 0 \text{ on } \partial\Omega \times [0,T],$$

$$u_h(x,y,0) = Q_h u^0(x,y) \text{ on } \bar{\Omega} \times \{0\}.$$

The WGrFEM for 2-D Burgers' problem with the fully discrete formulation utilizing backward difference form for the time variable is described as: for Eq. (2), seek $u_h = \{u_{h,0}, u_{h,b}\} \in S_h^0(l,m)$ such that satisfying

$$\left(\bar{\partial}_t u_h^n, v_0\right) + A_2(u_h^n, v) = (f^n, v_0) \text{ on } \Omega \times [0,T]\,, \forall\, v = \{v_0, v_b\} \in S_h^0(l,m), \tag{6}$$

$$u_h^n(x,y,t) = 0 \text{ on } \partial\Omega \times [0,T],$$

$$u_h(x,y,0) = Q_h u^0(x,y) \text{ on } \bar{\Omega} \times \{0\},$$

where

$$A_1(u_h^n, v) = \epsilon\left(\nabla_{W,r} u_h^n, \nabla_{W,r} \mathrm{v}\right) + \left(u_h^n . \nabla_{W,r} u_h^n\,, v_0\right),$$

$$A_2(u_h^n, v) = \epsilon\left(\nabla_{W,r} u_h^n, \nabla_{W,r} \mathrm{v}\right) - \tfrac{1}{2}\left((u_{0,h}^n)^2\,, \nabla_{W,r} v\right),$$

And $Q_h u \equiv \left\{Q_{h,0} u, Q_{h,b} u\right\}$ is the L^2-projection onto $W(\tau, l, m)$.

Lemma 1. Let $S_h^0(l,m)$ be the weak finite element space and $A_1(.,.)$ and $A_2(.,.)$ be the bilinear forms given in Eq. (5) and Eq. (6), respectively. There exists a positive constant C satisfying

$$A_1(v,v) \geq C\left\{\left\|\nabla_{W,r}\mathrm{v}\right\|^2 + \|v_0\|^2\right\},$$

$$A_2(v,v) \geq C\left\{\left\|\nabla_{W,r}\mathrm{v}\right\|^2 + \|v_0\|^2\right\}, \forall\, v \in S_h^0(l,m)\,.$$

Proof. Note that

$$A_1(v,v) = \epsilon\left\|\nabla_{W,r}\mathrm{v}\right\|^2 + (v.\nabla_{W,r} v\,, v_0),$$

$$A_2(v,v) = \epsilon\left\|\nabla_{W,r}\mathrm{v}\right\|^2 - \tfrac{1}{2}\left((v)^2\,, \nabla_{W,r} v\right),$$

applying Cauchy-Schwartz and Young's inequalities for second term in the right-hand sides in above equations, respectively, we have

$$\begin{aligned}(v.\nabla_{W,r}v\,, v_0) \le \|v\|\|\nabla_{W,r}v\|\|v_0\| &\le \tfrac{1}{2\lambda}\|v\|^2\|\nabla_{W,r}v\|^2 + \tfrac{\lambda}{2}\|v_0\|^2 \\ &= \tfrac{1}{2}\left\{\tfrac{1}{\lambda}\|v\|^2\|\nabla_{W,r}v\|^2 + \lambda\|v_0\|^2\right\}, \end{aligned} \quad (7)$$

$$\tfrac{1}{2}(v_0{}^2\,, \nabla_{W,r}v) \le \tfrac{1}{2}\|v_0{}^2\|\|\nabla_{W,r}v\| \le \tfrac{1}{4\lambda}\|v_0{}^2\|^2 + \tfrac{\lambda}{4}\|\nabla_{W,r}v\|^2 = \tfrac{1}{4}\left\{\tfrac{1}{\lambda}\|v_0{}^2\|^2 + \lambda\|\nabla_{W,r}v\|^2\right\}, \quad (8)$$

it follows from Eq. (7) and Eq. (8) respectively,

$$A_1(v,v) \ge \epsilon\|\nabla_{W,r}\mathrm{v}\|^2 - \tfrac{1}{2}\left\{\tfrac{1}{\lambda}\|v\|^2\|\nabla_{W,r}v\|^2 + \lambda\|v_0\|^2\right\} \ge C\left\{\|\nabla_{W,r}v\|^2 + \|v_0\|^2\right\},$$

$$A_2(v,v) \ge \epsilon\|\nabla_{W,r}\mathrm{v}\|^2 - \tfrac{1}{4}\left\{\tfrac{1}{\lambda}\|v_0{}^2\|^2 + \lambda\|\nabla_{W,r}v\|^2\right\} \ge C\left\{\|\nabla_{W,r}v\|^2 + \|v_0\|^2\right\}.$$

Theorem 1. The methods given in Eq. (5) and Eq. (6) respectively are stable over finite time, specifically, for any $N > 0$ and a time step Δt, as follows,

$$\|u_h^N\| \le \Delta t \sum_{n=1}^{N} f^n + \|u_h^0\|,$$

$$\|u_h^N\| \le \Delta t \textstyle\sum_{n=1}^{N} f^n + \|u_h^0\|.$$

Proof. By choosing $v = u_h^n$ in Eq. (5) and Eq. (6) respectively,

$$(\bar{\partial}_t u_h^n, u_h^n) + A_1(u_h^n, u_h^n) = (f^n, u_h^n),$$

$$(\bar{\partial}_t u_h^n, u_h^n) + A_2(u_h^n, u_h^n) = (f^n, u_h^n),$$

by Cauchy Schwarz inequality we have,

$$\tfrac{1}{\Delta t}\|u_h^n\|^2 + A_1(u_h^n, u_h^n) \le \|f^n\|\|u_h^n\| + \tfrac{1}{\Delta t}\|u_h^{n-1}\|\|u_h^n\|,$$

$$\tfrac{1}{\Delta t}\|u_h^n\|^2 + A_2(u_h^n, u_h^n) \le \|f^n\|\|u_h^n\| + \tfrac{1}{\Delta t}\|u_h^{n-1}\|\|u_h^n\|,$$

from Lemma 1, $A_1(u_h^n, u_h^n)$ and $A_2(u_h^n, u_h^n)$ are non-negative terms, we have,

$$\tfrac{1}{\Delta t}\|u_h^n\|^2 \le \|f^n\| + \tfrac{1}{\Delta t}\|u_h^{n-1}\|,$$

$$\tfrac{1}{\Delta t}\|u_h^n\|^2 \le \|f^n\| + \tfrac{1}{\Delta t}\|u_h^{n-1}\|.$$

Summing both sides from $n = 1$ to $n = N$, for above equations give,

$\|u_h^N\|^2 \leq \Delta t \sum_{n=1}^{N} \|f^n\| + \|u_h^0\|$,

$\|u_h^N\|^2 \leq \Delta t \sum_{n=1}^{N} \|f^n\| + \|u_h^0\|$.

4. Error Analysis

In this section we will derive a priori error estimate for the WGFEM and WGrFEM, where u_h^n is arising from Eq. (5) and Eq. (6) with assuming that the exact solutions of Eq. (1) and Eq. (2) are given by u. To discuss the approximation properties of the discrete weak space $S_h(l,m)$, we define the following projection operators, let R_h be the local L^2- projection onto $V\ (\tau, r)$. The last projection operator is Π_h such that $\Pi_h q \in H(div,\Omega)$, and on each $T \in \mathcal{T}_h$, $\Pi_h q \in V\ (\tau, r)$.

Lemma 2. [16] For any $q \in H(div,\Omega)$, we have,

$$\sum_{T\in\mathcal{T}_h} (-\nabla.\mathrm{q}, v_0)_T = \sum_{T\in\mathcal{T}_h} \left(\Pi_h q\, , \nabla_{W,r} v\right)_T , \qquad ,\forall\ v = \{v_0, v_b\} \in S_h^0(l,m).$$

From the definitions of local L^2- projection $R_h u$ and L^2- projection $Q_h u$, we note,

$(\mathrm{u} - R_h u, \mathrm{w})_T = 0\ ,\ \forall\, w \in V\ (\tau, r)$.

By the Bramble-Hilbert lemma, it is easy to prove that, [26,29]

$$\|\mathrm{u} - R_h u\|_{L^2(\Omega)} \leq Ch^s \|u\|_{H^s(\Omega)}\ , 0 \leq s \leq r+1. \tag{9}$$

It follows from Eq. (10) that, for $k \geq 0$,

$$\|\mathrm{u} - Q_h u\|_{L^2(\Omega)} \leq Ch^s \|u\|_{H^s(\Omega)}\ , 0 \leq s \leq k+1. \tag{10}$$

Moreover, the following identity is hold [14]

$$\nabla_{W,r}(Q_h u) = R_h(\nabla u),\ \forall\, u \in H^1(T). \tag{11}$$

Lemma 3. [25] For $u \in H^{1+k}(\Omega)$ with $k > 0$, we have,

$\|\Pi_h(\epsilon \nabla u^n) - \epsilon R_h(\nabla u^n)\| \leq Ch^k \|u\|_{k+1}$.

Lemma 4. [29] For $u \in H^{1+k}(\Omega)$ with $k > 0$, we have,

$\|u - \Pi_h u\| \leq Ch^k \|u\|_{k+1}$.

Theorem 2. Let u and u_h^n be the solutions of problems Eq. (1) and Eq. (5) respectively, then there exist a positive constant C such that,

$$\max_{1\le n\le N}\|u^n - u_h^n\| \le Ch^k \max_{1\le n\le N}\|u^n\|_{k+1}\,.$$

Proof. Let

$$u^n - u_h^n = \rho^n + e^n,$$

where,

$$\rho^n = u^n - Q_h u^n,\, e^n = Q_h u^n - u_h^n\,,$$

by the triangle inequality we have,

$$\max_{1\le n\le N}\|u^n - u_h^n\| \le \max_{1\le n\le N}\|\rho^n\| + \max_{1\le n\le N}\|e^n\|.$$

The error bound for ρ^n is determined by Eq. (10)

$$\max_{1\le n\le N}\|\rho^n\| \le Ch^k \max_{1\le n\le N}\|u^n\|_{k+1}. \tag{12}$$

Now, to estimate e^n, let $v = \{v_0, v_b\} \in S_h^0(l,m)$ be any test function. By testing Eq. (1) against v_0 with using the fully formulation using backward difference quotient for the time variable, we have,

$$(f^n, v_0)= \left(\bar{\partial}_t u^n, v_0\right) - \epsilon(\nabla.\nabla u^n, v_0) + (u^n.\nabla u^n\,,\ v_0),$$

by Lemma 2 we get,

$$(f^n, v_0)= \left(\bar{\partial}_t u^n, v_0\right) + \epsilon\left(\Pi_h(\nabla u^n), \nabla_{W,r} v\right) - \tfrac{1}{2}\left(\Pi_h(u^n)^2\,,\ \nabla_{W,r} v\right).$$

Adding and subtracting the term $A_2(Q_h u^n, v) \equiv\ \epsilon\left(\nabla_{W,r} Q_h u^n, \nabla_{W,r} v\right) - \tfrac{1}{2}\left(Q_{h,0}(u^n)^2\,,\ \nabla_{W,r} v\right)$ for the above equation with using the fact, $\left(\bar{\partial}_t Q_h u^n, v_0\right) = \left(\bar{\partial}_t u^n, v_0\right)$, we get,

$$(f^n, v_0)= \left(\bar{\partial}_t Q_h u^n, v_0\right) + \epsilon\left(\Pi_h(\nabla u^n) - \nabla_{W,r} Q_h u^n, \nabla_{W,r} v\right) - \tfrac{1}{2}\left(\Pi_h(u^n)^2 - \nabla_{W,r} Q_{h,0}(u^n)^2\,,\right.$$
$$\left.\nabla_{W,r} v\right)+ \epsilon\left(\nabla_{W,r} Q_h u^n, \nabla_{W,r} v\right) - \tfrac{1}{2}\left(Q_{h,0}(u^n)^2\,,\ \nabla_{W,r} v\right),$$

from Eq. (6) and using Eq. (11)

$$\left(\bar{\partial}_t u_h^n, v_0\right) + A_2(u_h^n, v)= \left(\bar{\partial}_t Q_h u^n, v_0\right) + \epsilon\left(\Pi_h(\nabla u^n) - R_h(\nabla u^n), \nabla_{W,r} v\right) - \tfrac{1}{2}\left(\Pi_h(u^n)^2 -\right.$$
$$\left.Q_{h,0}(u^n)^2\,,\ \nabla_{W,r} v\right)+ A_2(Q_h u^n, v)\,,$$

which rewrites as,

$$\left(\bar{\partial}_t Q_h u^n - \bar{\partial}_t u_h^n, v_0\right) + A_2(Q_h u^n - u_h^n, v)= \epsilon\left(R_h(\nabla u^n) - \Pi_h(\nabla u^n), \nabla_{W,r} v\right) + \tfrac{1}{2}\left(\Pi_h(u^n)^2 -\right.$$
$$\left.Q_{h,0}(u^n)^2\,,\ \nabla_{W,r} v\right),$$

so,

$$(\bar{\partial}_t e^n, v_0) + A_2(e^n, v) = \epsilon\big(R_h(\nabla u^n) - \Pi_h(\nabla u^n), \nabla_{W,r} v\big) + \tfrac{1}{2}\big(\Pi_h(u^n)^2 - Q_{h,0}(u^n)^2 ,\ \nabla_{W,r} v\big), \quad (13)$$

Substituting v in Eq. (13) by $e^n = \{e_0^n, e_b^n\} = \{Q_{h,0}u^n - u_{h,0}^n\, , Q_{h,b}u^n - u_{h,b}^n\}$ we have,

$$(\bar{\partial}_t e^n, e^n) + A_2(e^n, e^n) = \epsilon\big(R_h(\nabla u^n) - \Pi_h(\nabla u^n), \nabla_{W,r} e^n\big) + \tfrac{1}{2}\big(\Pi_h(u^n)^2 - Q_{h,0}(u^n)^2 ,\ \nabla_{W,r} e^n\big),$$

from Lemma 1, Cauchy-Schwarz and Young's inequalities,

$$\frac{\|e^n\|^2}{\Delta t} - \frac{\|e^{n-1}\|^2}{2\Delta t} - \frac{\|e^n\|^2}{2\Delta t} + C_2\left\{\left\|\nabla_{W,r} e^n\right\|^2 + \|e^n\|^2\right\} \le \frac{\epsilon}{2C_2}\|R_h(\nabla u^n) - \Pi_h(\nabla u^n)\|^2 + \frac{C_2}{2}\left\|\nabla_{W,r} e^n\right\|^2 + \frac{1}{4C_2}\left\|\Pi_h(u^n)^2 - Q_{h,0}(u^n)^2\right\|^2 + \frac{C_2}{2}\left\|\nabla_{W,r} e^n\right\|^2,$$

this implies,

$$\|e^n\|^2 + 2\Delta t C_2\left\|\nabla_{W,r} e^n\right\|^2 \le \frac{\epsilon\Delta t}{C_2}\|R_h(\nabla u^n) - \Pi_h(\nabla u^n)\|^2 + \Delta t C_2\left\|\nabla_{W,r} e^n\right\|^2 + \frac{\Delta t}{2C_2}\left\|\Pi_h(u^n)^2 - Q_{h,0}(u^n)^2\right\|^2 + \Delta t C_2\left\|\nabla_{W,r} e^n\right\|^2 + \|e^{n-1}\|^2 .$$

Summing both sides from $n = 1$ to $n = N$, noting that $e^0 = 0$,

$$\|e^N\|^2 \le \frac{\epsilon\Delta t}{C_2}\sum_{n=1}^{N}\|R_h(\nabla u^n) - \Pi_h(\nabla u^n)\|^2 + \frac{\Delta t}{2C_2}\sum_{n=1}^{N}\left\|\Pi_h(u^n)^2 - Q_{h,0}(u^n)^2\right\|^2,$$

then,

$$\max_{1\le n\le N}\|e^n\|^2 \le N\left\{\frac{\epsilon\Delta t}{C_2}\max_{1\le n\le N}\|R_h(\nabla u^n) - \Pi_h(\nabla u^n)\|^2 + \frac{\Delta t}{2C_2}\max_{1\le n\le N}\left\|\Pi_h(u^n)^2 - Q_{h,0}(u^n)^2\right\|^2\right\},$$

from Lemma 3 and Lemma 4 we have,

$$\max_{1\le n\le N}\|e^n\|^2 \le Ch^{2k}\max_{1\le n\le N}\|u^n\|_{k+1}^2 ,$$

thus, we have,

$$\max_{1\le n\le N}\|e^n\| \le Ch^{k}\max_{1\le n\le N}\|u^n\|_{k+1}. \quad (14)$$

Combining the bounds Eq. (14) and Eq. (12) the proof is completed.

Theorem 3. Let u and u_h^n be the solutions of problems Eq. (2) and Eq. (6) respectively, then there exist a positive constant C such that,

$$\max_{1\le n\le N}\|u^n - u_h^n\| \le Ch^k\max_{1\le n\le N}\|u^n\|_{k+1}$$

Proof. In the similar way as the proof of Theorem 1, we can prove this theorem.

5. The Numerical Results

In this section, we consider test examples to illustrate a WGFEM Eq. (5) and a WGrFEM Eq. (6) for 2-D Burgers' problem, respectively. We determine $f(x, y, t)$ according to the corresponding the exact solution of each example, where the boundary conditions equal to zero in all numerical examples. The domain Ω where the problems are to be solved is the unit square domain $\overline{\Omega} = [0,1] \times [0,1]$ and the time step $\Delta t = h$ where h is the mesh size parameter.

Example 1. In this example, we construct a WGFEM Eq. (5) and a WGrFEM Eq. (6) with the following initial conditions

$$u(x, y, 0) = x^2 (x - 1) y (y - 1),$$

and boundary condition

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0.$$

This problem has been considered in [13]. In Table 1, we compare the relative error $\frac{\|u-u_h\|_1}{\|u\|_1}$ of both our methods WGFEM and WGrFEM with those obtained in [13] for different grid of size, for $T = 1$ and $\epsilon = 1$. From tabular illustrations, we see that our result for both methods performed well and results agree with the exact solutions than that suggested by [13], for which the exact solution is given as

$$u(x, y, t) = (t + 1) x^2 (x - 1) y (y - 1).$$

Moreover, a WGrFEM yields more accurate and convergent to the exact solutions than WGFEM, solution profiles for grid of size 33×33 have been depicted in Figure 1, Figure 2 and Figure 3. From tabular illustrations and solution profiles, we see that our methods give better efficiency, accuracy and consistency numerical solutions with the exact solutions than those suggested in [13]. Moreover, we see that a WGrFEM is more likely to capture the actual evolution of the solution than a WGFEM.

Table 1
The error $\frac{\|u-u_h\|_1}{\|u\|_1}$ at $T = 1$ and $\epsilon = 1$

h	[13] S.S.	N.S.	O.S.	WGFEM	WGrFEM
$\frac{1}{4}$	0.465	0.465	0.465	$1.086e^{-02}$	$1.088e^{-02}$
$\frac{1}{8}$	0.241	0.241	0.241	$4.151e^{-03}$	$4.151e^{-03}$
$\frac{1}{16}$	0.122	0.122	0.122	$1.508e^{-03}$	$1.499e^{-03}$
$\frac{1}{32}$	0.061	0.061	0.061	$5.506e^{-04}$	$5.330e^{-04}$
$\frac{1}{64}$	0.030	0.030	0.030	$2.239e^{-04}$	$1.886e^{-04}$

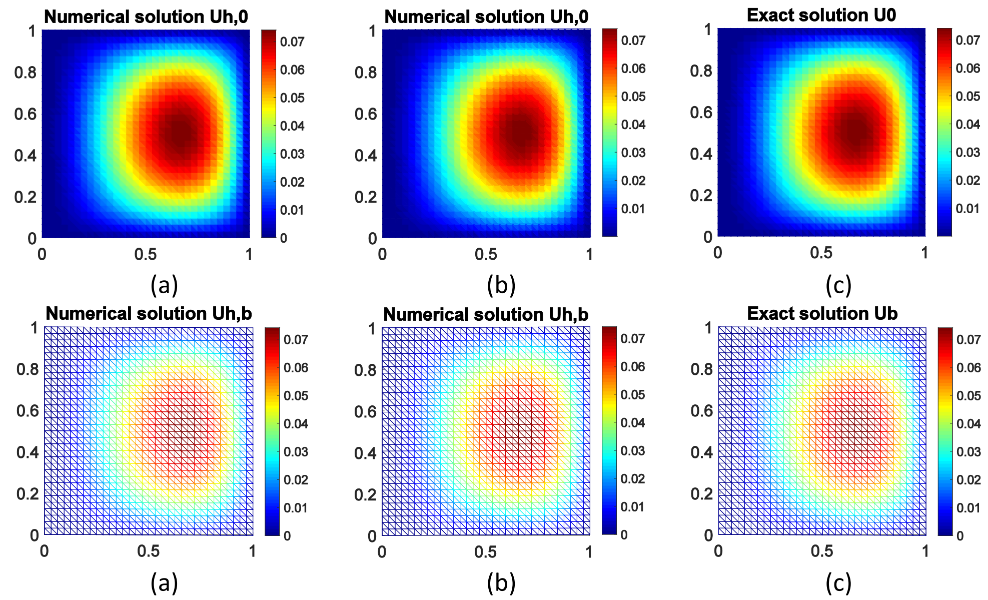


Fig. 1. Numerical solutions for (a) WGFEM, (b) WGrFEM for 2-D Burgers' problem and (c) exact solutions of u_0 and u_b respectively at $\epsilon = 1,\ h = \frac{1}{32}$, $N = 33$ and $T = 1$

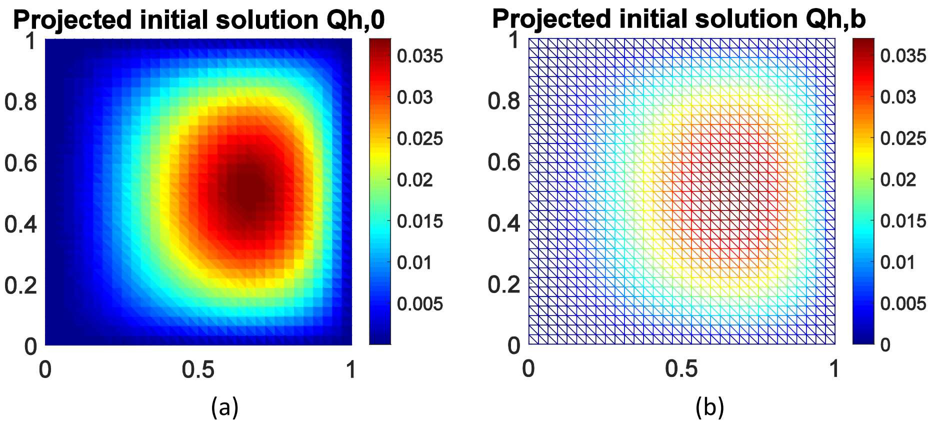


Fig. 2. Numerical projected initial solution (a) $Q_{h,0}u^0$, (b) $Q_{h,b}u^0$ at $\epsilon = 1,\ h = \frac{1}{32}$, $N = 33$ and $T = 0$

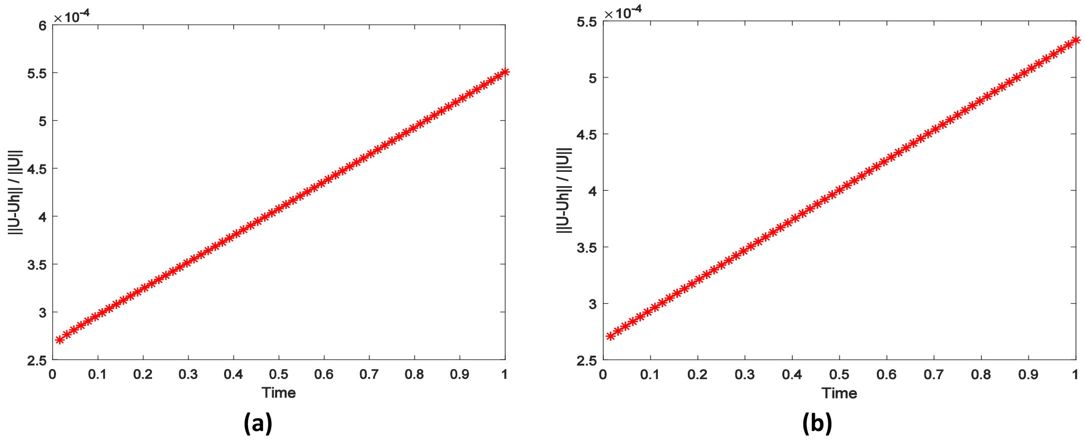


Fig. 3. Error estimate $\|\mathrm{u} - \mathrm{u}_h\|_{w,1}$ of (a) WGFEM, (b) WGrFEM for 2-D Burgers' problem at $\epsilon = 1,\ h = \frac{1}{32}$, $N = 33$ and $T = 1$

Example 2. In this example, we construct a WGFEM Eq. (5) and a WGrFEM Eq. (6) with the following initial conditions

$$u(x,y,0) = x(x-1)y(y-1),$$

and boundary conditions

$$u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0.$$

This problem has been considered in [12]. In Table 2, Table 3 and Table 4, we compare the relative error $\frac{\|u-u_h\|_1}{\|u\|_1}$ of both our methods WGFEM and WGrFEM with those obtained in [12] for different grid of size for $T = 1$ and $\epsilon = 1, 0.1, 0.01$. The tabulated results show that the proposed methods produce better results than [12], for which the exact solution is given as

$$u(x,y,t) = cos(t)\, x\,(x-1)\, y\,(y-1).$$

Moreover, a WGrFEM yields more accurate and convergent to the exact solutions than WGFEM, solution profiles for grid of size 37×37 have been depicted in Figure 4, Figure 5 and Figure 6. From tabular illustrations and solution profiles, we see that our methods give better efficiency and accuracy, consistency numerical solutions with the exact solutions than those suggested in [12]. Moreover, we see that a WGrFEM is more likely to capture the actual evolution of the solution than a WGFEM.

Table 2
The error $\frac{\|u-u_h\|_1}{\|u\|_1}$ at $T = 1$ and $\epsilon = 1$

h	[12]			
	1-G.M.	2-G.M.	WGFEM	WGrFEM
$\frac{1}{16}$	0.102	0.104	$4.694e^{-04}$	$4.681e^{-04}$
$\frac{1}{36}$	0.045	0.046	$1.465e^{-04}$	$1.44e^{-04}$
$\frac{1}{64}$	0.026	0.026	$7.594e^{-05}$	$7.24e^{-05}$
$\frac{1}{100}$	0.016	0.016	$5.709e^{-05}$	$5.302e^{-05}$
$\frac{1}{144}$	0.011	0.011	$5.159e^{-05}$	$4.740e^{-05}$

Table 3
The error $\frac{\|u-u_h\|_1}{\|u\|_1}$ at $T = 1$ and $\epsilon = 0.1$

h	[12]			
	1-G.M.	2-G.M.	WGFEM	WGrFEM
$\frac{1}{16}$	0.102	0.105	$7.474e^{-04}$	$5.619e^{-04}$
$\frac{1}{36}$	0.046	0.047	$4.132e^{-04}$	$1.174e^{-04}$
$\frac{1}{64}$	0.026	0.027	$3.721e^{-04}$	$1.058e^{-04}$
$\frac{1}{100}$	0.017	0.017	$3.712e^{-04}$	$1.554e^{-04}$
$\frac{1}{144}$	0.011	0.0122	$3.053e^{-04}$	$1.458e^{-04}$

Table 4
The error $\frac{\|u-u_h\|_1}{\|u\|_1}$ at $T = 1$ and $\epsilon = 0.01$

h	[12] 1-G.M.	2-G.M.	WGFEM	WGrFEM
$\frac{1}{16}$	0.105	0.228	$2.306e^{-03}$	$1.168e^{-03}$
$\frac{1}{36}$	0.048	0.118	$2.015e^{-03}$	$3.281e^{-04}$
$\frac{1}{64}$	0.029	0.070	$1.986e^{-03}$	$1.420e^{-04}$
$\frac{1}{100}$	0.019	0.046	$1.956e^{-03}$	$1.334e^{-04}$
$\frac{1}{144}$	0.014	0.033	$1.850e^{-03}$	$1.023e^{-04}$

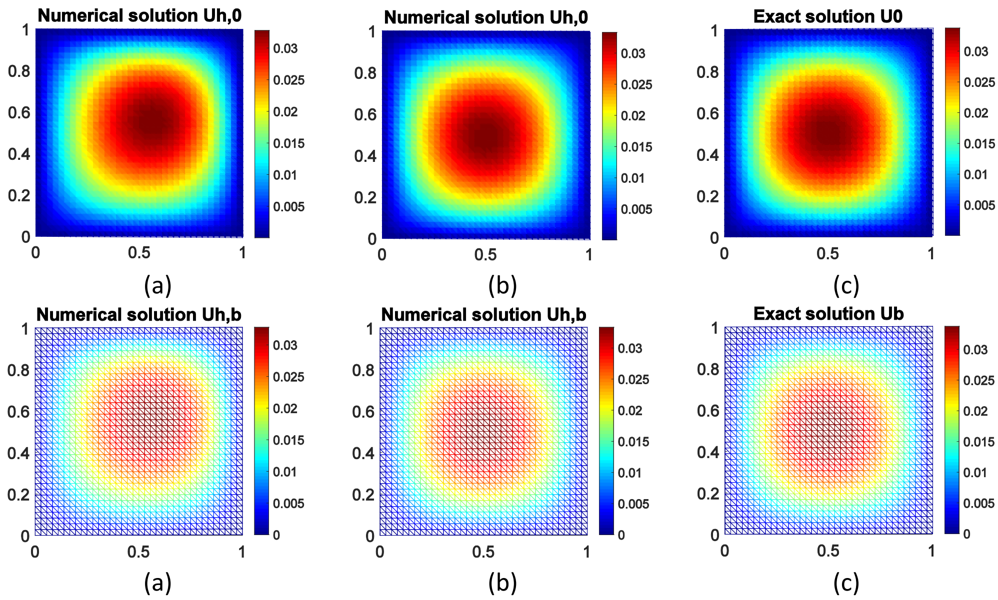


Fig. 4. Numerical solutions for (a) WGFEM (b) WGrFEM for 2-D Burgers' problem and (c) exact solutions of u_0 and u_b respectively at $\epsilon = 0.01, h = \frac{1}{36}, N = 37$ and $T = 1$

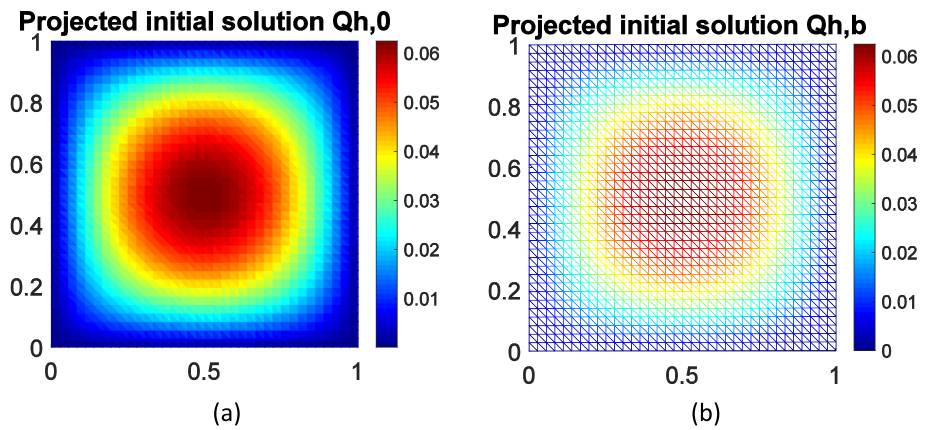
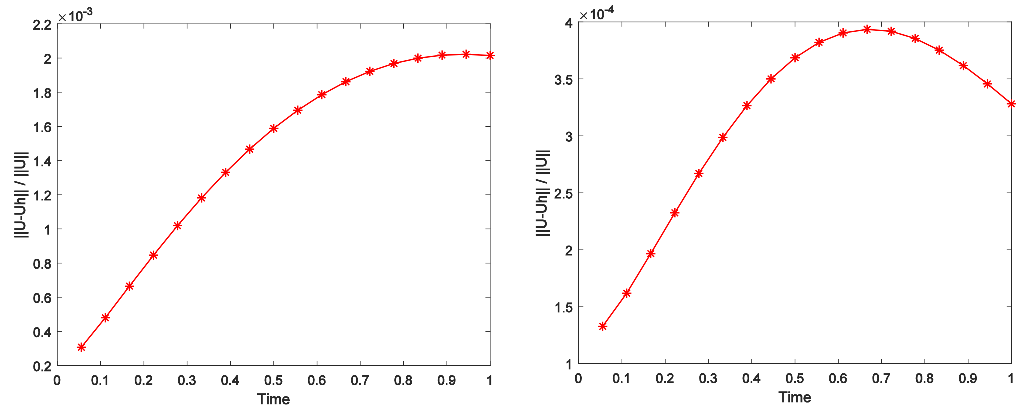


Fig. 5. Numerical projected initial solution (a) $Q_{h,0}u^0$, (b) $Q_{h,b}u^0$ at $\epsilon = 0.01$, $h = \frac{1}{36}, N = 37$ and $T = 0$



(a) (b)

Fig. 6. Error estimate $\|u - u_h\|_{w,1}$ of (a) WGFEM, (b) WGrFEM for 2-D Burgers' problem at $\epsilon = 0.01$, $h = \frac{1}{36}$, $N = 37$ and $T = 1$

Example 3. In this example, we construct a WGFEM Eq. (5) and a WGrFEM Eq. (6) with the following initial conditions

$$u(x, y, 0) = sin(2\pi x)sin(2\pi y),$$

and boundary conditions

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0.$$

This problem has been considered in [12]. In Table 5 we compare the relative error $\frac{\|u - u_h\|_1}{\|u\|_1}$ of both our methods WGFEM and WGrFEM with those obtained in [12] for different grid of size for $T = 1$ and $\epsilon = 1$, From tabular illustrations, we see that our result for both methods performed a better agreement than that obtained in [12], for which the exact solution is given as

$$u(x, y, t) = (t^2 + 1)\sin(2\pi\, x)\sin(2\pi\, y).$$

Moreover, the accuracy of a WGrFEM is better than a WGFEM, solution profiles for grid of size 145×145 have been depicted in Figure 7, Figure 8 and Figure 9. From tabular illustrations and solution profiles, we see that our methods give better results and consistent numerical solutions with the exact solutions than those suggested in [12]. Moreover, we see that a WGrFEM is more accurate than a WGFEM.

Table 5
The error $\frac{\|u - u_h\|_1}{\|u\|_1}$ at $T = 1$ and $\epsilon = 1$

h	[12]			
	1-G.M.	2-G.M.	WGFEM	WGrFEM
$\frac{1}{16}$	0.163	0.218	$8.061e^{-02}$	$5.150e^{-02}$
$\frac{1}{36}$	0.073	0.093	$5.194e^{-02}$	$1.458e^{-02}$
$\frac{1}{64}$	0.041	0.054	$3.804e^{-02}$	$6.068e^{-03}$
$\frac{1}{100}$	0.026	0.035	$1.847e^{-02}$	$1.752e^{-04}$
$\frac{1}{144}$	0.018	0.025	$1.653e^{-02}$	$8.495e^{-05}$

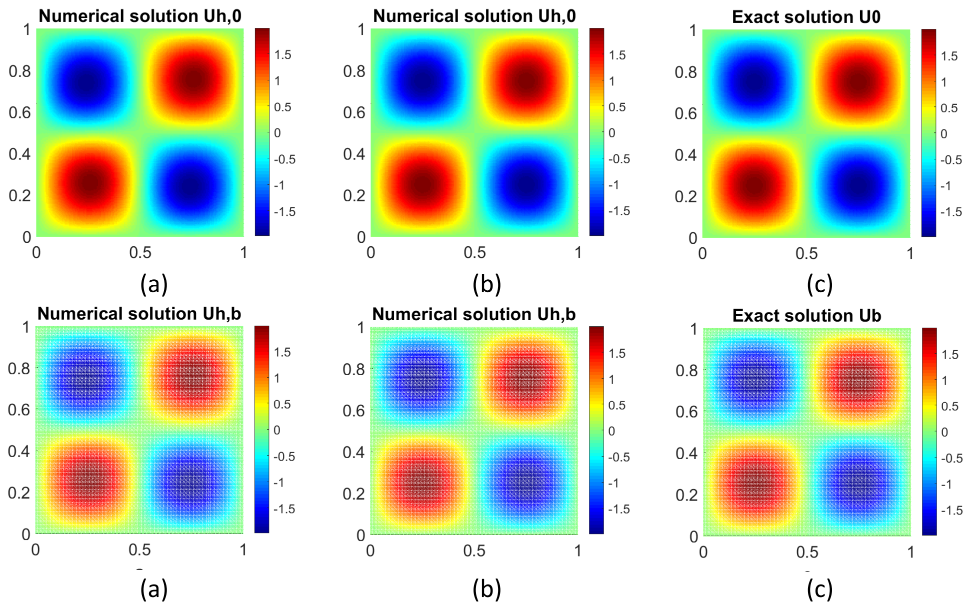


Fig. 7. Numerical solutions for (a) WGFEM, (b) WGrFEM for 2-D Burgers' problem and (c) exact solutions of u_0 and u_b respectively at $\epsilon = 1,\ h = \frac{1}{144}, N = 145$ and $T = 1$

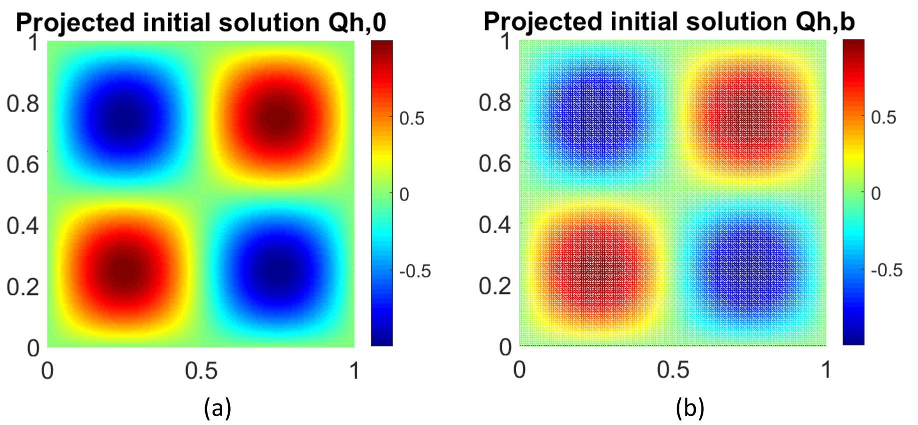


Fig. 8. Numerical projected initial solution (a) $Q_{h,0}u^0$, (b) $Q_{h,b}u^0$ at $\epsilon = 1$, $h = \frac{1}{144}, N = 145$ and $T = 0$

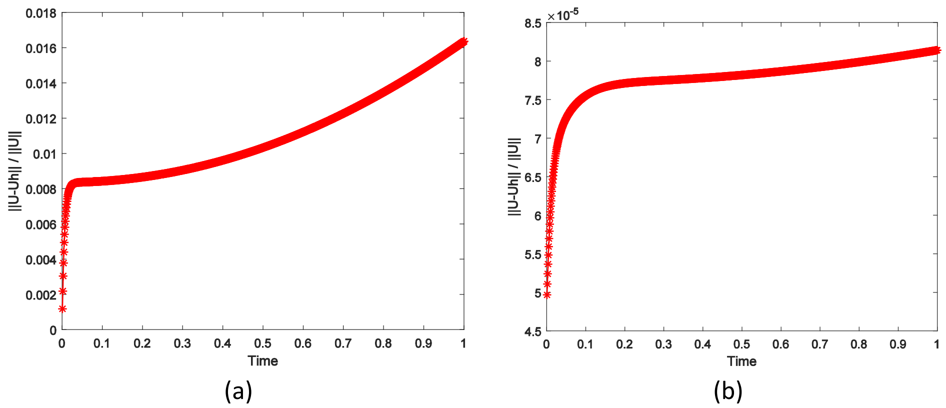


Fig. 9. Error estimate $\|u - u_h\|_{w,1}$ of (a) WGFEM, (b) WGrFEM for 2-D Burgers' problem at $\epsilon = 1,\ h = \frac{1}{144}, N = 145$ and $T = 1$

Example 4. In this example, we construct a WGFEM Eq. (5) and a WGrFEM Eq. (6) with the following initial conditions

$u(x, y, 0) = 10\, x\, y\, (x-1)(y-1),$

and boundary conditions

$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0.$

This problem has been considered in [3]. In Table 6 we compare the relative error $\frac{\|u-u_h\|_0}{\|u\|_0}$ of a WGFEM and a WGrFEM for different grid of size for $T = 10$ and $\epsilon = 0.01$, from tabular illustrations, we see that our result for both methods achieve satisfactory results. Moreover, a WGrFEM yields more accurate than a WGFEM, for which the exact solution is given as

$$u(x, y, t) = 10\, x\, y\, (x-1)(y-1)[\sin(2xt)\, e^{\frac{-t}{2}} + \cos(yt)\, e^{\frac{-t}{4}} + \sin(xyt)\, e^{-t}].$$

Solution profiles for grid of size 65×65 have been depicted in Figure 10 and Figure 11, which illustrate a better agreement of a WGFEM and a WGrFEM. In Figure 12, for $T \in (0,10]$, we note the WGrFEM yields more accurate and a better agreement than the WGFEM and that obtained in Figure 2 and Figure 3 [3]. From tabular illustrations and solution profiles, we see that our methods give better results and consistent numerical solutions with the exact solutions than those suggested in [3]. Moreover, we see that a WGrFEM is more accurate than a WGFEM.

Table 6
The error $\frac{\|u-u_h\|_1}{\|u\|_1}$ at $T = 10$ and $\epsilon = 0.01$

h	WGFEM	WGrFEM
$\frac{1}{4}$	$1.020e^{-02}$	$1.301e^{-02}$
$\frac{1}{8}$	$6.840e^{-03}$	$5.828e^{-03}$
$\frac{1}{16}$	$6.163e^{-03}$	$2.481e^{-03}$
$\frac{1}{32}$	$6.108e^{-03}$	$1.034e^{-03}$
$\frac{1}{64}$	$6.109e^{-03}$	$4.540e^{-04}$

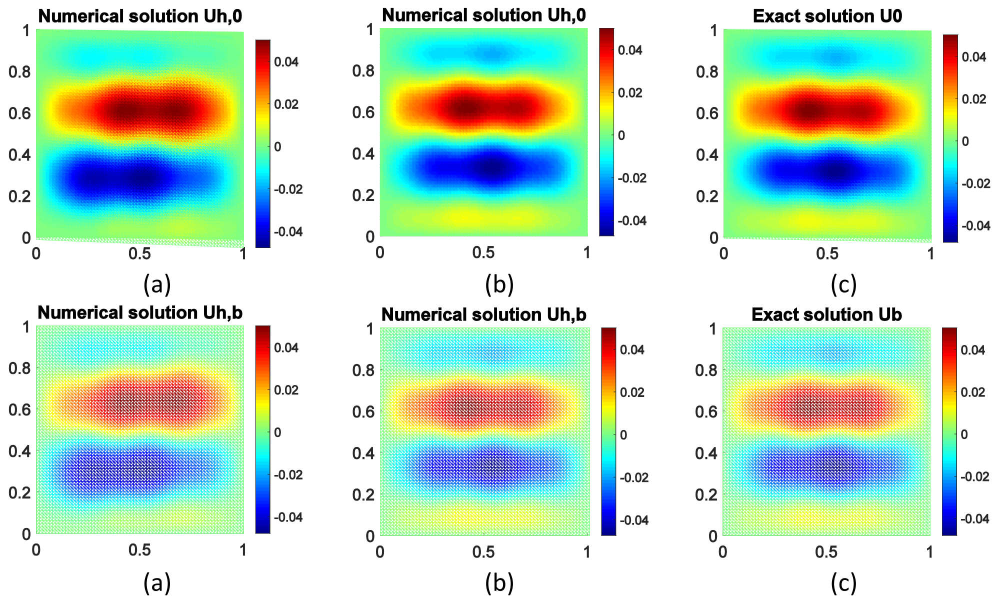


Fig. 10. Numerical solutions for (a) WGFEM, (b) WGrFEM for 2-D Burgers' problem and (c) exact solutions of u_0 and u_b respectively at $\epsilon = 0.01,\ h = \frac{1}{64}, N = 65$ and $T = 10$

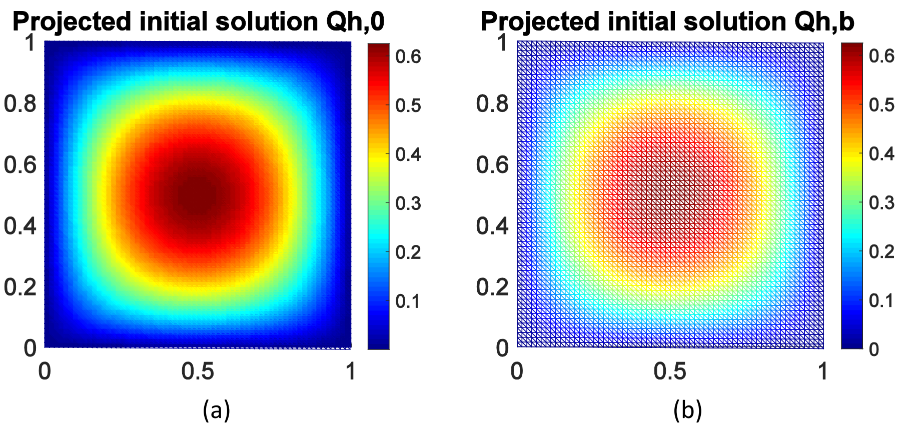


Fig. 11. Numerical projected initial solution (a) $Q_{h,0}u^0$, (b) $Q_{h,b}u^0$ at $\epsilon = 0.01,\ h = \frac{1}{64}, N = 65$ and $T = 0$

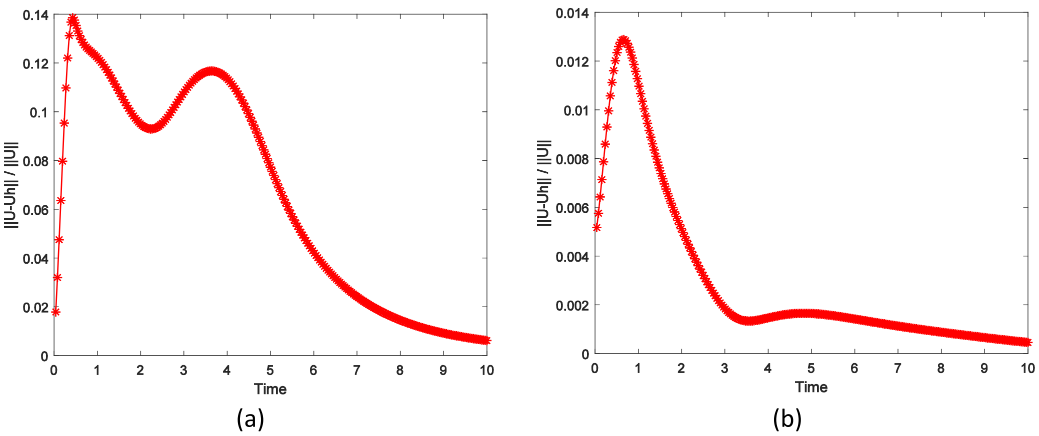


Fig. 12. Error estimate $\|u - u_h\|_{w,1}$ of (a) WGFEM, (b) WGrFEM for 2-D Burgers' problem at $\epsilon = 0.01,\ h = \frac{1}{64}, N = 65$ and $T = 10$

6. Conclusions

In this work, we have applied the WGFEM and the WGrFEM for 2-D Burgers' problem in the fully discrete case utilizing the backward-difference form for the time variable. From the theoretical analysis and the numerical results, we can conclude that the stability condition of a WGFEM and a WGrFEM are satisfied. The theoretical study is shown that the error estimate of a WGFEM and a WGrFEM are $O(h^k)$. we have compared the relative error with other available literature and the numerical solution has been compared with the exact solution and has been illustrate graphically. We conclude that the WGFEM and the WGrFEM provide convergent and consistency approximations in different cases and successfully provide accurate solutions. The results obtained are more satisfactory than the results available in the other literature. The WGrFEM is found to give generally more accurate approximation for 2-D Burgers' problem in comparison with the WGFEM. Finally, the WGFEM and the WGrFEM are appropriate to deal with other nonlinear PDEs which is appeared in various applications of science and engineering.

References

[1] Harris, Anna, and Stephen Harris. "Superconvergence of weak Galerkin finite element approximation for second order elliptic problems by L^2-projections." *Applied Mathematics and Computation* 227 (2014): 610-621. https://doi.org/10.1016/j.amc.2013.11.065

[2] Jiang, B. N. *The Least-Squares Finite Element Method*. Springer-Verlag, Berlin, 1998. https://doi.org/10.1007/978-3-662-03740-9

[3] Dickinson, Benjamin T., and John R. Singler. "Nonlinear Model Reduction Using Group Proper Orthogonal Decomposition." *International Journal of Numerical Analysis & Modeling* 7, no. 2 (2010): 356-372.

[4] Tripathi, Brajesh, and Sandipan GhoshMoulic. "Numerical Evaluation of Inclined Ceiling Diffuser on Buoyancy and Airflow Patterns in an Enclosed Space." *CFD Letters* 3, no. 1 (2011): 48-64.

[5] Fletcher, Clive A. J. "The group finite element formulation." *Computer Methods in Applied Mechanics and Engineering* 37, no. 2 (1983): 225-244. https://doi.org/10.1016/0045-7825(83)90122-6

[6] Fletcher, Clive A. J. "Fundamental and General Techniques." In *Computational Techniques for Fluid Dynamics I*. Springer-Verlag, 1991.

[7] Ning, Chen, and Haiming Gu. "Weak Galerkin Finite Element Method for the Unsteady Stokes Equation." *American Journal of Computational Mathematics* 8, no. 01 (2018): 108-119. https://doi.org/10.4236/ajcm.2018.81009

[8] Chen, Chuan-Miao, Stig Larsson, and Nai-Ying Zhang. "Error estimates of optimal order for finite element methods with interpolated coefficients for the nonlinear heat equation." *IMA Journal of Numerical Analysis* 9, no. 4 (1989): 507-524.
https://doi.org/10.1093/imanum/9.4.507

[9] Ghiasi, E. Khoshrouye, and Reza Saleh. "Analytical and numerical solutions to the 2D Sakiadis flow of Casson fluid with cross diffusion, inclined magnetic force, viscous dissipation and thermal radiation based on Buongiorno's mathematical model." *CFD Letters* 11, no. 1 (2019): 40-54.

[10] Gao, Fuzheng, and Lin Mu. "On L² Error Estimate for Weak Galerkin Finite Element Methods for Parabolic Problems." *Journal of Computational Mathematics* 23 (2014): 195-204. https://doi.org/10.4208/jcm.1401-m4385

[11] Gao, Fuzheng, and Xiaoshen Wang. "A modified weak Galerkin finite element method for a class of parabolic problems." *Journal of Computational and Applied Mathematics* 271 (2014): 1-19. https://doi.org/10.1016/j.cam.2014.03.028

[12] Hu, Xiaohui, Pengzhan Huang, and Xinlong Feng. "Two-Grid Method for Burgers' Equation by a New Mixed Finite Element Scheme." *Mathematical Modelling and Analysis* 19, no. 1 (2014): 1-17. https://doi.org/10.3846/13926292.2014.892902

[13] Hu, Xiaohui, Pengzhan Huang, and Xinlong Feng. "A new mixed finite element method based on the Crank-Nicolson scheme for Burgers' equation." *Applications of Mathematics* 61, no. 1 (2016): 27-45. https://doi.org/10.1007/s10492-016-0120-3

[14] Christie, Ian, David F. Griffiths, A. Ronald Mitchell, and Jesús Marıa Sanz-Serna. "Product approximation for non-linear problems in the finite element method." *IMA Journal of Numerical Analysis* 1, no. 3 (1981): 253-266. https://doi.org/10.1093/imanum/1.3.253
[15] Wang, Junping, and Xiu Ye. "A weak Galerkin mixed finite element method for second order elliptic problems." *Mathematics of Computation* 83, no. 289 (2014): 2101-2126. https://doi.org/10.1090/S0025-5718-2014-02852-4
[16] Wang, Junping, and Xiu Ye. "A weak Galerkin finite element method for second-order elliptic problems." *Journal of Computational and Applied Mathematics* 241 (2013): 103-115. https://doi.org/10.1016/j.cam.2012.10.003
[17] Wang, Junping, and Xiu Ye. "A weak Galerkin finite element method for the Stokes equations." *Advances in Computational Mathematics* 42, no. 1 (2016): 155-174. https://doi.org/10.1007/s10444-015-9415-2
[18] Zhang, Jiachuan, Kai Zhang, Jingzhi Li, and Xiaoshen Wang. "A weak Galerkin finite element method for the Navier-Stokes equations." *Commun. Comput. Phys* 23 (2018): 706-746. https://doi.org/10.4208/cicp.OA-2016-0267
[19] Chen, Long, Junping Wang, and Xiu Ye. "A posteriori error estimates for weak Galerkin finite element methods for second order elliptic problems." *Journal of Scientific Computing* 59, no. 2 (2014): 496-511. https://doi.org/10.1007/s10915-013-9771-3
[20] Mu, Lin, Junping Wang, and Xiu Ye. "A weak Galerkin finite element method with polynomial reduction." *Journal of Computational and Applied Mathematics* 285 (2015): 45-58. https://doi.org/10.1016/j.cam.2015.02.001
[21] Mu, Lin, Xiaoshen Wang, and Xiu Ye. "A modified weak Galerkin finite element method for the Stokes equations." *Journal of Computational and Applied Mathematics* 275 (2015): 79-90. https://doi.org/10.1016/j.cam.2014.08.006
[22] Mu, Lin, Junping Wang, Guowei Wei, Xiu Ye, and Shan Zhao. "Weak Galerkin methods for second order elliptic interface problems." *Journal of Computational Physics* 250 (2013): 106-125. https://doi.org/10.1016/j.jcp.2013.04.042
[23] Mu, Lin, Junping Wang, and Xiu Ye. "Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes." *Numerical Methods for Partial Differential Equations* 30, no. 3 (2014): 1003-1029. https://doi.org/10.1002/num.21855
[24] Yang, Min. "Couplings of mixed finite element and weak Galerkin methods for elliptic problems." *Journal of Applied Mathematics and Computing* 47, no. 1-2 (2015): 327-343. https://doi.org/10.1007/s12190-014-0778-1
[25] Li, Qiaoluan H., and Junping Wang. "Weak Galerkin finite element methods for parabolic equations." *Numerical Methods for Partial Differential Equations* 29, no. 6 (2013): 2004-2024. https://doi.org/10.1002/num.21786
[26] Zhang, Ran, and Qilong Zhai. "A weak Galerkin finite element scheme for the biharmonic equations by using polynomials of reduced order." *Journal of Scientific Computing* 64, no. 2 (2015): 559-585. https://doi.org/10.1007/s10915-014-9945-7
[27] Larsson, Stig, Vidar Thomée, Nai-Ying Zhang, and W. Wendland. "Interpolation of coefficients and transformation of the dependent variable in finite element methods for the non-linear heat equation." *Mathematical Methods in The Applied Sciences* 11, no. 1 (1989): 105-124. https://doi.org/10.1002/mma.1670110108
[28] Zhang, Tie, and Lixin Tang. "A weak finite element method for elliptic problems in one space dimension." *Applied Mathematics and Computation* 280 (2016): 1-10. https://doi.org/10.1016/j.amc.2016.01.018
[29] Thomée, Vidar. *Galerkin finite element method for parabolic problems. (Springer Series in Computational Mathematics)*. Berlin, Springer-Verlag, New York, Inc., 2006.
[30] Liu, Xin, Jian Li, and Zhangxin Chen. "A weak Galerkin finite element method for the Navier–Stokes equations." *Journal of Computational and Applied Mathematics* 333 (2018): 442-457. https://doi.org/10.1016/j.cam.2017.11.010
[31] Chen, Yanli, and Tie Zhang. "A weak Galerkin finite element method for Burgers' equation." *arXiv preprint arXiv:1607.05622* (2016).
[32] Tourigny, Yves. "Product approximation for nonlinear Klein-Gordon equations." *IMA journal of Numerical Analysis 10, no. 3 (1990): 449-462.* https://doi.org/10.1093/imanum/10.3.449