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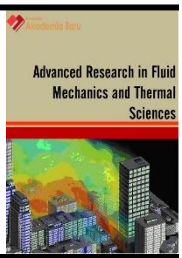

Simulation of Internal Undular Bores of Depression Propagating over a Slowly Varying Region

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ABSTRACT

Internal undular bores have been observed in many parts of the world. This paper intends to look at the variable topographic effects on the evolution of the internal undular bores of depression. Here, the internal undular bores is considered to be propagating in a two-layer fluid system. The topography is assumed to be slowly varying. Therefore, the appropriate mathematical model is the variable-coefficient extended Korteweg-de Vries equation. The governing equation is solved numerically using the method of lines. We are especially interested in looking at the transformation of two types of internal undular bore of depression, which are Korteweg de Vries-type and table-top internal undular bore. Our numerical results show that the internal undular bore of depression transformed into a positive undular bore as it propagates over a slowly increasing slope and when it involves polarity change. In front of the transformed internal bore, a series of isolated solitary waves or a solitary wavetrain is generated as a non-adiabatic response to the interaction between the internal undular bore with the changing bottom surface. The solitary wavetrain is observed to be climbing the negative pedestal. As time increases, the amplitude of the individual solitary wavetrain is decreasing and finally the solitary wavetrain will die out due to the pedestal. However, if there is no polarity change, the internal undular bores deforms adiabatically and its amplitude decreases slowly until it reaches new limiting amplitude value due the increasing bottom surface. On the other hand, when the slope is decreasing slowly, the internal undular bore deforms adiabatically where its amplitude increases slowly. There is a multi-phase behaviour is observed during the evolution as the results of the interaction between the internal bore and the varying slope. In this case, there is no polarity change. The transformation of internal undular bores of depression depends on the nature of the topographic change.

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1. Introduction

The study of the internal wave can be traced back to the observation of internal gravity wave at the interface between oil and water in a glass tumbler [1]. The first observation of internal waves in nature have been attributed by Nansen during his expedition in the Arctic ocean in 1893 when his boat was sailing on the surface of water but his boat experienced a great resistance to against it moving forward [2]. Later, Ekman studied Nansen's observation to prove that there is an existence of internal wave hidden entirely in the layer of dead water [3]. Until the 1960s, the interest of internal wave grew up after World War 2 when the United States navy suffered difficulty in controlling their depth detection inside submarine [4]. Until now, internal wave often attracts the attention of the scientific community due to it is important in influencing the ocean environment and human activities.

The existence of internal wave can affect the propagation of the acoustic wave in the deep ocean such that causing interference with the acoustic field [5]. Meanwhile, it can reduce the acoustic signal when internal waves make the temporal and spatial changes of medium density [6]. It is also a major danger to marine activities, submarine navigation and fishery activities due to large amplitude during propagation implying that a large amount of kinetic energy can be transferred in between the ocean layers [7]. When the high energy hits on the physical structures, e.g. deep-sea drilling station [8-9] and blockwork structures near the coastal areas [10], the great power has sufficient energy to scrape away the surface of them. Besides, it can affect the sea ice evolution in the frozen sea [11] and the growth of corals due to the occurrence of temperature fluctuations between the ocean layers [12-13].

Large internal waves in the ocean usually have a negative polarity which has been proved by the observation of internal wave packets in the deep ocean [14-16]. In nature, the general structure of large-amplitude internal wave in the interior of coastal ocean commonly presents in the form of unsteady undular bores [17] and it has been observed in some part of the coastal regions, i.e. Australian North West Shelf [18], Japan/East Sea shelf-coastal region [19] and Peter the Great Bay [20]. In nature, internal waves are always propagating across an uneven sea-bottom in the ocean region. Therefore, we would like to study the evolution of internal undular of negative polarity or depression propagating over slowly varying region topography.

In the following section, we will present the problem formulation and followed by the numerical methods adopted in our problem in section 3. The numerical results are presented and discussed in section 4 and the conclusion is in the final section.

2. Problem Formulation

The Korteweg-de Vries (KdV) equation has been widely used to solve nonlinear long wave problems. It was first used to describe the shallow water waves by Benney [21] and Benjamin [22], and subsequently by many others [23-25]. However, the KdV equation is not appropriate for describing the behaviour of internal wave due to nearly vanishing quadratic nonlinear effect in the KdV equation under certain condition such as large amplitude internal wave [26]. Therefore, the derivation of the KdV equation is extended such that a cubic nonlinear term is included to increase the nonlinearity effect for dynamic balancing with the dispersion effect [25]. Hence, the extended KdV equation is obtained.

In this study, we consider the internal wave is propagating in a two-layer fluid system. Thus, the appropriate mathematical model is the variable-coefficient eKdV (veKdV) equation [27],

$$A_t + cA_x - \frac{cQ_x}{Q}A + \mu A A_x + \mu_1 A^2 A_x + \delta A_{xxx} = 0 \qquad (1)$$

where $A(x, t)$ refers to the amplitude of the wave, x and t represent the spatial and temporal variables respectively. The coefficient $c(x)$ denotes the relevant linear long wave speed and $Q(x)$ is the linear modification factor, so that $Q^{-2}A^2$ becomes the wave action flux for linear long waves [26]. The coefficients $\mu(x)$, $\mu_1(x)$, and $\delta(x)$ are the coefficients of the nonlinear and the dispersive terms which are determined by the properties of the basic state of the fluid. All these coefficients are slowly varying functions of x.

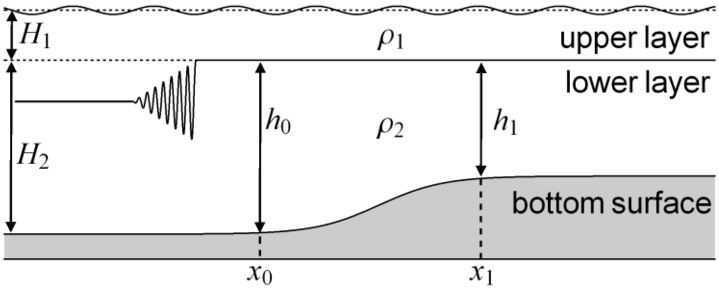


Fig. 1. Schematic illustration of internal undular bore of depression propagating in a two-layer fluid of different density over a slowly decreasing slope region

The schematic of our problem is illustrated in Figure 1. Here, we let the depths of upper and lower layers are represented by H_1 and H_2 respectively. Also, we consider the densities of the fluid for both layers are constants and denoted by ρ_1 and ρ_2 respectively. Thus, the coefficients of nonlinear and dispersion terms, i.e. μ, μ_1, and δ in the veKdV Eq. (1) are given by

$$\mu = \frac{3c\left(\rho_2 H_1{}^2 - \rho_1 H_2{}^2\right)}{2H_1 H_2\left(\rho_2 H_1 + \rho_1 H_2\right)},$$

$$\mu_1 = \frac{-3c}{8\left(H_1{}^2 H_2{}^2\right)\left(\rho_2 H_1 + \rho_1 H_2\right)^2}\left[\left(\rho_1 H_2{}^2 - \rho_2 H_1{}^2\right)^2 + 8\rho_1\rho_2 H_1 H_2\left(H_1 + H_2\right)^2\right],$$

$$\delta = \frac{cH_1 H_2\left(\rho_1 H_1 + \rho_2 H_2\right)}{6\left(\rho_2 H_1 + \rho_1 H_2\right)},$$

where

$$c = \sqrt{\frac{g\left(\rho_2 - \rho_1\right)H_1 H_2}{2\rho_1 H_2}}, \quad \text{and} \quad Q = \sqrt{\frac{1}{2g\left(\rho_2 - \rho_1\right)c}}. \qquad (2)$$

By following the usual oceanic condition, the density difference between upper and lower layers to be very small so that $\rho_2 - \rho_1 \ll \rho_2$ [28]. In this paper, we consider initially $H_1 > H_2$ so that we have an internal undular bore of depression. We also consider that the upper layer has a constant depth for all x and the depth of lower layer varies monotonically from h_0 to h_1 in the interval $x_0 \leq x \leq x_1$ where $x_1 - x_0 \gg 1$.

The first two terms in veKdV Eq. (1) are the dominant terms. Thus, the veKdV Eq. (1) can be transformed by introducing new variables [26],

$$A = QU, \qquad T = \int^x \frac{dx}{c}, \qquad X = c(T - t). \tag{3}$$

By substituting the new variables (3) into veKdV Eq. (1) gives the following equation, to the same leading order of approximation where veKdV Eq. (1) holds

$$U_T + \alpha U U_X + \beta U^2 U_X + \lambda U_{XXX} = 0, \tag{4}$$

where

$$\alpha = Q\mu, \qquad \beta = Q^2 \mu_1, \qquad \lambda = \delta. \tag{5}$$

In terms of the new variables $U(X, T)$, $H_1(T)$ is constant for all T and $H_2(T)$ varies monotonically through a monotonic function, $f(T)$ from h_0 to h_1 in the interval $T_0 \leq T \leq T_1$. In this study, we consider two types of variable topography, i.e. slowly increasing slope and slowly decreasing slope regions.

The structure of undular bore is an oscillatory transition between two different basic states. It can be generated from a simple unit step using the Heaviside function [29]. Hence, we consider the initial condition for the veKdV Eq. (4) to be in the form of a sharp step,

$$U = U_0 P(-X),$$

where $U_0 < 0$ and P is a Heaviside function (6) to generate a hydraulic jump that connects two different constant depths.

$$P(X) = \begin{cases} 1, & \text{if } X > 0, \\ 0, & \text{if } X < 0. \end{cases} \tag{6}$$

3. Numerical Method

To solve the veKdV Eq. (4) numerically, we apply the method of lines (MOL). In previous studies, the MOL is widely used in solving many KdV-type equations, e.g. constant-coefficient eKdV equation [30], variable-coefficient eKdV equation [31], forced KdV equation [32], and forced KdV-Burgers equation [33]. First, the veKdV Eq. (4) is rewritten as follows

$$U_T = -\alpha U U_X - \beta U^2 U_X - \lambda U_{XXX}.$$

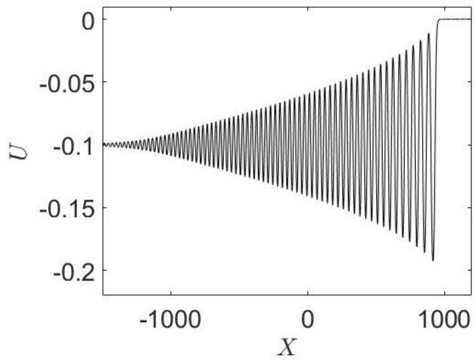
Then, the spatial derivatives are discretized using central finite difference formulae as follows

$$U_X \approx \frac{U_{j+1} - U_{j-1}}{2\Delta X}, \quad \text{and} \quad U_{XXX} \approx \frac{U_{j+2} - 2U_{j+1} + 2U_{j-1} - U_{j-2}}{2(\Delta X)^3},$$

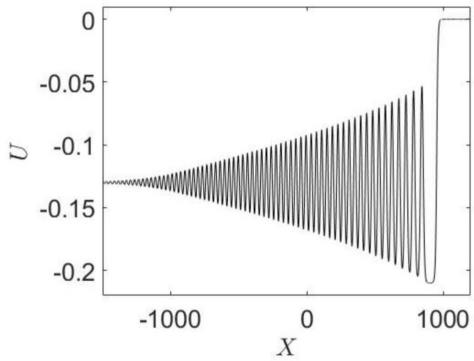
where *j* indicates the position along spatial axis and ΔX represents the increment value of the spatial axis. Therefore, the MOL approximation of the veKdV Eq. (4) is given by

$$\frac{\partial U_j}{\partial T} = -(\alpha U_j + \beta U_j^{\,2})\frac{U_{j+1} - U_{j-1}}{2\Delta X} - \lambda \frac{U_{j+2} - 2U_{j+1} + 2U_{j-1} - U_{j-2}}{2(\Delta X)^3} = f(U_j).$$

To solve for the time integration, we adopt the classical fourth-order Runge-Kutta method.



(a)



(b)

Fig. 2. The structure of internal undular bore of depression: (a) KdV-type internal undular bore where *b* = –0.10, (b) Table-top internal undular bore where *b* = –0.13

The initial condition of veKdV Eq. (4) is taken as

$$U(X) = \frac{b}{2}\left(1 - \tanh\left(\frac{X}{20}\right)\right),$$

where *b* denotes the height of the sharp step. Here, we consider two values for *b*, i.e. *b* = –0.10 and *b* = –0.13 so that we would have a KdV-type solitary wave and a table-top solitary wave as the leading wave of the internal undular bore of depression (refer to Figure 2(a) and Figure 2(b)). The values for H_1 and H_2 are chosen to be 1.0 and 1.5 respectively. The amplitude of leading solitary wave of KdV-type internal undular bore is $U_{lim0} \approx 2b = -0.20$ and the leading solitary wave of table-top internal undular bore has a limiting amplitude, i.e.,

$$U_{\lim 0} = \frac{-\alpha}{\beta} = -0.21.$$

We would like to observe how the varying depth would affect the transformation of the internal undular bore especially the leading solitary wave when it propagates over the slowly varying slope.

4. Numerical Results

The numerical results are divided into two categories, i.e., one that involves polarity change and one that without polarity change.

4.1 Transformation of Internal Undular Bore Involves Polarity Change

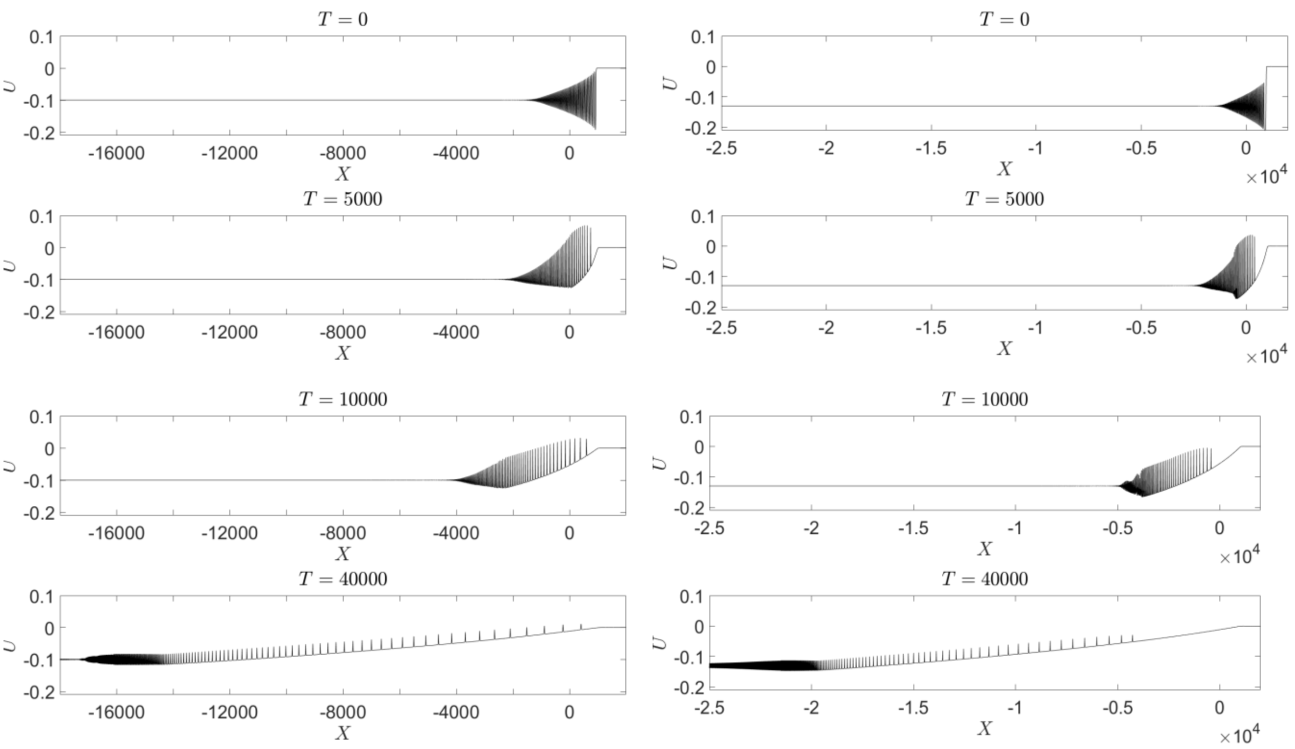
We observed that the polarity of the internal undular bore changes from negative to polarity when it propagates over a slowly increasing slope where the depth of the lower layer after the slope is less than or equal to the depth of the upper layer, i.e. $h_1 \leq H_1$.

During the shoaling process, the internal wave of depression normally involves polarity change when it propagates from the deep-sea region to the coastal area [34–36]. This polarity conversion can be well-explained by the KdV theory which stated that the polarity changes for the wave solutions is based on the change of sign of coefficient in the quadratic nonlinear term, α [37–38]. When the internal undular bore of depression is propagating on the slowly increasing slope region, the negative coefficient of quadratic nonlinear term, α in Eq. (4) is slowly increasing toward positive value. Once the polarity of the internal undular bore has changed to a positive value, the leading solitary wave deforms adiabatically so that the amplitude of the leading solitary wave increases slowly. At the same time, there is a non-adiabatic respond where there is a generation of a set of isolated solitary waves with different amplitude or a solitary wavetrain in front of the transformed bore as the result of the interaction of the undular bore with the slowly varying bottom surface. The solitary wavetrain is seen to be riding on negative pedestal due to the rarefaction wave propagating in the background caused by the polarity change. As the result, the isolated solitary wave is diminishing as it climbs the pedestal and the amplitude of each isolated solitary wave in the wavetrain is slowly decreasing on the negative pedestal. Figure 3 shows the transformation of the internal undular bore over the slope where $h_1 < H_1$ after the slope. Figure 4 shows the contour and 3D plots of the evolution of the internal undular bore. The depth profile of lower layer is given by

$$H_2(T) = \begin{cases} 1.5 & : 0 \leq T < 100, \\ -0.00016T + 1.516 & : 100 \leq T < 5100, \\ 0.7 & : T \geq 5100, \end{cases}$$

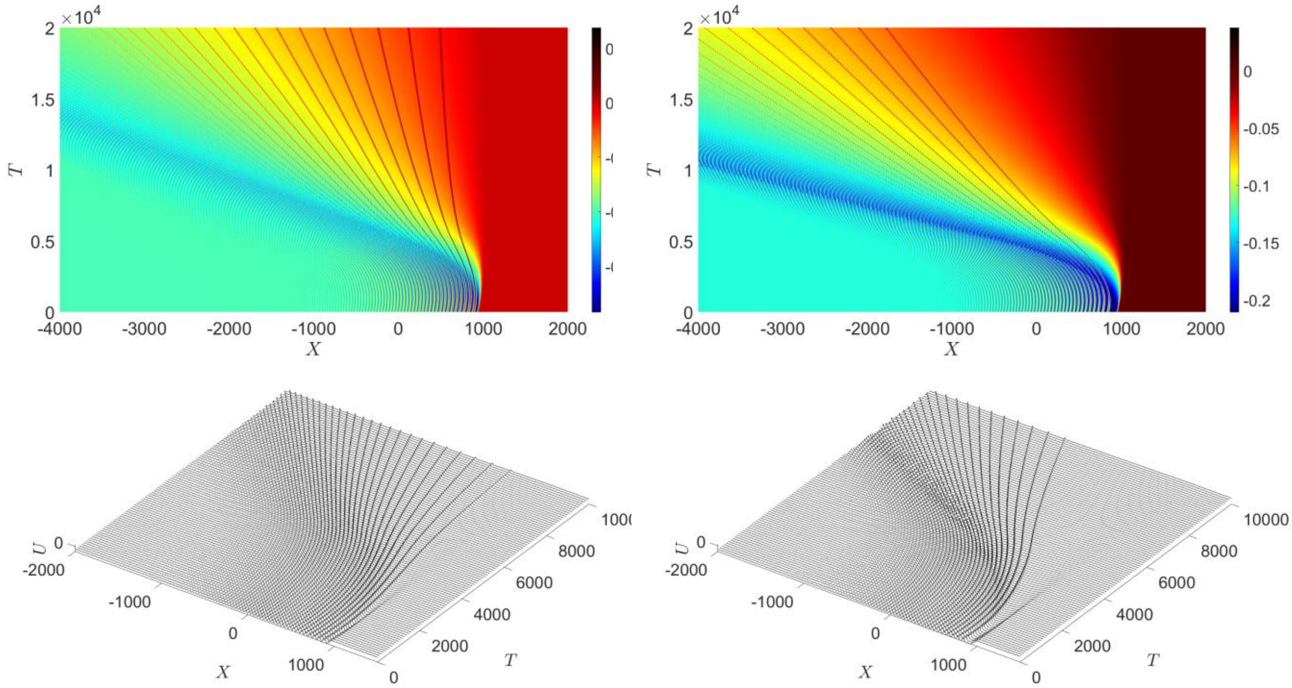
Figure 5 shows the transformation of the internal undular bore over the slope where $h_1 = H_1$ after the slope. Figure 6 shows the contour and 3D plots of the evolution of the internal undular bore. The depth profile of lower layer is

$$H_2(T) = \begin{cases} 1.5 & : \ 0 \leq T < 100, \\ -0.0001T \ + \ 1.51 & : \ 100 \leq T < 5100, \\ 1.0 & : \ T \geq 5100. \end{cases}$$



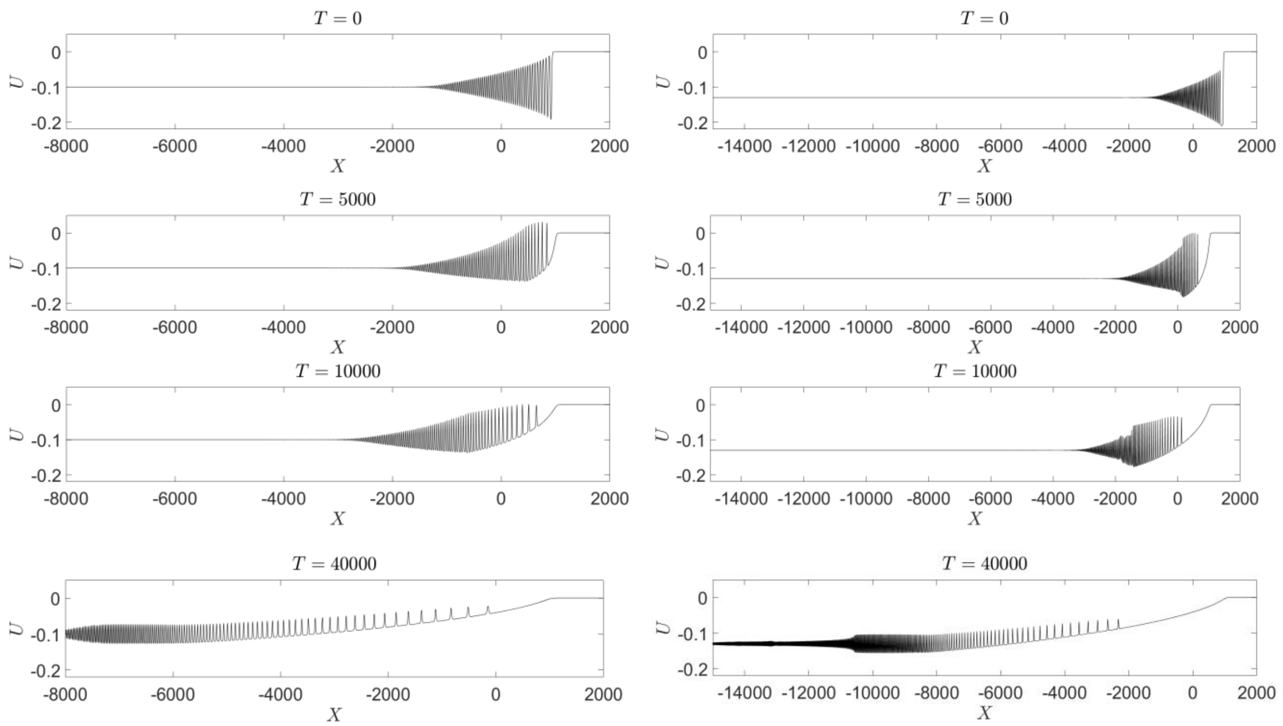
(a) (b)

Fig. 3. 2D plots for the propagation of internal undular bores of depression over a slowly increasing slope where $h_1 < H_1$: (a) KdV-type internal undular bore where b = –0.10 and (b) Table-top internal undular bore where b = –0.13



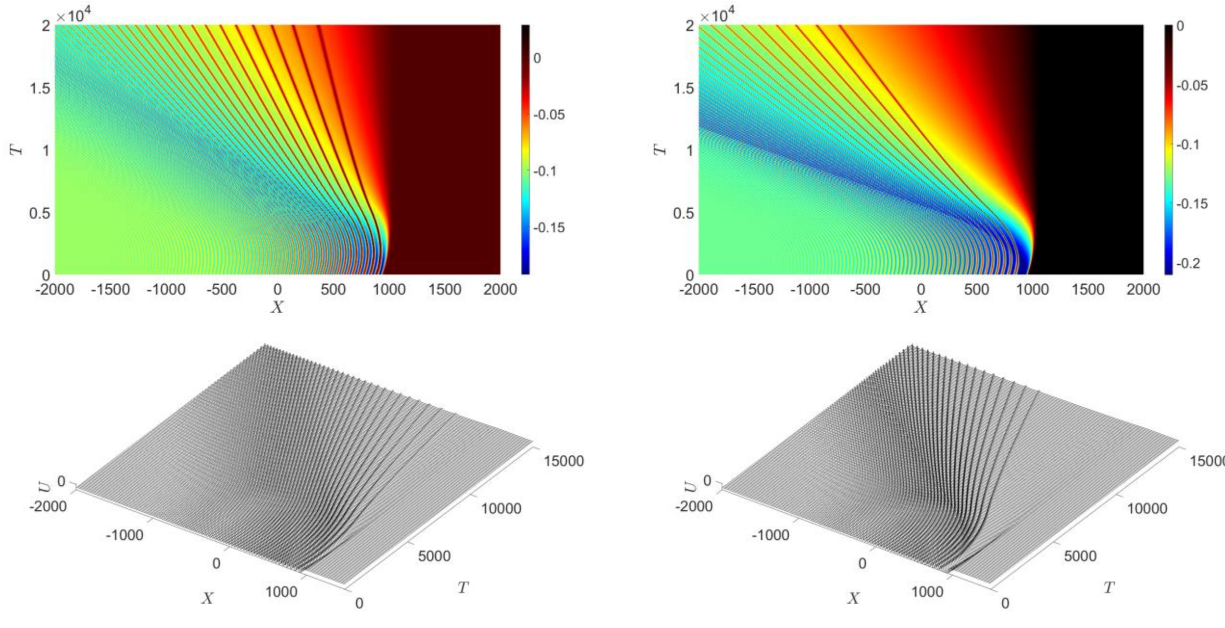
(a) (b)

Fig. 4. Contour plot (upper panel) and 3D plot (lower panel) for the propagation of internal undular bores of depression over a slowly increasing slope where $h_1 < H_1$: (a) KdV-type internal undular bore where b = –0.10 and (b) Table-top internal undular bore where b = –0.13



(a) (b)

Fig. 5. 2D plots for the propagation of internal undular bores of depression over a slowly increasing slope where $h_1 = H_1$: (a) KdV-type internal undular bore where b = –0.10 and (b) Table-top internal undular bore where b = –0.13



(a) (b)

Fig. 6. Contour plot (upper panel) and 3D plot (lower panel) for the propagation of internal undular bores of depression over a slowly increasing slope where $h_1 = H_1$: (a) KdV-type internal undular bore where b = –0.10 and (b) Table-top internal undular bore where b = –0.13

4.2 Transformation of Internal Undular Bore Without Polarity Change

When the depth after the increasing slope is still greater than the depth of the upper layer, i.e. $h_1 > H_1$, the polarity of the internal undular bore remains unchanged throughout the entire propagation. The leading solitary wave deforms adiabatically and its amplitude decreases as it propagates over the slope. However, the value of the limiting amplitude changes to a new value, i.e. $U_{lim1} = -0.0557$, which is a smaller value compared to the limiting amplitude value before the slope. Therefore, we observe the formation of a step-like wave of negative polarity. Figure 7 shows the transformation of the internal undular bore over the slope where $h_1 > H_1$ after the slope. Figure 8 shows the contour and 3D plots of the evolution of the internal undular bore. The depth profile of lower layer is given by

$$H_2(T) = \begin{cases} 1.5 & : \ 0 \le T < 100, \\ -0.00006T + 1.506 & : \ 100 \le T < 5100, \\ 1.2 & : \ T \ge 5100. \end{cases}$$

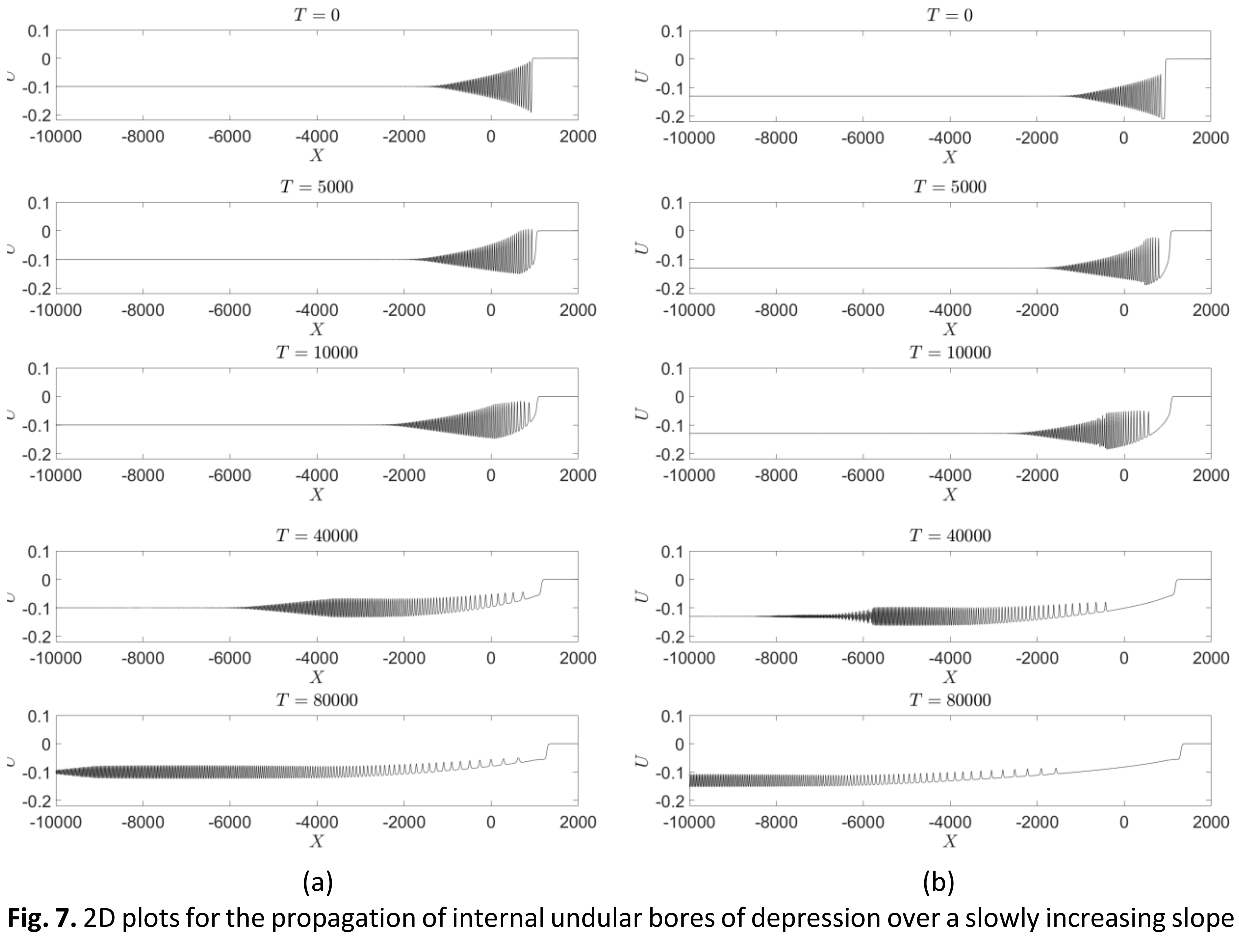


Fig. 7. 2D plots for the propagation of internal undular bores of depression over a slowly increasing slope where $h_1 > H_1$: (a) KdV-type internal undular bore where $b = -0.10$ and (b) Table-top internal undular bore where $b = -0.13$

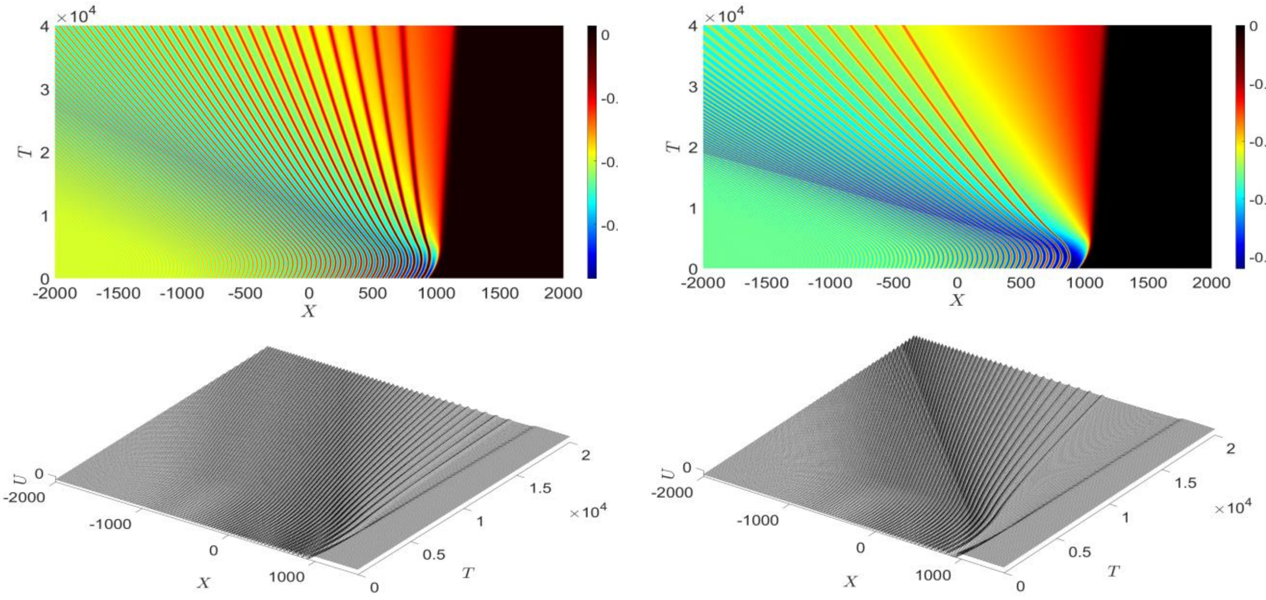


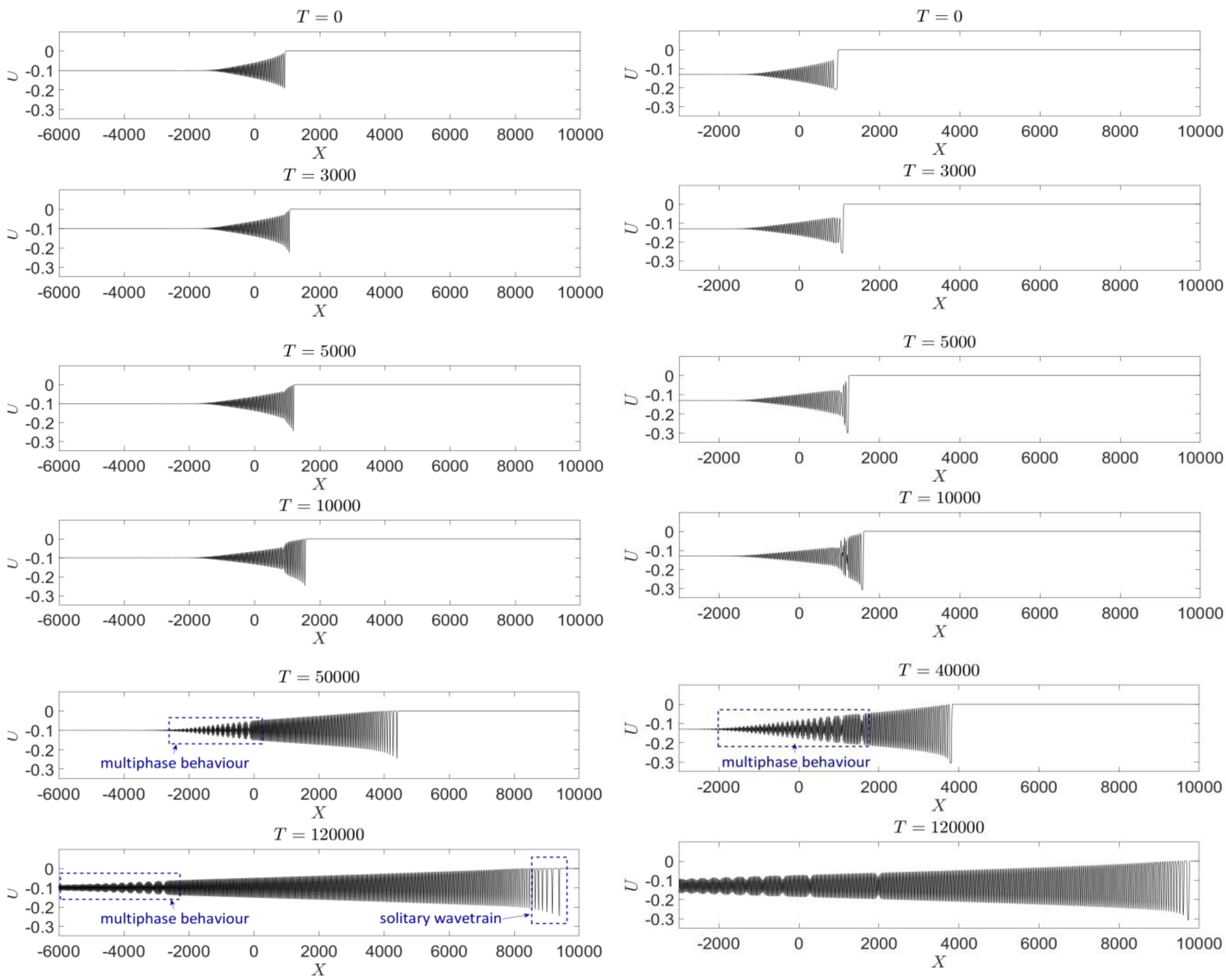
Fig. 8. Contour plot (upper panel) and 3D plot (lower panel) for the propagation of internal undular bores of depression over a slowly increasing slope where $h_1 > H_1$: (a) KdV-type internal undular bore where b = –0.10 and (b) Table-top internal undular bore where b = –0.13

Now, we look at the case where the bottom slope is decreasing slowly. For the KdV-type internal undular bore, i.e., when b = –0.10, the leading solitary wave of the internal undular bore behaves as an isolated solitary wave and it deforms adiabatically. Its amplitude increases as it enters the varying depth region. Once it reaches new constant depth region, the amplitude stops increasing and remains constant throughout the propagation. It also has a non-adiabatic respond where a solitary wavetrain is generated. This phenomenon has been observed as well when a surface undular bore evolves over a slowly varying region [39]. Also, a multi-phase interaction is observed during the evolution. The multi-phase interaction carries on for quite some time and it diminishes after the transformed bore has settled down on the new constant region.

Figure 9(a) shows the evolution of the KdV-type internal undular bore over the slowly decreasing slope region. The depth of lower layer varies according to the following

$$H_2(T) = \begin{cases} 1.5 & : \ 0 \le T < 100, \\ 0.00004T + 1.496 & : \ 100 \le T < 5100, \\ 1.7 & : \ T \ge 5100. \end{cases}$$

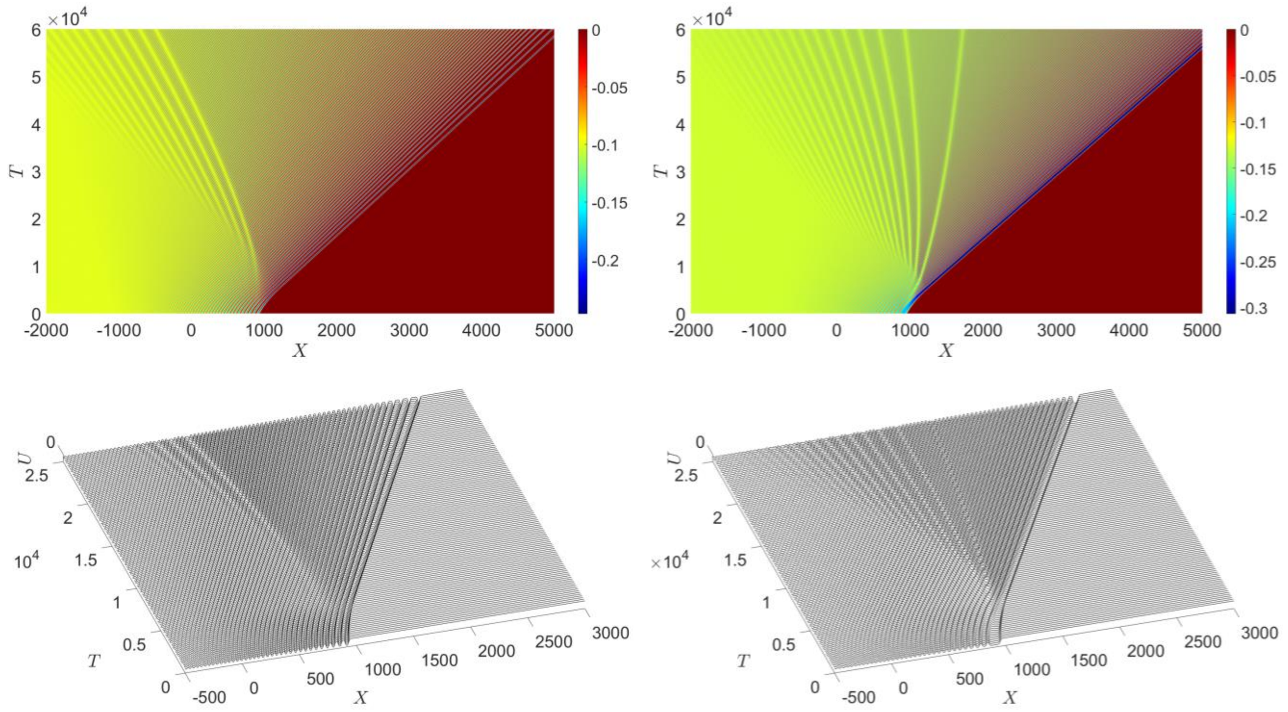
Figure 10(a) shows the contour and 3D plots of the evolution of the internal undular bore. However, we do not observe the generation of solitary wavetrain for the table-top internal undular bore, i.e., when b = –0.13. The leading table-top solitary wave also behaves as an isolated solitary wave such that it deforms adiabatically and reach a new limiting amplitude, i.e., $U_{\lim 2}$ = –0.3077. We also observe that the occurrence of multi-phase behaviour due (see Figure 9(b)). The contour plot and 3D plots of the transformation of internal undular bore over a slowly decreasing slope are shown in Figure 10(b).



(a) (b)

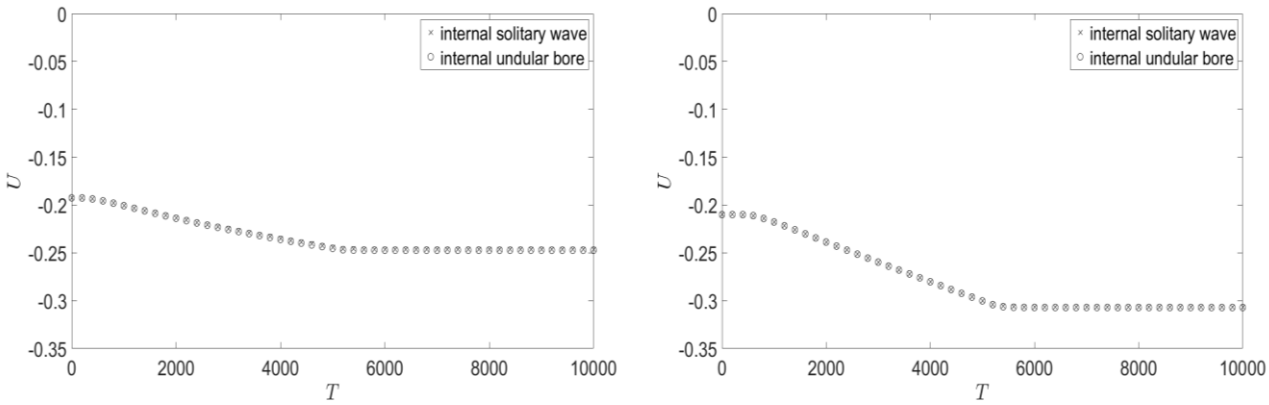
Fig. 9. 2D plots for the propagation of internal undular bores of depression over a slowly decreasing slope: (a) KdV-type internal undular bore where *b* = –0.10 and (b) Table-top internal undular bore where *b* = –0.13

Figure 11 shows the comparison of the amplitude variation between the leading solitary wave of the internal undular bore and an isolated solitary wave propagating over a slowly decreasing region. This shows that the leading solitary wave is indeed behaving as an isolated solitary wave during the evolution.



(a) (b)

Fig. 10. Contour plot (upper panel) and 3D plot (lower panel) for the propagation of internal undular bores of depression over a slowly decreasing slope: (a) KdV-type internal undular bore where *b* = –0.10 and (b) Table-top internal undular bore where *b* = –0.13



(a) (b)

Fig. 11. Comparison of the amplitude variation between the leading solitary wave of internal undular bore and an isolated solitary wave: (a) *b* = –0.10 (b) *b* = –0.13

5. Conclusions

In this paper, we have simulated the propagation of internal undular bore of negative polarity with two types of leading wave, i.e., KdV-type and table-top over different types of slowly varying topography in a two-layer fluid system. When the internal undular bores propagates over a slowly increasing region, the polarity of the internal undular bore may change depending on the depth of the constant region after the slope. If the depth after the slope is less than or equal to the depth of the upper layer, then the polarity of the internal undular bore changes from negative polarity to a positive polarity. In this case, we notice that there is a generation of solitary wavetrain riding on a negative pedestal. At larger time-scale, the amplitude of the solitary waves in the solitary wave train decreases and diminishes due to the pedestal. However, if there is no polarity change during transformation, the internal undular bore deforms adiabatically until the amplitude of leading wave reaches the new limiting amplitude. This leads to the formation of a step-like wave at the front of the transformed bore. When the internal undular bore propagates over a slowly decreasing slope, the leading wave of the KdV-type internal undular bore behaves as an isolated solitary wave and deforms adiabatically. A non-adiabatically respond in the form of a solitary wavetrain is observed at the front of the transformed bore. However, no solitary wavetrain is generated during the transformation of table-top internal undular bore even though the leading wave also behaves as an isolated solitary wave. These simulation results can be used to provide a useful insight to understand the evolution of internal undular bore of depression propagating in ocean layers for the prevention of oceanic disasters.

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